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Finite-Time Convergence Control for Acyclic Persistent Formations

Myoung-Chul Park†, Zhiyong Sun‡, Kwang-Kyo Oh†, Brian D. O. Anderson‡, and Hyo-Sung Ahn†

***Abstract*— In this paper, we propose a distance-based control law for acyclic persistent formations of mobile agents. The proposed normalized gradient law, which can be implemented distributively by using local measurements, allows agents to achieve their desired formation shape specified by inter-agent distance constraints in finite time, with local but not global convergence. We show some local finite-time convergence properties including an upper bound for the convergence time. Further the existence of attractive incorrect equilibrium formations is demonstrated. Simulation results are provided for illustration.**

I. Introduction

Formation control of mobile autonomous agents has attracted a lot of research interest because it is a fundamental task in distributed multi-agent systems. For example, it is necessary to form a formation to move in the plane without collision among the agents, to avoid obstacles, or to conduct a surveillance in a cooperative way. In particular, various solutions for displacement-based formation control are provided in [1]–[3] by virtue of consensus algorithms.

Unlike displacement-based formation control, we can obtain a certain formation shape by controlling inter-agent distances, and such control strategies are called distance-based formation control. If the task of maintaining the distance between two agents is given to both of the agents, the formations are modeled by undirected graphs [4]–[8], while if the task is given to one of two agents, the formations are modeled by directed graphs [9]–[12]. One can view the former case as a particular instance of the latter case.

The aforementioned existing results ensure only asymptotic convergence of formation shapes. Distinguished from such results, Cao et al. have shown that finite-time convergence to the desired formation shape can be achieved by using a sliding mode estimator in displacement-based formation control [13]. In the case of the distance-based control approach, Sun et al. have revealed that finite-time convergence of a formation associated with a minimally rigid graph is achieved by modifying the existing gradient control law proposed in [4], [14].

In this paper, we deal with finite-time control of formations with directed acyclic persistent graphs by using distance-based formation control. By taking advantage of acyclicity of the formation graph, we show that we can make an ordered sequence of subformations converge successively. We use a normalized gradient control law which is motivated from [4] and [15]. Further we show by way of an explicit example that there can exist an attractive incorrect equilibrium formation even if the shape of the desired formation is uniquely determined by given inter-agent distance constraints.

The rest of the paper is organized as follows. In Section II, we review background knowledge on graph theory and some results on acyclic persistent graphs. The dynamic model of each agent and the control law are also introduced in this section. Next, in Section III, we formulate the main results on finite-time convergence of formations with an acyclic persistent graph. Simulation results verifying our analysis are contained in Section IV. Finally, we summarize the paper in Section V.

†School of Mechatronics, Gwangju Institute of Science and Technology (GIST), 123 Cheomdan-gwagiro, Buk-gu, Gwangju, Republic of Korea. E-mail: mcpark@gist.ac.kr, kwangkyo.oh@gmail.com, hyosung@gist.ac.kr

‡Zhiyong Sun is with Shandong Computer Science Center (SCSC), Jinan, China; Brian D. O. Anderson was a visiting expert with SCSC. Zhiyong Sun and Brian D. O. Anderson are with National ICT Australia and Research School of Engineering, The Australian National University, Canberra ACT 0200, Australia. {zhiyong.sun, brian.anderson}@anu.edu.au

II. Preliminaries

In the rest of the paper, we use the following notations:

- $\mathbb{R}^n$: n-dimensional Euclidean space
- $|\mathcal{S}|$: the cardinality of a set $\mathcal{S}$
- I_n: the n by n identity matrix
- $\underline{\lambda}_{>0}(M)$: the smallest positive eigenvalue of a symmetric matrix M
- $\|\mathbf{x}\|$: the Euclidean norm of a vector $\mathbf{x}$
- $\nabla_{\mathbf{x}} f = [\partial f / \partial \mathbf{x}]^\top$ for a differentiable scalar function f
- critical $f(\mathbf{x}) = \{\mathbf{x} \in \mathbb{R}^n : \nabla f(\mathbf{x}) = \mathbf{0},\ f \text{ is differentiable}\}$
- $\text{dist}(\mathbf{x}, \mathcal{A}) = \inf_{\mathbf{y} \in \mathcal{A}} \|\mathbf{x} - \mathbf{y}\|$
- $\text{Hess}_{\mathbf{x}} f = \left[\partial^2 f / \partial \mathbf{x}^2\right]$ for a second-order differentiable scalar function f

A. Formation and graph representation

We use directed graphs to represent formations by relating the vertices to agents. For a given graph, each vertex of the graph represents the corresponding agent, and the set of all vertices is given by $\mathcal{V} = \{1, \ldots, N\}$. An edge (i, j) of the graph denotes that agent i measures the relative position of agent j and controls its distance from agent j. The set of all edges is denoted by $\mathcal{E}$. When $(i, j) \in \mathcal{E}$, we call agent j a neighbor of agent i. Hence, the set of all neighbors of agent i is defined by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The direction of an edge shows which agent is supposed to maintain the edge length. Note that if $(i, j) \in \mathcal{E}$, not only is agent i responsible for maintaining the prescribed distance from agent j but also agent j does not care about its distance from agent i.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a graph describing the given formation. We use $\mathbf{p}_i = [x_i \quad y_i]^\top \in \mathbb{R}^2$ to represent the position vector of agent i, $\forall i \in \mathcal{V}$, and the concatenated vector of the position vectors is denoted by $\mathbf{p} = [\mathbf{p}_1^\top \quad \ldots \quad \mathbf{p}_N^\top]^\top \in \mathbb{R}^{2N}$.

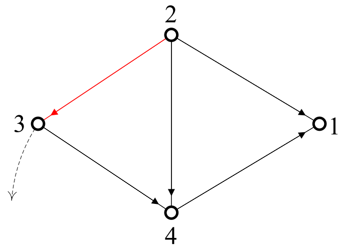


Fig. 1. An example of rigid but non-persistent graph mentioned in [16]

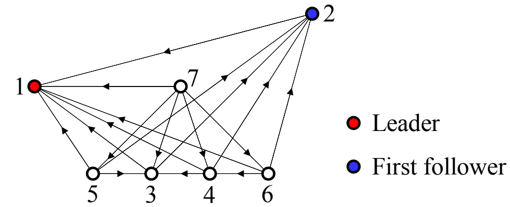


Fig. 2. An example of acyclic persistent graph: a trilateration graph

We say $\mathbf{p}$ is a *realization* of $\mathcal{G}$ in $\mathbb{R}^2$ and call the pair $(\mathcal{G},\mathbf{p})$ a *framework*. Two realizations $\mathbf{p}$ and $\mathbf{p}'$ of $\mathcal{G}$ are said to be *congruent* if $\|\mathbf{p}_i-\mathbf{p}_j\| = \|\mathbf{p}'_i-\mathbf{p}'_j\|$ for all $i,j\in\mathcal{V}$. Let a realization $\bar{\mathbf{p}}\in\mathbb{R}^{2N}$ be a representative of the desired formation shape. Then our objective is to make $\mathbf{p}$ converge to a realization which is congruent to $\bar{\mathbf{p}}$ by using the relative position measurements for which the sensing topology is given by $\mathcal{G}$. In the rest of the paper, all frameworks considered are supposed to be in 2-dimensional space.

In distance-based formation control, a necessary condition for the underlying graph topology to achieve the desired formation shape by achieving the desired inter-agent distances is graph rigidity. For example, the shape of a ring graph with four vertices cannot be maintained only by preserving the edge lengths in 2-dimensional space in general while the shape of another ring graph with three vertices is uniquely determined by fixed edge lengths provided that the triangle inequality is satisfied. On the other hand, if the formation graph has directed topology, we may not be able to preserve all the edge lengths. For example, suppose that all edge lengths are equal to the desired lengths in Fig. 1 at initial time. If agent 3 moves on the circle centered at the location of agent 4 satisfying the prescribed distance from agent 4, then agent 2 cannot maintain all three edge lengths from agents 1, 3 and 4 anymore. Hendrickx, Yu, Anderson, and their colleagues propose graph *persistence* to characterize the graphs of which the shape can be maintained with directed topology based on distance preservation [16]–[18].

The main approach that we are going to take is to drive each agent to the location in which the agent maintains the desired distances from its neighbors provided that all those neighbors have maintained their desired distances already. Hence we assume the graph of the desired formation is an *acyclic persistent graph*. The acyclic property allows the neighbor property just mentioned and the persistence property guarantees rigidity of the formation under the directed controls associated with each edge. An example of such an acyclic persistent graph is shown in Fig. 2. More information on acyclic persistent graphs is found in [16, Section 5].

From the fact that the formation graph is acyclic, we can order the indices of the vertices so that $\mathcal{N}_i\subseteq\{1,\dots,i-1\}$ for each $i\geq 2$. In this case, we call agent 1 the *leader*, agent 2 the *first follower*, and the other agents the *ordinary followers*. A necessary and sufficient condition for an acyclic graph to be persistent is given in the following theorem.

Theorem 1 (Theorem 5 in [16]): An acyclic graph having more than one vertex is persistent in $\mathbb{R}^2$ if and only if

- One vertex (called the leader) has an out-degree 0.
- One vertex (called the first follower) has an out-degree 1 and the corresponding edge is incident to the leader.
- Every other vertex has an out-degree larger or equal to 2.

Therefore, each ordinary follower of an acyclic persistent graph has at least two outgoing edges (equivalently two neighbors). We assume that any pair of two agents is not collocated in the initial time. We additionally assume that, for each ordinary follower i, at least two relative displacements to its neighbors in $\bar{\mathbf{p}}$ are linearly independent, namely at least two elements in $\{\mathbf{x}\in\mathbb{R}^2\colon \mathbf{x}=\bar{\mathbf{p}}_j-\bar{\mathbf{p}}_i, j\in\mathcal{N}_i\}$ are linearly independent. The independence condition will be required for the proof of finite-time convergence.

We further assume that the vertices are ordered corresponding to a *topological sort*, which is a linear ordering of the vertices such that if there is a directed edge (u,v) from vertex u to vertex v, then u comes after v in the ordering. (Conventionally, a topological sort would require vertex u to come before vertex v, but this is simply a matter of convention). The leader node is the first node, and the first follower is the second node.

B. Equations of motion

We define squared-distance errors by $e_{ij}=\|\mathbf{p}_i-\mathbf{p}_j\|^2-\|\bar{\mathbf{p}}_i-\bar{\mathbf{p}}_j\|^2$ for all $(i,j)\in\mathcal{E}$. For each $i\in\mathcal{V}$, let $\mathbf{q}_i$ denote a concatenated vector defined by $\mathbf{q}_i=[\dots\ \ \mathbf{p}_j^\top\ \ \dots]^\top\in\mathbb{R}^{2|\mathcal{N}_i|}$, and $\bar{\mathbf{q}}_i=[\dots\ \ \bar{\mathbf{p}}_j^\top\ \ \dots]^\top\in\mathbb{R}^{2|\mathcal{N}_i|}$, $\forall j\in\mathcal{N}_i$. Then a local potential function ϕ_i of agent i is defined by $\phi_i(\mathbf{p}_i,\mathbf{q}_i)=\frac{1}{4}\sum_{j\in\mathcal{N}_i}e_{ij}^2$. We use a single integrator model to describe the motion of each agent. Thus we have $\dot{\mathbf{p}}_i=\mathbf{u}_i$, $\forall i\in\mathcal{V}$, where $\mathbf{u}_i$ is the control input which is supposed to be constructed from local measurements.

In the literature some researchers use the gradient-descent algorithm to minimize the local potential functions [4], [5], [8], namely they use

$$\mathbf{u}_i=-\nabla_{\mathbf{p}_i}\phi_i=\sum_{j\in\mathcal{N}_i}(\mathbf{p}_j-\mathbf{p}_i)e_{ij},\quad \forall i\in\mathcal{V}, \tag{1}$$

and the results in [4], [5], [8] show the asymptotic convergence of $\mathbf{p}$ to the desired formation shape. Beyond the results of [4], [5], [8], we aim at achieving finite-time convergence to the desired formation shape. Thus, our objective is to achieve $\|\mathbf{p}_i(t)-\mathbf{p}_j(t)\|=\|\bar{\mathbf{p}}_i-\bar{\mathbf{p}}_j\|$, $\forall t\in[T_f,\infty)$, $\forall i,j\in\mathcal{V}$,

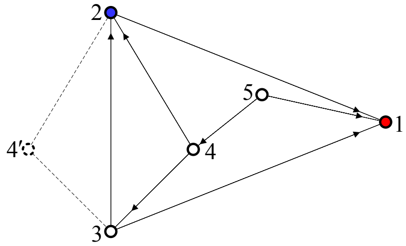


Fig. 3. Possible unrealizability due to flip ambiguity

and $\mathbf{p}(t) = \mathbf{p}^\infty$, $\forall t \in [T_f, \infty)$, where T_f is a positive constant, and $\mathbf{p}^\infty$ is a realization that is congruent to $\bar{\mathbf{p}}$ with $\|\mathbf{p}^\infty\| < \infty$.

To achieve finite-time convergence, we modify the control law in (1) by normalization as follows:

$$\dot{\mathbf{p}}_i = \mathbf{u}_i = -\frac{\nabla_{\mathbf{p}_i}\phi_i}{\|\nabla_{\mathbf{p}_i}\phi_i\|}, \quad \forall i \in \mathcal{V}. \tag{2}$$

For each agent, the right side in (2) is discontinuous at $\mathbf{p}_i$ for which $\|\nabla_{\mathbf{p}_i}\phi_i\| = 0$, so we understand the solution $\mathbf{p}_i$ in the sense of Filippov [15], [19].

We now explain the need to restrict the initial conditions. Consider a formation with acyclic persistent graph shown in Fig. 3. Suppose that all agents satisfy the desired inter-agent distances from their neighbors. Now we think of a situation such that agent 4 is positioned in the location of $4'$ satisfying the distance constraints. In this situation, agent 4 still satisfies two distance constraints from agents 2 and 3, but agent 5 cannot satisfy its two distance constraints if the sum of two distances of $(5,1)$ and $(5,4)$ is less than the distance of $(4',1)$. Therefore, depending on the initial conditions, we may not achieve the desired formation shape so we can consider only local stability. One may think that we could avoid the preceding situation by inserting more distance constraints in the graph, for example a constraint on $(4,1)$. However, there is another problem in which we cannot achieve the desired formation shape under the proposed control laws, which will be explained in Section III-C.

Consequently, we need to assume that the initial formation shape is sufficiently close to the desired one. Since the real issue is whether the shapes are close, we can without loss of generality work with a translated and rotated version of $\bar{\mathbf{p}}$. Accordingly, suppose that (after translation and rotation if necessary), $\mathbf{p}_1(0) = \bar{\mathbf{p}}_1$ and $\mathbf{p}_2(0) - \mathbf{p}_1(0)$ is aligned with $\bar{\mathbf{p}}_2 - \bar{\mathbf{p}}_1$. We assume that there is a sufficiently small positive constant ε such that $\|\mathbf{p}(0) - \bar{\mathbf{p}}\| < \varepsilon$. Let $\boldsymbol{\delta}_i = \mathbf{p}_i(0) - \bar{\mathbf{p}}_i$. Then we have $\|\boldsymbol{\delta}_i\| < \varepsilon$ for all $i \in \mathcal{V}$, and $\|\boldsymbol{\delta}_1\| = 0$ evidently.

III. Main Analysis

Since the leader does not have any neighbor, it is not required to maintain any prescribed distance. Thus the leader is supposed to stay in the initial location, which results in $\dot{\mathbf{p}}_1 = \mathbf{0}$ and $\mathbf{p}_1(t) = \bar{\mathbf{p}}_1$, $\forall t \in [0, \infty)$.

A. Motion of the first follower

From the fact that $\dot{\mathbf{p}}_2 = (\mathbf{p}_1 - \mathbf{p}_2)e_{12}/\|(\mathbf{p}_1 - \mathbf{p}_2)e_{12}\|$, we know that the first follower moves on the line through $\mathbf{p}_1(0)$ and $\mathbf{p}_2(0)$ if $\mathbf{p}_1(0) \neq \mathbf{p}_2(0)$. Let $w = \|\mathbf{p}_1 - \mathbf{p}_2\| - \|\bar{\mathbf{p}}_1 - \bar{\mathbf{p}}_2\|$. Then, from the geometric interpretation, we have

$$\dot{w} = -\operatorname{sgn} w, \tag{3}$$

where sgn is the signum function. Thus, w converges to zero in finite time, which means that $\mathbf{p}_2$ converges to $\bar{\mathbf{p}}_2$ in finite time. The convergence time τ_2 is given as $|\|\mathbf{p}_2(0) - \mathbf{p}_1(0)\| - d_{12}|$. From the fact that $\|\mathbf{p}_2(0) - \mathbf{p}_1(0)\| = \|(\bar{\mathbf{p}}_2 - \bar{\mathbf{p}}_1) + (\boldsymbol{\delta}_2 - \boldsymbol{\delta}_1)\|$, and $\|\boldsymbol{\delta}_i\| < \varepsilon$ for all $i \in \mathcal{V}$, we have

$$d_{12} - \varepsilon < \|\mathbf{p}_2(0) - \mathbf{p}_1(0)\| < d_{12} + \varepsilon,$$

(with $\|\boldsymbol{\delta}_1\| = 0$) which results in $\tau_2 < \varepsilon$.

Remark 1: Due to characteristics of the signum function, the solution of (3) will exhibit some chattering behavior in case there is some time delay from the input to the output of the function though we do not consider the case.

To formulate finite-time convergence, for each $i \in \mathcal{V}$, we define t_i^∞ as the time such that $\mathbf{p}_i(t) = \bar{\mathbf{p}}_i$ for all $t \in [t_i^\infty, \infty)$ so $t_1^\infty = 0$ evidently. Additionally, for each $i \in \{2, \ldots, N\}$, we define τ_i as $\tau_i = t_i^\infty - \max_{j \in \mathcal{N}_i} t_j^\infty$, i.e., τ_i is the elapsed time from the moment that all neighbors of agent i are located at the desired locations to when agent i is located at its desired location.

B. Motion of the ordinary followers

Now we turn our focus to the motion of the ordinary followers. Since the formation graph is acyclic, we are going to adopt an inductive argument. More precisely, we are going to show that each agent converges to the desired location in finite time if all neighbors of the agent achieve the desired distances in finite time. First, we show that, for all $i \in \mathcal{V}$, finite-time convergence of $\mathbf{p}_j$ to $\bar{\mathbf{p}}_j$, $\forall j \in \mathcal{N}_i$, implies finite-time convergence of $\mathbf{p}_i$ to $\bar{\mathbf{p}}_i$ with respect to (2). Unlike the case of the first follower ($i = 2$), the motion of the ordinary followers is not confined in a line and thus equations of the errors are not reduced to a simple form such as (3). Further, similar to Remark 1, the solution trajectory of the system in (2) will exhibit a chattering behavior in practice, i.e., the solution trajectory will zigzag near the local minimum of the potential function. In the sequel, however, we are going to consider only the ideal case. To handle the discontinuous right side in (2), we invoke the following theorem;

Theorem 2 (Theorem 8 in [15]): Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a second-order differentiable function. Let $\mathbf{x}_0 \in \mathcal{S} \subset \mathbb{R}^n$, with $\mathcal{S}$ compact and strongly invariant for

$$\dot{\mathbf{x}} = -\frac{\nabla_{\mathbf{x}} f(\mathbf{x})}{\|\nabla_{\mathbf{x}} f(\mathbf{x})\|}. \tag{4}$$

Assume there exists a neighborhood $\mathcal{X}$ of critical $f \cap \mathcal{S}$ in $\mathcal{S}$ where either one of the following conditions hold:

i) for all $\mathbf{x} \in \mathcal{X}$, $\mathrm{Hess}_{\mathbf{x}} f(\mathbf{x})$ is positive definite; or
ii) for all $\mathbf{x} \in \mathcal{X} \setminus (\text{critical} f \cap \mathcal{S})$, $\mathrm{Hess}_{\mathbf{x}} f(\mathbf{x})$ is positive semidefinite, the multiplicity of the eigenvalue 0 is constant, and $\nabla_{\mathbf{x}} f(\mathbf{x})$ is orthogonal to the eigenspace of $\mathrm{Hess}_{\mathbf{x}} f(\mathbf{x})$ corresponding to 0.

Then the solution of (4) starting from $\mathbf{x}_0$ converges in finite time to a critical point of f. Furthermore, if $\mathcal{X} = \mathcal{S}$,

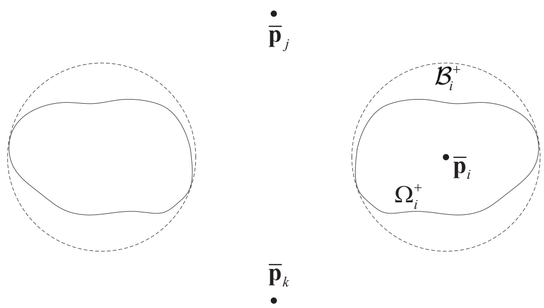


Fig. 4. Illustration of $\mathcal{B}_i^+$ and Ω_i^+, where $j,k \in \mathcal{N}_i$.

then the convergence time of the solutions of (4) starting from $\mathbf{x}_0$ is upper bounded by $\|(\nabla_{\mathbf{x}} f)(\mathbf{x}_0)\|/\lambda_0$, where $\lambda_0 = \min_{\mathbf{x}\in\mathcal{S}} \underline{\lambda}_{>0}(\mathrm{Hess}_{\mathbf{x}} f(\mathbf{x}))$.

Lemma 1: For $i \geq 2$, suppose that $\mathbf{p}_j = \bar{\mathbf{p}}_j$ for all $j \in \mathcal{N}_i$. Let $\mathcal{D}_i = \{\mathbf{p}_i \in \mathbb{R}^2\colon \phi_i(\mathbf{p}_i, \bar{\mathbf{q}}_i) = 0\}$. Then, $\mathrm{Hess}_{\mathbf{p}_i} \phi_i$ is positive definite for all $\mathbf{p}_i \in \mathcal{D}_i$, and there exists a neighborhood $\mathcal{B}_i(\gamma_i)$ of $\mathcal{D}_i$ such that $\mathrm{Hess}_{\mathbf{p}_i} \phi_i$ is positive definite for all $\mathbf{p}_i \in \mathcal{B}_i(\gamma_i)$, where $\mathcal{B}_i(\gamma_i) = \{\mathbf{p}_i \in \mathbb{R}^2\colon \mathrm{dist}(\mathbf{p}_i, \mathcal{D}_i) < \gamma_i\}$.

Proof: The Hessian of ϕ_i is given by

$$\mathrm{Hess}_{\mathbf{p}_i} \phi_i = \sum_{j\in\mathcal{N}_i} \left(2 \begin{bmatrix} (x_i - x_j)^2 & (x_i - x_j)(y_i - y_j) \\ (x_i - x_j)(y_i - y_j) & (y_i - y_j)^2 \end{bmatrix} + e_{ij} I_2 \right) = 2A_i^\top A_i + \sum_{j\in\mathcal{N}_i} e_{ij} I_2,$$

where

$$A_i = \begin{bmatrix} (\mathbf{p}_i - \mathbf{p}_{j_1})^\top \\ \vdots \\ (\mathbf{p}_i - \mathbf{p}_{j_{|\mathcal{N}_i|}})^\top \end{bmatrix} \in \mathbb{R}^{|\mathcal{N}_i| \times 2}, \quad \{j_1, \ldots, j_{|\mathcal{N}_i|}\} = \mathcal{N}_i.$$

For $|\mathcal{N}_i| \geq 2$, since $A_i^\top A_i$ is positive definite if and only if A_i has full column rank, $A_i^\top A_i$ is positive definite if and only if at least two of the outgoing edges of agent i are linearly independent. Thus, for all $\mathbf{p}_i \in \mathcal{D}_i$, we have $e_{ij} = 0$, $\forall j \in \mathcal{N}_i$, and $\mathrm{Hess}_{\mathbf{p}_i} \phi_i$ is positive definite. Since the eigenvalues of $\mathrm{Hess}_{\mathbf{p}_i} \phi_i$ are continuous in $\mathbf{p}_i$, there is a positive constant γ_i such that $\mathrm{Hess}_{\mathbf{p}_i} \phi_i$ is positive definite for all $\mathbf{p}_i \in \mathcal{B}_i(\gamma_i)$. ■

Using the positive definiteness of $\mathrm{Hess}_{\mathbf{p}_i} \phi_i$ at the equilibrium point, we can show finite-time convergence of agent i under the assumption that all neighbors of agent i are at the desired locations. The set $\mathcal{B}_i(\gamma_i)$ defined in Lemma 1 either would be a connected set or could be partitioned into two separated subsets. In the latter case, let $\mathcal{B}_i^+(\gamma_i)$ be the subset including $\bar{\mathbf{p}}_i$, and $\mathcal{B}_i^-(\gamma_i) = \mathcal{B}_i(\gamma_i) \setminus \mathcal{B}_i^+(\gamma_i)$. If $\mathcal{B}_i(\gamma_i)$ is a connected set, then let $\mathcal{B}_i^+(\gamma_i) = \mathcal{B}_i(\gamma_i)$ (see Fig. 4).

Theorem 3: For $i \geq 2$, suppose that $\mathbf{p}_j(t) = \bar{\mathbf{p}}_j$, $\forall j \in \mathcal{N}_i$, $\forall t \in [\nu_i, \infty)$, where $\nu_i = \max_{j\in\mathcal{N}_i} t_j^\infty$. Let $\Omega_i(\rho_i) = \{\mathbf{p}_i \in \mathbb{R}^2\colon \phi_i(\mathbf{p}_i, \bar{\mathbf{q}}_i) \leq \rho_i\}$ and choose ρ_i such that $\Omega_i(\rho_i) \subseteq \mathcal{B}_i(\gamma_i)$. Let $\Omega_i^+(\rho_i) = \Omega_i(\rho_i) \cap \mathcal{B}_i^+(\gamma_i)$. If $\mathbf{p}_i(\nu_i)$ is in $\Omega_i^+(\rho_i)$, then $\mathbf{p}_i$ converges in finite time to $\bar{\mathbf{p}}_i$.

Proof: Since the direction of $-\nabla_{\mathbf{p}_i}\phi_i$ is the steepest descent direction reducing ϕ_i at the level surface $\{\mathbf{p}_i\colon \phi_i(\mathbf{p}_i, \bar{\mathbf{q}}_i) = \rho_i\}$, the set $\Omega_i^+(\rho_i)$ is compact and strongly invariant for the flow in (2) on $t \in [\nu_i, \infty)$. From the fact that $\mathcal{B}_i^+(\gamma_i)$ is the neighborhood of $\mathcal{D}_i \cap \Omega_i^+(\rho_i)$, the condition i) in Theorem 2 is satisfied. Therefore, $\mathbf{p}_i$ converges in finite time to $\bar{\mathbf{p}}_i$, and the convergence time τ_i is upper bounded by

$$\tau_i \leq \frac{\|(\nabla_{\mathbf{p}_i}\phi_i)(\mathbf{p}_i(\nu_i), \bar{\mathbf{q}}_i)\|}{\min\limits_{\mathbf{p}_i\in\Omega_i^+(\rho_i)} \underline{\lambda}_{>0}(\mathrm{Hess}_{\mathbf{p}_i} \phi_i(\mathbf{p}_i, \bar{\mathbf{q}}_i))} \tag{5}$$

■

In Theorem 3, the results are established on the assumption that all neighbors of agent i are at the desired locations. However, those neighbors take some time to converge to the desired locations. Thus we need to investigate whether or not $\mathbf{p}_i$ stays in $\Omega_i^+(\rho_i)$ until all neighbors of agent i converge to the desired locations. Since the speed of each agent is bounded by 1, the maximal distance away from the initial position is bounded by the elapsed time. Hence, $\mathbf{p}_i(t)$ exists on $\{\mathbf{p}_i \in \mathbb{R}^2\colon \|\bar{\mathbf{p}}_i - \mathbf{p}_i\| \leq \varepsilon_i\}$ for each $t \in [0, \nu_i]$, where $\varepsilon_i = \|\boldsymbol{\delta}_i\| + \nu_i$. Let $\nu_i = \max_{j\in\mathcal{N}_i} t_j^\infty$ as we did in Theorem 3. From the definition of t_j^∞, we have

$$\begin{aligned} \nu_i &= \tau_j + \max_{k\in\mathcal{N}_j} t_k^\infty, \quad j = \arg\max_{h\in\mathcal{N}_i} t_h^\infty \\ &= \tau_j + \tau_k + \max_{l\in\mathcal{N}_k} t_l^\infty, \quad j = \arg\max_{h\in\mathcal{N}_i} t_h^\infty, \quad k = \arg\max_{h\in\mathcal{N}_j} t_h^\infty \\ &\ \ \vdots \\ &\leq \sum_{m=2}^{j} \tau_m. \end{aligned}$$

For all $m \in \{3, \ldots, j\}$, τ_m is bounded by a term multiplied by $\|(\nabla_{\mathbf{p}_m}\phi_m)(\mathbf{p}_m(\nu_m))\|$ which converges to 0 as $\varepsilon \to 0$, and $\tau_2 < \varepsilon$. Moreover, we know that $\lim_{\varepsilon\to 0} \|\boldsymbol{\delta}_i\| = 0$ from the fact that $\|\mathbf{p}(0) - \bar{\mathbf{p}}\| < \varepsilon$. Therefore, we can let ε_i be small enough by taking sufficiently small ε such that $\{\mathbf{p}_i \in \mathbb{R}^2\colon \|\bar{\mathbf{p}}_i - \mathbf{p}_i\| \leq \varepsilon_i\} \subseteq \Omega_i^+(\rho_i)$.

Theorem 4: For $i \in \mathcal{V}$, if the finite-time convergence of $\mathbf{p}_j$ to $\bar{\mathbf{p}}_j$ for all $j \in \mathcal{N}_i$ implies the finite-time convergence of $\mathbf{p}_i$ to $\bar{\mathbf{p}}_i$ with respect to (2), then the overall system of N agents converges to $\bar{\mathbf{p}}$ in finite time.

Proof: Since $\mathcal{N}_i \subseteq \{1, \ldots, i-1\}$ for all $i \in \{2, \ldots, N\}$, and $\mathbf{p}_1 = \bar{\mathbf{p}}_1$, the proposition is true by induction. ■

We then present the main result:

Theorem 5: Under the control law given in (2), the overall system with N agents achieves the desired formation shape in finite time with sufficiently small ε, and the convergence time is upper bounded by

$$t^\infty < \varepsilon + \sum_{i=3}^{N} \frac{\|(\nabla_{\mathbf{p}_i}\phi_i)(\mathbf{p}_i(\nu_i), \bar{\mathbf{q}}_i)\|}{\min\limits_{\mathbf{p}_i\in\Omega_i^+(\rho_i)} \underline{\lambda}_{>0}(\mathrm{Hess}_{\mathbf{p}_i} \phi_i(\mathbf{p}_i, \bar{\mathbf{q}}_i))}.$$

where t^∞ is the time such that $\mathbf{p}(t) = \bar{\mathbf{p}}$ for all $t \in [t^\infty, \infty)$.

Proof: From Theorem 3, if all neighbors of agent i converge in finite time ν_i to their desired locations, then agent i also converges to its desired location within $\nu_i + \tau_i$. Therefore, by Theorem 4, the overall system of N agents converge to $\bar{\mathbf{p}}$ with sufficiently small ε. Moreover, t^∞ is less than or equal to the sum of each elapsed time τ_i, i.e., $t^\infty \leq \sum_{i=2}^{N} \tau_i$, which results in the upper bound of the finite convergence time of the overall system. ■

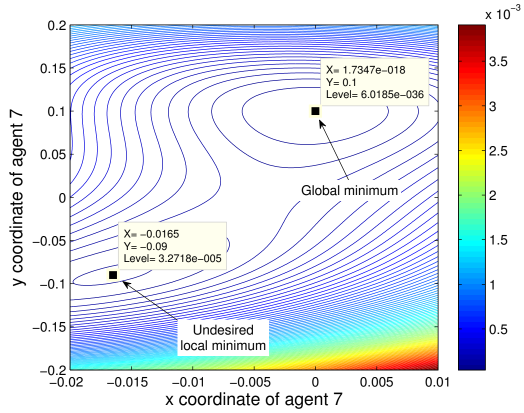


Fig. 5. Contour plot of ϕ_7 in Section III-C

Remark 2: The upper bound for the finite convergence time t^∞ is conservative. Suppose that, for some $i, j \in \mathcal{V}$, $i = \arg\max_{h\in\mathcal{V}} t_h^\infty$, j is less than i, and j is not in any path from i to the leader. If $t_j^\infty < t_i^\infty$, then τ_j does not contribute to t^∞. Accordingly, the actual convergence time would not be greater than $\sum_{i=2}^N \tau_i$.

C. Remark on the incorrect equilibrium formation

Under the control laws in (1) or (2), for certain initial conditions, some agents may converge to an undesired local minimum where their potential functions are not equal to zero. Consider a trilateration graph $\mathcal{G}_T$ shown in Fig. 2. Suppose that the desired formation shape is given by a representative realization $\bar{\mathbf{p}}$ such that

$$\bar{\mathbf{p}}_1 = [-1 \;\; 0.1]^\top, \quad \bar{\mathbf{p}}_2 = [1 \;\; 0.5]^\top, \quad \bar{\mathbf{p}}_3 = [-0.1 \;\; 0]^\top,$$
$$\bar{\mathbf{p}}_4 = [0.1 \;\; 0]^\top, \quad \bar{\mathbf{p}}_5 = [-0.2 \;\; 0]^\top, \quad \bar{\mathbf{p}}_6 = [0.2 \;\; 0]^\top,$$
$$\bar{\mathbf{p}}_7 = [0 \;\; 0.1]^\top.$$

Then the framework $(\mathcal{G}_T, \bar{\mathbf{p}})$ is *globally rigid*[1]. Consider an initial condition $\mathbf{p}(0)$ such that $\mathbf{p}_i(0) = \bar{\mathbf{p}}_i$ for all $i \in \{1, \ldots, 6\}$ and $\mathbf{p}_7(0) = [-0.01 \;\; -0.05]^\top$. If we draw a contour plot of ϕ_7, then we get Fig. 5, which shows an undesired local minimum. As a consequence, there could be an attractive incorrect equilibrium formation under the proposed control law even if the framework of the formation is globally rigid.

IV. Simulations

We provide two simulation results. In both of the simulations, the formation graph is the same as the trilateration graph shown in Fig. 2, and the desired formation shape is of $\bar{\mathbf{p}}$ which is given in Section III-C (see Fig. 6). In the first simulation, the initial formation shape is close enough to the desired formation shape and thus the agents finally achieve the desired formation shape in finite time.

In the second simulation, we assume that some agents are located near the undesired local minima of their potential functions. We can observe that some squared-distance errors do not converge to zero in Fig. 8(b) and the final formation shape in Fig. 8(a) is different from the one of Fig. 7(a).

[1]For a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and its realization $\mathbf{p}$, if the framework $(\mathcal{G}, \mathbf{p})$ is globally rigid, every realization $\mathbf{p}'$ such that $\|\mathbf{p}_i - \mathbf{p}_j\| = \|\mathbf{p}'_i - \mathbf{p}'_j\|$, $\forall (i, j) \in \mathcal{E}$, is congruent to $\mathbf{p}$. Hence, the shape of the framework is uniquely determined up to congruence. Refer to [20] for global rigidity.

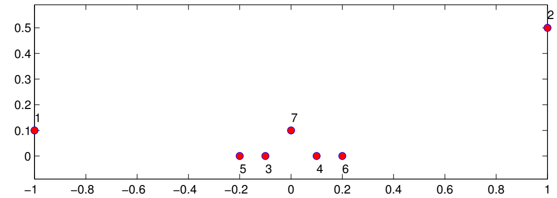
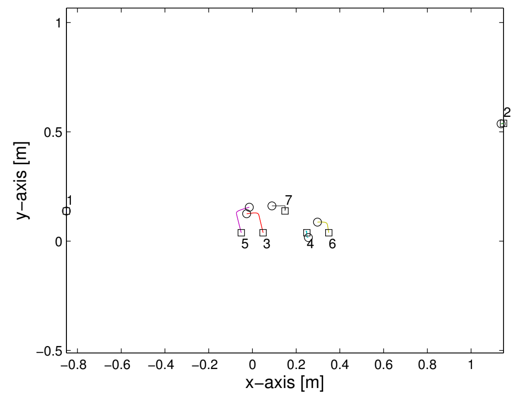
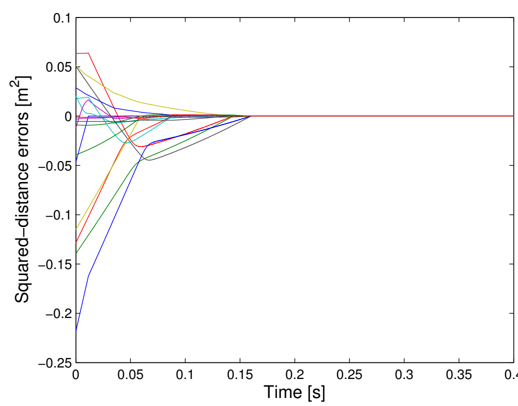


Fig. 6. Desired formation shape for simulations



(a) The initial and the final locations of the agents are denoted by circles and squares, respectively.

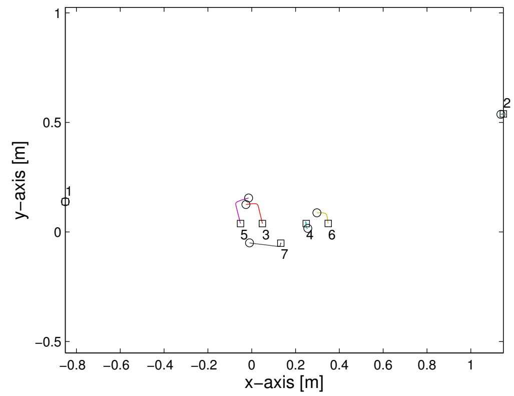


(b) Corresponding squared-distance errors

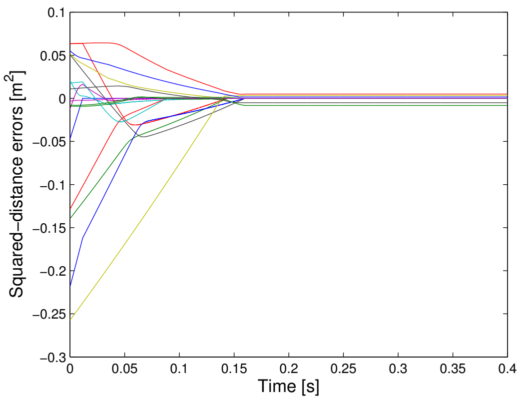
Fig. 7. Convergence to the desired formation shape

V. Conclusion

We proposed a distance-based control law for formations with a directed acyclic persistent graph. The proposed control law, which is a normalized gradient law, allows agents to achieve their desired formation shape in finite time by minimizing their local potential functions of inter-agent distances. We showed that finite-time convergence was achieved if the initial formation shape was sufficiently close to the desired one. We further revealed using an example that there may exist an attractive incorrect equilibrium formation with



(a) The initial and the final locations of the agents are denoted by circles and squares, respectively.



(b) Corresponding squared-distance errors

Fig. 8. Convergence to an incorrect formation shape: note that agent 7 is located in the undesired local minimum of ϕ_7 so it cannot approach the location in which $\phi_7 = 0$. Compare the final location of agent 7 in (a) to the final location of agent 7 in Fig. 7(a).

trilateration graph even when the desired formation shape is uniquely determined by the inter-agent distance constraints.

An immediate further research direction is to develop finite-time distance-based control law that can remove such an attractive incorrect equilibrium formation. Since the proposed control law was developed under the assumption of ideal behavior of the agents, to develop a solution to reduce the chattering in real implementation would be also meaningful future work.

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