Convergence Analysis for Rigid Formation Control with Unrealizable Shapes: The 3 Agent Case

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***Abstract*— We study the outcome of using a gradient descent control law for a minimally rigid formation consisting of N agents, in which each agent is modeled by a single integrator and the desired interagent distances are specified though they are not realizable. We first formulate the problem for formations of $N \geq 3$ agents and derive a condition in terms of the rigidity matrix which the final formation must satisfy. Special attention will be given to the triangular formation for which the desired distances fail to satisfy the triangle inequality. In this case, we show the formation converges to a straight line. Detailed analysis is provided to describe the stability properties in the unrealizable triangle shape control problem.**

I. INTRODUCTION

Formation control of networked multi-agent systems has attracted a lot of attention in recent years due to its growing applications in both military and civilian areas [1]. One of the key problems in this field that receives particular interest is how to design a controller to maintain a geometrical shape for the formation. Graph rigidity theory [2], [3] ensures that, by controlling a certain set of interagent distances, a desired whole formation shape can be achieved. Early efforts on using graph rigidity to maintain a formation shape can be found in [4], [5]. Along this direction, one popular control method is the gradient descent control law derived by specifying potential functions which involve the squared errors between actual distances and desired distances [6]. The stability and convergence analysis of this gradient control has been studied extensively in the literature (see e.g. [7], [8], [9], [10]), with all references assuming that the formation shape with the desired distance set is realizable in the given ambient dimension.

In this paper we consider what will happen when the formation shape is specified by a certain set of interagent distances that cannot be realized in the plane. We note that in [11] a similar problem was discussed to show what happens if *interagent relative positions*, rather than *interagent distances*, are to be stabilized for infeasible formation shapes. We note that the control method in [11] is in fact the displacement-based control, which can be transformed to a consensus problem that is linear, and more easily therefore analyzed. In that control framework setting, the authors in [11] demonstrated an interesting connection between formation infeasibility and velocity alignment, and showed that the infeasibility of the formation shape may lead to flocking movements. In this paper we will focus on the stabilization control of rigid formation shapes when the shape is unrealizable for the given set of interagent distances. We will show that in this distance-based gradient control framework, all the agents will converge to form a stationary shape rather than moving formation. Furthermore, a convergence property relating to the rigidity matrix will be given for the final formation shape. Then the main focus will be given to the 3-agent triangular shape, in which detailed convergence results will be discussed.

It is becoming well known that if realizable distances for a triangle are specified, then from any initial condition in which agents are not collinear, there will be convergence to a correctly dimensioned triangle [12]. If agents are initially collinear however, they will remain collinear, and convergence to an equilibrium will occur but not an equilibrium in which the desired distances are realized. Given that collinear formations are in some sense nongeneric, this is not altogether surprising. What is a little surprising in the result of this paper is that when one commences at a generic initial condition, one always converges to a nongeneric equilibrium state (where the agents are collinear).

The rest of this paper is organized as follows. In Section II, the problem description and motion equations are presented for this rigid formation control problem. In Section III, we show general convergence results of the gradient flow when formation shape is not realizable. In Section IV, we focus on the 3-agents triangular case, and provide detailed analysis to identify different types of equilibria. Finally, concluding remarks are provided in Section V.

II. PROBLEM FORMULATION

Consider an undirected graph of a formation consisting of $N \geq 3$ autonomous agents in the plane, labeled as $1, 2, ..., N$. The kinematic dynamics of each agent i are described in global coordinates by

$$\dot{p}_i = u_i, \quad i = 1, \ldots, N, \tag{1}$$

where $p_i = [p_{ix}, p_{iy}]^\top \in \mathbb{R}^2$ denotes the position of agent i and u_i describes the control input. Each agent i has the task

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of maintaining desired distances away from certain other agents called its *neighbors*. Let $\mathcal{N}_i$ denote the set of agent i's neighbors. Assume each agent i is able to measure the relative position $p_j - p_i$ of each of its neighbors $j \in \mathcal{N}_i$, and this can be done by using agent i's *own coordinate basis* [6]. It is standard to describe such models using notions from graph theory. The neighbor relationship is described by an undirected graph Γ with a set of N vertices, denoted as $\mathcal{V} = \{1,2,...,N\}$ and a set of M edges, denoted as $\mathcal{E} = \{1,2,...,M\}$, such that there is an edge (i,j) in Γ if and only if i and j are neighbors in the formation. In the following we let (p,Γ) denote the formation, where the positions of all agents are encoded by the column vector

$$p = \mathrm{col}(p_1,\ldots,p_N) := \left[p_1^\top, p_2^\top, \cdots, p_N^\top\right]^\top \in \mathbb{R}^{2N}. \quad (2)$$

To describe the dynamics we orient each edge in Γ with a specific direction from the tail vertex to the head vertex. Let $\mathcal{E}_i^+, \mathcal{E}_i^-$ denote the set of labels of all edges with i as the head vertex and the tail vertex, respectively. For the kth edge in the oriented graph with i as the tail and j as the head, $k \in \mathcal{E}$, we define an *edge vector*

$$z_k = p_j - p_i.$$

Consider any vector $d^* = \mathrm{col}(d_1^*, \cdots, d_M^*)$ of desired positive distances $d_k^* > 0$ that we want to achieve for the formation. Let

$$e_k = ||z_k||^2 - (d_k^*)^2$$

denote the error between the mutual distances of the actual formation and the desired ones. Define $e = \mathrm{col}(e_1,\ldots,e_M) \in \mathbb{R}^M$. We consider the following potential function

$$V(p_1,\ldots,p_N) = \frac{1}{4}e^\top e = \frac{1}{4}\sum_{k=1}^{M} e_k^2. \quad (3)$$

In order to drive the formation to achieve the desired distances we use the standard steepest descent control law

$$\dot{p}_i = u_i = -\sum_{k\in\mathcal{E}_i^+} z_k e_k + \sum_{k\in\mathcal{E}_i^-} z_k e_k, \quad i \in \mathcal{V} \quad (4)$$

as introduced by [6]. Note that the right hand side of (4) coincides with the negative gradient $-\nabla V$. Several authors, see e.g. [6], [7], [8], [9], [13], [12] have investigated the convergence properties of this gradient flow in the case where the desired distances are realized by a two dimensional formation $p_1^*, \ldots, p_N^*$. The goal of this paper is to find out what happens under the above gradient control if the given desired distances are *not realizable*, that is, there does not exist a formation (p,Γ) such that $||z_k|| = d_k^*$ for all $k \in \mathcal{E}$. Specifically, much focus of this paper will be given to the triangular formations, in which the given desired distance set does not satisfy the triangle inequality.

By way of notation, we define the rigidity matrix associated with the formation (p,Γ) as $R(p)$. Definitions and discussions on rigidity matrix can be found in [2], [3]. The formation is termed infinitesimally rigid when the rigidity matrix R has rank $2N-3$. When the matrix R for generic vertex positions p_i has rank $2N-3$, the associated graph is termed rigid, even though there may be particular formations for which the rank property is lost (e.g. one where all agents are collinear). The graph is termed minimally rigid when $M = 2N-3$. Throughout this paper we will assume that Γ is minimally rigid, although arbitrary rigid graphs are of obvious interest, too.

III. Convergence Analysis

To write equation (4) in a compact form, we let $H = [h_{ki}]_{M\times N}$ with entries given by

$$h_{ki} = \begin{cases} -1 & \text{if} \quad i \text{ is the tail of the } k\text{th edge} \\ 1 & \text{if} \quad i \text{ is the head of the } k\text{th edge} \\ 0 & \quad \text{otherwise} \end{cases}$$

Then H is the transpose of the incidence matrix. The rigidity matrix R can be expressed as

$$R = Z^\top (H \otimes I_2)$$

where $Z = \mathrm{diag}\,(z_1, z_2, ..., z_M)$. More information on the construction of the rigidity matrix can be found in [3] and [14]. Thus system (4) is equivalent to the gradient flow

$$\dot{p} = -R^\top(p)e \quad (5)$$

from which can obtain the *distance error system*

$$\dot{e} = 2Z^\top (H\otimes I_2)\dot{p} = 2R(p)\dot{p} = -2R(p)R^\top(p)e \quad (6)$$

It is of interest to explore the issue when a gradient flow converges to a single equilibrium point. This property is not generally satisfied if the potential function has a continuum of critical points and therefore should not be taken for granted. The following result establishes a very useful convergence property for real analytic gradient flows. It has been first revealed and proven by [15] in the special case of real analytic gradient flows in Euclidean space $\mathbb{R}^n$; the simple extension as formulated in the proposition is due to [16].

***Proposition** 1:* Let $\mathcal{M}$ denote a real analytic manifold endowed with a real analytic Riemannian metric. Let V denote an arbitrary real analytic function on a real analytic Riemannian manifold $\mathcal{M}$ such that all sublevel sets $\{p \in \mathcal{M} \mid V(p) \le c\}, c \in \mathbb{R}$ are compact. Let $\mathrm{grad}\,V(p)$ denote the gradient of V, associated with the Riemannian metric on $\mathcal{M}$. Then each solution $p(t)$ of the negative gradient flow

$$\dot{p} = -\,\mathrm{grad}\,V(p)$$

exists for all $t \ge 0$ and converges for $t \to +\infty$ to a single equilibrium point p^* which satisfies $\mathrm{grad}\,V(p^*) = 0$.

In addition to this it has been shown by [17] that the local minima of such real analytic functions V coincide with the locally stable equilibrium points of the gradient flow. Note that in [13] the Lojasiewicz inequality from [15] has been used to show the asymptotic stability of the gradient descent control around each local minimum of V for distance-based undirected rigid formation stabilization in N dimensional space.

A. Main convergence result

Our first convergence result is stated as follows.

Theorem 1: Assume that the graph Γ is minimally rigid, i.e. $M = 2N-3$. Assume further that the desired distances $d^* \in \mathbb{R}^M_+$ are not realizable by a formation of N-agents in $\mathbb{R}^2$.

1) The centroid of the formation remains constant in time.
2) The gradient descent flow (5) converges from any initial condition to a single equilibrium point, denoted by p_∞. The formation described by the limiting equilibrium point p_∞ is not infinitesimally rigid.
3) Let $N = 3$. All three agents converge to collinear positions. For generic desired distances $d \in \mathbb{R}^3$ there are exactly 5 collinear equilibrium formations.

Proof 1) Let $\mathbf{1} \in \mathbb{R}^N$ denote the vector with all entries equal to 1. Using (5), one has

$$\dot{p} = -(H^\top \otimes I_2)Ze.$$

Note that $H\mathbf{1} = 0$ and therefore

$$(\mathbf{1}^\top \otimes I_2)\dot{p} = 0.$$

Let $\bar{p}$ denote the position of the centroid of all N agents, that is,

$$\bar{p} = \frac{1}{N}\sum_{i=1}^{N} p_i = \frac{1}{N}(\mathbf{1}^\top \otimes I_2)p$$

Then

$$\dot{\bar{p}} = \frac{1}{N}(\mathbf{1}^\top \otimes I_2)\dot{p} = 0$$

which shows that the formation centroid is stationary.

2) The convergence property follows from Proposition 1 by first observing that the potential function V is real analytic on the Riemannian manifold $\mathbb{R}^{2N}$, endowed with the standard Euclidean metric. Second, note that $V(p)$ depends only on the difference $\delta = p - \bar{p}$. Define $E_0 := \{\delta \in \mathbb{R}^{2N} \mid \delta_1 + \cdots + \delta_N = 0\}$ and observe that for any solution $p(t)$ of (5) we have $\delta(t) \in E_0$. The invertible linear transformation $\mathscr{F} : \mathbb{R}^{2N} \longrightarrow E_0 \times \mathbb{R}^2, p \mapsto (\delta_1, \ldots, \delta_N, \bar{p})$, maps solutions of the gradient flow (5) bijectively onto solutions of the decoupled system

$$\dot{\delta} = -(I_{2N} - \frac{1}{N}\mathbf{1}\mathbf{1}^\top \otimes I_2)\nabla V(\delta), \quad \dot{\bar{p}} = 0$$

on $E_0 \times \mathbb{R}^2$. Note further that the real analytic function $\delta \mapsto V(\delta)$ has compact sublevel sets on E_0. Moreover, $(I_{2N} - \frac{1}{N}\mathbf{1}\mathbf{1}^\top \otimes I_2)\nabla V(\delta)$ defines the gradient of the function $\delta \mapsto V(\delta)$ with respect to the Riemannian metric that is obtained by restricting the standard Euclidean metric onto E_0. Finally, using $(\mathbf{1}^\top \otimes I_2)\nabla V(\delta) = 0$, we see that $\nabla V(\delta) = 0$ if and only if $(I_{2N} - \frac{1}{N}\mathbf{1}\mathbf{1}^\top \otimes I_2)\nabla V(\delta) = 0$ holds. Thus by applying Proposition 1 to the restriction $V|E_0$, we conclude that $\delta(t)$ converges to a single equilibrium point $\delta_\infty \in E_0$ satisfying $\nabla V(\delta_\infty) = 0$. Since $\bar{p}(t)$ is constant this is equivalent to the claim that any solution $p(t)$ converges to a single equilibrium point p_∞ and in particular, $\lim_{t\to\infty} e(t) = e_\infty$ therefore exists.

Since $\dot{p} = -R^\top(p)e$ we have that $R^\top(p_\infty)e_\infty = 0$. Since d^* is assumed to be not realizable we have $e_\infty \neq 0$. By minimal rigidity of the graph, $M = 2N-3$. Suppose p_∞ were infinitesimally rigid. Then $R^\top(p_\infty)$ has full column rank $2N-3$. Thus $R^\top(p_\infty)e_\infty = 0$ implies $e_\infty = 0$, which is in contradiction to our assumption. Thus the final formation described by p_∞ is not infinitesimally rigid.

3) In the triangle case with $N = 3$, the formation loses its infinitesimal rigidity (i.e. $R(p_\infty)$ has rank less than 3) if and only if the positions of all the agents are collinear. Hence, by using the gradient descent control to stabilize the unrealizable triangular shape, the three agents will converge to a line formation. Further, in [18] it has been shown that, for generic choices of d^* and up to arbitrary translations, there are exactly 5 equilibrium formations on the line $\mathbb{R}$. This implies that, for generic choices of d^* and up to arbitrary translation and rotation in the plane, there are exactly 5 collinear equilibrium formations in $\mathbb{R}^2$. This completes the proof. ∎

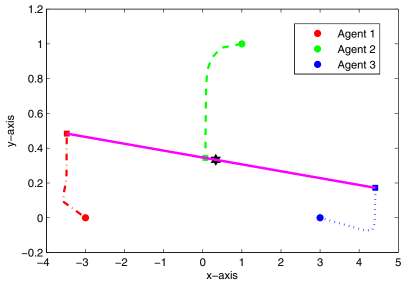

Fig. 1. Simulation on an unrealizable 3-agent triangle formation shape. The initial and final positions are denoted by circles and squares, respectively. The black star denotes the formation centroid.

B. Simulations

In this subsection we provide two simulations to show the behavior of agent's formations with an unrealizable desired distance set. Firstly consider a triangular formation shape, with the desired distances given as $d^*_{12} = 3$, $d^*_{23} = 4$, $d^*_{13} = 8$. Note these distances violate the triangle inequality and hence cannot be realized. The initial conditions for each agent are chosen as $p_1(0) = [-3, 0]^\top$, $p_2(0) = [1, 1]^\top$ and $p_3(0) = [3, 0]^\top$. The trajectories of each agent and the final shape are depicted in Fig. 1, which shows that the three agents converge to a line formation, with the formation centroid remaining invariant.

Then we consider a 4-agent quadrilateral formation with 5 edges, with the desired distance set as $d^*_{12} = 3$, $d^*_{23} = 7$, $d^*_{13} = 9$, $d^*_{34} = 2$, $d^*_{14} = 5$. Note that these distances cannot be realized to obtain a genuine quadrilateral formation, since the distances in the triangle formed by agents 1, 4 and 3 violate the triangle inequality. The initial conditions for each agent are chosen as $p_1(0) = [4, -1]^\top$, $p_2(0) = [5, 5]^\top$, $p_3(0) = [-7, 6]^\top$ and $p_4(0) = [-4, -2]^\top$. The trajectories of each agent and the final shape are depicted in Fig. 2, in which three agents (agents 1, 3 and 4) converge to a line and the infinitesimal rigidity of the formation is lost.

In the preceding analysis we have shown that all 3 agents in the triangular formation become asymptotically collinear, if the desired distances are not realizable. We next offer

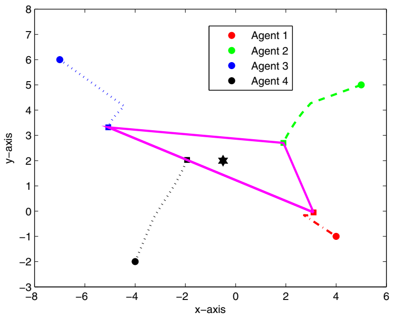

Fig. 2. Simulation on an unrealizable 4-agent quadrilateral formation shape. The initial and final positions are denoted by circles and squares, respectively. The black star denotes the formation centroid.

a local stability analysis around such a collinear formation before going on to look at global properties. In the papers [19] and [18] we have analyzed the critical formations on the line, proving that there are exactly 5 critical formations for 3 agents' formations on the line, of which two are local minima, two are saddle points and one is a local maximum. However, the analysis in [18] was strictly limited to formations confined to a one-dimensional ambient space, i.e. in the real line $\mathbb{R}$, and not to formations in the plane $\mathbb{R}^2$ which happened to be collinear. The problem remains to study the local stability properties of these five critical formations in the plane. This is done in the following section.

IV. Stability Analysis for a Triangle

A. Equations of motion and Hessian of the potential function

The stabilization and convergence of triangle formation control using a gradient descent law has been discussed in [12], [8], in which the triangle shapes are assumed to be realizable. In this section we will provide detailed convergence analysis for unrealizable triangular shapes when the gradient descent law is employed. In the triangle case, let $z_{12} = p_1 - p_2, z_{23} = p_2 - p_3, z_{31} = p_3 - p_1$. By gradient control, one has

$$\begin{aligned} \dot{p}_1 &= -e_{12}z_{12} + e_{31}z_{31} \\ \dot{p}_2 &= -e_{23}z_{23} + e_{12}z_{12} \\ \dot{p}_3 &= -e_{31}z_{31} + e_{23}z_{23} \end{aligned} \quad (7)$$

Note that here the subscripts for e_{ij}, z_{ij} and d^*_{ij} indicates the relationship between agents i and j, which are slightly different to the notations in Section II. In the following, we will also use d_{ij} to denote the actual distances that agents i and j converge to. For a compact form, the above p-system can be written as

$$\dot{p} = -R^\top(p)e(p) = -(E(p)\otimes I_2)p \quad (8)$$

where the rigidity matrix $R(p)$ is of size 3×6, and the matrix $E(p)$ is given as

$$E = \begin{bmatrix} e_{12}+e_{13} & -e_{12} & -e_{13} \\ -e_{12} & e_{12}+e_{23} & -e_{23} \\ -e_{13} & -e_{23} & e_{13}+e_{23} \end{bmatrix} \quad (9)$$

In this section we aim to investigate the local stability of the equilibrium set

$$\mathscr{L} = \{p \in \mathbb{R}^6 | R^\top(p)e(p) = 0\}$$

To this end, we need to linearize the p-system (8) around the equilibrium point. Note that the equilibrium points of the p-system (8) are the same as the critical points of the potential function V defined in (3). The Jacobian of the right side of the p-system (8) is the same as the negative of the Hessian of V, which is denoted as $H_V(p)$ and can be computed to be (see [9])

$$H_V(p) = 2R^\top(p)R(p) + (E(p)\otimes I_2) \quad (10)$$

The nature of an equilibrium (of being a local minimum, a saddle point or a local maximum) can be determined by the signs of the eigenvalues of the Hessian at that equilibrium. Firstly, we introduce a transformation matrix T, which is used to transform the rigidity matrix: $RT = [R_x, R_y] = \bar{R}$, where R_x and R_y are the matrices whose columns consist of the columns of R corresponding to the coordinates x and y, respectively. Then one can obtain a transformed Hessian matrix $\bar{H}_V(p)$ as

$$\begin{aligned} \bar{H}_V(p) &= T^\top H_V(p) T \\ &= 2\begin{bmatrix} R_x^\top \\ R_y^\top \end{bmatrix} [R_x\ R_y] + \begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix} \end{aligned} \quad (11)$$

which is congruent to $H_V(p)$.

Further note that the stability of an equilibrium point is independent of rotation and translation of the formation shape. Thus, without loss of generality, we will analyze the stability of the collinear equilibria in the triangle case under the assumption that the final formation converges to the x axis. That is, $R_y = 0$ and one obtains

$$\bar{H}_V = \begin{bmatrix} 2R_x^\top R_x + E & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix} \quad (12)$$

The above expression of the transformed Hessian matrix will be the key to identify different types of equilibria.

B. Properties of local minima and saddle points

Note that for an unrealizable triangle with the given desired distances d^*_{12}, d^*_{23}, d^*_{13}, there are three cases for which the triangle inequality is violated. Without loss of generality, we assume that the desired distances satisfy

$$d^*_{12} + d^*_{23} < d^*_{13} \quad (13)$$

We first show an important result concerning the rank of the matrix E when the formation converges to a line.

***Lemma** 1:* The matrix $E(p^*)$ is of rank 1 at a collinear equilibrium p^* at which not all agents are collocated. .

Proof It is evident that the vector $\mathbf{1} = [1\ 1\ 1]^\top$ is a null vector of E. At the equilibrium point of (8) there holds $(E(p^*)\otimes I_2)p^* = 0$. Thus one can write

$$E(p^*)\begin{bmatrix} 1 & p^*_{1x} & p^*_{1y} \\ 1 & p^*_{2x} & p^*_{2y} \\ 1 & p^*_{3x} & p^*_{3y} \end{bmatrix} = 0$$

Because the agents are not all collocated, i.e. not all p_i^* are the same, the matrix on the right has rank at least two. Because the agents are collinear, it has rank precisely two. Hence the matrix E should be of rank 0 or 1. Further note that due to the infeasibility of the formation shape, at least one e_{ij} is not zero. Then $E(p^*)$ has rank 1 at the collinear equilibrium p^*. ∎

Note that there is a special equilibrium p^* in which all the agents are collocated (i.e. $p_1^* = p_2^* = p_3^*$). This equilibrium is in fact a maximum in which $E(p^*)$ has rank 2. The maximum property follows immediately by simply considering the change in V when agents are moved from an all-collocated configuration.

In the following, we show some properties of the local minima of the collinear formations.

***Theorem** 2:* The local minima correspond to the line position ordering with agent 2 between agents 1 and 3, so that

$$d_{12} + d_{23} = d_{13} \tag{14}$$

At such collinear equilibrium, E is of rank 1 and nonnegative definite, which indicates that the equilibrium of this type considered in $\mathbb{R}^2$ is stable.

Proof The rank 1 property of the matrix E is proved from Lemma 1. Note that since E is also symmetric, it is either nonnegative definite or nonpositive definite. Since $E = \alpha z z^\top$ for some nonzero constant α and some nonzero vector z, the diagonal entries of E must all be nonnegative or nonpositive, and not all zero, depending on the sign of α. We shall show that the (2-2) entry of E is positive, which establishes that E is nonnegative definite.

Suppose, in order to obtain a contradiction, that there holds $e_{12} < 0$. By considering the equilibrium of each of agents 1 and 2, we can easily obtain that $e_{13} > 0, e_{23} < 0$. These three signs imply that

$$d_{12} < d_{12}^*, \;\; d_{23} < d_{23}^*, \;\; d_{13} > d_{13}^* \tag{15}$$

and then we have

$$d_{12}^* + d_{23}^* > d_{12} + d_{23} = d_{13} > d_{13}^* \tag{16}$$

This contradicts the given distance inequality in (13). Hence $e_{12} > 0$. Then considering the equilibrium of agent 2, we see that $e_{23} > 0$, and so the (2-2) entry of E is positive as claimed. By considering the expression of the Hessian in (12), this equilibrium is locally stable as considered in $\mathbb{R}^2$. ∎

***Remark** 1:* Theorem 2 shows that at the collinear equilibrium p^* with the position ordering of (14), the Hessian $\bar{H}_V(p^*)$ is positive semidefinite. Conditions for exponential convergence to the equilibrium positions and exponential stability of the inter agent distances are more difficult and need further investigation, in which the Center Manifold Theorem [20] may come into play. This will be done elsewhere.

According to Theorem 1-3), the only two stable equilibria should be those with the position ordering shown in (14) by considering position reflection. Note that the only maximum is the one that all three agents are collocated. Then the remaining two equilibria, which are saddle points, should correspond to line formations with two but not three agents collocated, or correspond to line formations with some position orderings other than (14). Also, at the saddle points, the matrix E is non-positive definite and has a negative eigenvalue, which further indicates that the Hessian is indefinite. In the following, we show that this is the case.

***Lemma** 2:* Suppose there is an equilibrium in which two agents are collocated. Then this equilibrium is a saddle point.

Proof We first show that such an equilibrium exists only when the desired distances satisfy $d_{12}^* = d_{23}^*$ and agents 1 and 3 are collocated. From (7), if agents 1 and 2 are collocated in the equilibrium, then $e_{12} < 0$ and $e_{31} = e_{23} = 0$, which indicates that $d_{23} = d_{23}^* = d_{13} = d_{13}^*$. But this contradicts the relationship of the desired distance set in (13). The same reasoning can further show that agents 2 and 3 cannot be collocated at an equilibrium point. Let us consider the last case that agents 1 and 3 are collocated at an equilibrium. This implies that $e_{13} < 0$ and $e_{12} = e_{23} = 0$ (this happens under the condition that $d_{12}^* = d_{23}^*$). Then the matrix E is non-positive definite. According to the expression of the Hessian $\bar{H}_V$ in (12), the 11 block of the Hessian $\bar{H}_V$ cannot be negative semidefinite. This follows because $R_x^\top R_x$ has rank 2 and E has rank 1, so there must be at least one positive eigenvalue for $\bar{H}_V$. Hence the collinear equilibrium in this case should be saddle point. ∎

The following two Lemmas further confirm that if an equilibrium point with the position ordering other than (14) exists, then it should be a saddle point.

***Lemma** 3:* Suppose there is an equilibrium with agent 1 between agents 2 and 3, so that

$$d_{12} + d_{13} = d_{23} \tag{17}$$

Then at this collinear equilibrium, E is of rank 1 and nonpositive definite, and such an equilibrium is a saddle point.

Proof The proof for the rank 1 property of E follows from Lemma 1. We also use a similar argument to that of Theorem 2 to show that $e_{12} + e_{13} < 0$. Suppose that there holds $e_{12} > 0$. Then from the equilibrium equations for agent 1 and 3, there must hold $e_{13} > 0$, $e_{23} < 0$, which implies that $d_{12} > d_{12}^*$, $d_{13} > d_{13}^*$ and $d_{23} < d_{23}^*$. These further imply that $d_{12}^* + d_{13}^* < d_{12} + d_{13} = d_{23} < d_{23}^*$. But this contradicts the condition that $d_{12}^* + d_{23}^* < d_{13}^*$. Thus one has $e_{12} < 0$, $e_{13} < 0$, $e_{23} > 0$, which implies that $e_{12} + e_{13} < 0$. Then the rank 1 matrix E is nonpositive definite, and further the Hessian is indefinite. Hence the collinear equilibria in this case are saddle points. ∎

***Lemma** 4:* Suppose there is an equilibrium with agent 3 between agents 1 and 2, such that

$$d_{13} + d_{23} = d_{12} \tag{18}$$

Then at this collinear equilibrium, E is of rank 1 and nonpositive definite, and the equilibrium with this position ordering is a saddle point.

Proof The proof is similar, in that one can show that there should hold $e_{13} < 0$, $e_{12} > 0$ and $e_{23} < 0$. Hence one can conclude that $e_{13}+e_{23}<0$ and the rank 1 matrix E is negative semidefinite. We omit the detailed steps here. ■

C. Convergence analysis

Based on the above analysis, one can show the following convergence result.

***Theorem** 3:* When the three agents start with a non-collinear formation and approaches collinearity, then the formation will not end at a saddle point. That is, if the given distances satisfy (13), then the final line formation will satisfy the position ordering described in (14). Furthermore, a trajectory starting off non-collinear will not become collinear at a finite time.

Proof From above we have shown that E is nonpositive definite at the saddle points in the line formations. Thus with the extra dimension added, one cannot have a stability in the y-dimension. Hence there can be no trajectory which starts with a non-collinear formation and approaches collinearity as t goes to infinity and ends at a saddle point.

We then prove the second statement. From the equations of motion in (7), it follows that

$$\begin{aligned}\dot{z}_{12} &= -2e_{12}z_{12}+e_{31}z_{31}+e_{23}z_{23}\\ \dot{z}_{23} &= -2e_{23}z_{23}+e_{12}z_{12}+e_{31}z_{31}\end{aligned} \tag{19}$$

By substituting $z_{31}=-z_{12}-z_{23}$ into the above equations, one has

$$\begin{aligned}\dot{z}_{12} &= -(2e_{12}+e_{31})z_{12}+(e_{23}-e_{31})z_{23}\\ \dot{z}_{23} &= -(2e_{23}+e_{31})z_{23}+(e_{12}-e_{31})z_{12}\end{aligned} \tag{20}$$

Let $w=z_{12}^{\top}Gz_{23}$ where

$$G=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

Then

$$\begin{aligned}\dot{w} &= z_{12}^{\top}G\dot{z}_{23}+\dot{z}_{12}^{\top}Gz_{23}\\ &= -2(e_{12}+e_{23}+e_{31})w\end{aligned} \tag{21}$$

Note that the magnitude of w is 2 times the area of the triangle, so that the three agents are collinear if and only if $w=0$ (i.e. a zero triangle area). From (21), one sees that w can not reach 0 in finite time if $w(0)\neq 0$. Thus the three agents are not collinear in a finite time if they are initially not collinear. ■

***Remark** 2:* In the special case that $d_{12}^{*}=d_{23}^{*}<\frac{1}{2}d_{13}^{*}$, based on the above analysis one can show that the formations will converge to a collinear shape in which $d_{12}=d_{23}=\frac{1}{2}d_{13}$ if they start in a non-collinear formation. In fact, by specifying the desired distance set as $d_{12}^{*}=d_{23}^{*}=a$, $d_{13}^{*}=b$, the formation will converge to a line formation in which

$$d_{12}=d_{23}=\frac{1}{3}\sqrt{a^2+2b^2},\ d_{13}=\frac{2}{3}\sqrt{a^2+2b^2} \tag{22}$$

V. Conclusions

In this paper, we have investigated the stabilization problem of minimally rigid formations in the case that the formation shape with the given interagent distances cannot be realized in the plane. We have shown that the final shape will converge to the one that the formation loses its infinitesimal rigidity. Then we focus on the triangle case in which the given distances do not satisfy the triangle inequalities. By confirming that the final shape will converge to a line, we have further shown the local stability of different equilibria by investigating the properties of the Hessian matrix. Future research will focus on the convergence analysis of more general formation shapes.

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