**Preprints of the 19th World Congress**
**The International Federation of Automatic Control**
**Cape Town, South Africa. August 24-29, 2014**

# Finite Time Distance-based Rigid Formation Stabilization and Flocking

**Z. Sun** **,* **S. Mou** *** **M. Deghat** ** **B. D. O. Anderson** **,*
**A.S. Morse** ***

* *Shandong Computer Science Center (SCSC), Jinan, China*
** *National ICT Australia and Research School of Engineering, The Australian National University, Canberra ACT 0200, Australia. (e-mail: {zhiyong.sun, mohammad.deghat, brian.anderson}@anu.edu.au).*
*** *Department of Electrical Engineering, Yale University, New Haven, CT, USA (e-mail: {shaoshuai.mou, as.morse}@yale.edu)*

**Abstract:** Most of the existing results on distributed distance-based rigid formation control establish asymptotic and often exponentially asymptotic convergence. To further improve the convergence rate, we explain in this paper how to modify existing controllers to obtain finite time stability. For point agents modeled by single integrators, the controllers proposed in this paper drive the whole formation to converge to a desired shape with finite settling time. For agents modeled by double integrators, the proposed controllers allow all agents to both achieve the same velocity and reach a desired shape in finite time. All controllers are totally distributed. Simulations are also provided to validate the proposed control.

*Keywords:* Formation control; finite time stability; graph rigidity; flocking.

## 1. INTRODUCTION

Formation control of networked multi-agent systems has received considerable attention in recent years due to its extensive applications, the underlying objective of which is to maintain a pre-specified geometric shape for a group of agents Anderson et al. (2007),Yu et al. (2009). According to different measurements and actively controlled variables, the existing formation control strategies can be classified as position-based methods, displacement-based methods and distance-based methods Oh et al. (2012). In the *distance-based* control approach, the desired formation shape is specified by a certain set of inter-agent distances, though each agent requires the relative position measurements in order to control the distances. We adopt the notation convention as in Dimarogonas and Johansson (2009) and Oh et al. (2012) and call it the distance-based formation stabilization approach. This approach attracts particular interest since it does not require a common sense of orientation for each individual agent. Along this direction, Krick et al. (2009), Dimarogonas and Johansson (2009), Dorfler and Francis (2010), Cao et al. (2011), Mou et al. (2011) and Oh and Ahn (2013) have studied the distance-based formation control for different formation shapes, all with asymptotic convergence (or sometimes exponentially asymptotic convergence).

One objective of this paper is to design controllers to stabilize rigid formation shapes with a finite settling time. Finite time convergence brings about many benefits, which include not only a faster convergence rate, but also improved disturbance rejection and robustness properties Bhat and Bernstein (2000), Hui et al. (2008). We modify the commonly-used gradient control law and devise a simple finite time controller which can be easily implemented in a decentralized way. Compared with the finite time formation control proposed by Xiao et al. (2009), which can be transformed into a linear consensus problem and requires a global coordinate system for controlling relative positions, the finite time controller devised in this paper does not require a global coordinate system for each agent and thus can be implemented in a totally distributed way.

Another aim of this paper is to design a novel flocking controller for agents modeled by double integrators such that all agents achieve the same velocity and the whole formation converges to a desired shape. This problem was solved in Anderson et al. (2012) by combining the consensus protocol and distance-based shape control. In this paper we modify this kind of control to achieve the desired goal in finite time.

The paper is organized as follows. In Section 2, preliminary concepts on graph theory, rigidity theory and finite time stability are introduced. In Section 3, the modified gradi-

★ This work was supported by NICTA Ltd, Overseas Expert program of Shandong Province, a grant from Shandong Academy of Science Development Fund for Science and Technology, the Pilot Project for Science and Technology in Shandong Academy of Science, the Open Research Project of the State Key Laboratory of Industrial Control Technology, Zhejiang University (No.ICT 1427) and the National Natural Science Foundation of China (61375072). Z. Sun was also supported by the Prime Minister's Australia Asia Incoming Endeavour Postgraduate Award. B. D. O. Anderson was also supported by the ARC under grant DP110100538 and DP130103610. The research of S. Mou and A.S. Morse was supported by the US Air Force Office of Scientific Research and the National Science Foundation.

ent control law is proposed which can achieve finite time formation stabilization. In Section 4, we further consider the distance-based flocking problem when agents are modelled by double integrators. Some simulations are provided in Section 5. Finally, Section 6 concludes this paper.

*Notations.* The notations used in this paper are fairly standard. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices. A matrix or vector transpose is denoted by a superscript $T$. The rank, image and null space of a matrix $M$ are denoted by $rank(M)$, $Im(M)$ and $ker(M)$, respectively. When $m$ is an $n$-tuple vector, the symbol $|m|^\alpha$ ($\alpha > 0$) represents $\sum_{i=1}^n |m_i|^\alpha$, where $m_i$ is the $i$-th element of $m$. As in Xiao et al. (2009), the function $sig(x)^\alpha$ is defined as $sig(x)^\alpha = sign(x)|x|^\alpha$ with $\alpha \in (0,1)$, $x \in \mathbb{R}$, and $sign(\cdot)$ is the signum function. If $x$ is a real vector, then $sig(x)^\alpha$ is a vector function which is defined in a componentwise way. The notation $\text{diag}\{x\}$ denotes a diagonal matrix with the vector $x$ on its diagonal, and $span\{v_1, v_2, \cdots, v_k\}$ represents the subspace spanned by a set of vectors $v_1, v_2, \cdots, v_k$. The symbol $I_n$ denotes the $n \times n$ identity matrix, and $\mathbf{1}_n$ denotes a $n$-tuple column vector of all ones. We use $\otimes$ to denote the Kronecker product.

## 2. PRELIMINARIES

### *2.1 Basic Concepts on Graph and Rigidity Theory*

Consider an undirected graph with $m$ edges and $n$ vertices, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \cdots, n\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The neighbor set $\mathcal{N}_i$ of node $i$ is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$. The matrix relating the nodes to the edges is called the incidence matrix $H = \{h_{ij}\} \in \mathbb{R}^{m\times n}$, whose entries are defined as (with arbitrary edge orientations)

$$h_{ij} = \begin{cases} 1, & \text{the } i\text{-th edge sinks at node } j \\ -1, & \text{the } i\text{-th edge leaves node } j \\ 0, & \text{otherwise} \end{cases}$$

The adjacency matrix $A(\mathcal{G})$ is a symmetric $n \times n$ matrix encoding the vertex adjacency relationships, with entries $A_{ij} = 1$ if $\{i,j\} \in \mathcal{E}$, and $A_{ij} = 0$ otherwise. Another important matrix representation of a graph $\mathcal{G}$ is the Laplacian matrix $L(\mathcal{G})$, which is defined as $L(\mathcal{G}) = H^T H = \text{diag}\{A\mathbf{1}_n\} - A$. For a connected undirected graph, one has $rank(L) = n - 1$ and $ker(L) = ker(H) = span\{\mathbf{1}_n\}$.

Let $p_i \in \mathbb{R}^d$ where $d = \{2, 3\}$ denote a point that is assigned to $i \in \mathcal{V}$. The stacked vector $p = [p_1^T, p_2^T, \cdots, p_n^T]^T \in \mathbb{R}^{dn}$ represents the realization of $\mathcal{G}$ in $\mathbb{R}^d$. The pair $(\mathcal{G}, p)$ is said to be a framework of $\mathcal{G}$ in $\mathbb{R}^d$. By introducing the matrix $\bar{H} := H \otimes I_d \in \mathbb{R}^{dm\times dn}$, one can construct the relative position vector as an image of $\bar{H}$ from the position vector $p$:

$$z = \bar{H}p \tag{1}$$

where $z = [z_1^T, z_2^T, \cdots, z_m^T]^T \in \mathbb{R}^{dm}$, with $z_k \in \mathbb{R}^d$ being the relative position vector for the vertex pair defined by the $k$-th edge.

Given an arbitrary ordering of the edges in $\mathcal{E}$, the rigidity function $r_\mathcal{G}(p) : \mathbb{R}^{dn} \to \mathbb{R}^m$ associated with the framework $(\mathcal{G}, p)$ is given as:

$$r_\mathcal{G}(p) = \frac{1}{2}\left[\cdots, \|p_i - p_j\|^2, \cdots\right]^T, \ (i,j) \in \mathcal{E} \tag{2}$$

where the norm is the standard Euclidean norm, and the $k$-th component in $r_\mathcal{G}(p)$, $\|p_i - p_j\|^2$, corresponds to the squared length of the relative position vector $z_k$ which connects the vertices $i$ and $j$.

The rigidity of frameworks is then defined as follows.

*Definition 1.* (Asimow and Roth (1979)) A framework $(\mathcal{G}, p)$ is rigid in $\mathbb{R}^d$ if there exists a neighborhood $\mathbb{U}$ of $p$ such that $r_\mathcal{G}^{-1}(r_\mathcal{G}(p)) \cap \mathbb{U} = r_\mathcal{K}^{-1}(r_\mathcal{K}(p)) \cap \mathbb{U}$ where $\mathcal{K}$ is the complete graph with the same vertices as $\mathcal{G}$.

In the following, the set of all frameworks $(\mathcal{G}, p)$ which satisfies the distance constraints is referred to as the *target formation*. One useful tool to characterize the rigidity property of a framework is the rigidity matrix $R \in \mathbb{R}^{m\times dn}$, which is defined as

$$R(p) = \frac{\partial r_\mathcal{G}(p)}{\partial p} \tag{3}$$

It is not difficult to see that each row of the rigidity matrix $R$ takes the following form

$$[\mathbf{0}_{1\times d}, \cdots, (p_i - p_j)^T, \cdots, \mathbf{0}_{1\times d}, \cdots, (p_j - p_i)^T, \cdots, \mathbf{0}_{1\times d}] \tag{4}$$

In the following, we indicate a simple expression for the rigidity matrix which involves both the network topology and position configuration. Recall (1), which shows that the relative position vector lies in the image of $\bar{H}$. The rigidity function is a map from the node positions to the squared edge lengths. Thus we can redefine the rigidity function, $g_\mathcal{G}(z) : Im(\bar{H}) \to \mathbb{R}^m$ as $g_\mathcal{G}(z) = \frac{1}{2}\left[\|z_1\|^2, \|z_2\|^2, \cdots, \|z_m\|^2\right]^T$. From (1) and (3), one can obtain the following simple form for the rigidity matrix

$$\begin{aligned} R(p) = \frac{\partial r_\mathcal{G}(p)}{\partial p} &= \frac{\partial g_\mathcal{G}(z)}{\partial z}\frac{\partial z}{\partial p} \\ &= \begin{pmatrix} z_1^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_m^T \end{pmatrix} \bar{H} \\ &= Z^T \bar{H} \end{aligned} \tag{5}$$

where $Z = \text{diag}\{z_1, z_2, \cdots, z_m\}$.

The rigidity matrix will be used to determine the infinitesimal rigidity of the framework, as shown in the following definition.

*Definition 2.* (Hendrickson (1992)) A framework $(\mathcal{G}, p)$ is infinitesimally rigid in $d$-dimensional space if

$$rank(R(p)) = dn - d(d+1)/2 \tag{6}$$

Specifically, the framework $(\mathcal{G}, p)$ is infinitesimally rigid in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) if and only if $rank(R(p)) = 2n - 3$ (resp. $rank(R(p)) = 3n - 6$). Obviously, in order to have an infinitesimally rigid framework, the graph should have at least $2n - 3$ (resp. $3n - 6$) edges in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$). If the framework is infinitesimally rigid in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) and has exactly $2n - 3$ (resp. $3n - 6$) edges, then it is called a minimally and infinitesimally rigid framework.

Also, if $(\mathcal{G}, p)$ is infinitesimally rigid, so is $(\mathcal{G}, p')$ for a generic (open and dense) set of $p'$. Generally speaking, infinitesimal rigidity implies rigidity, but the converse is not true. From the definition of infinitesimal rigidity, one could easily prove the following lemma:

*Lemma 1.* If the framework $(\mathcal{G}, p)$ is minimally and infinitesimally rigid in the $d$-dimensional space, then the matrix $R(p)R(p)^T$ is positive definite.

Another useful observation shows that there exists a smooth function which maps the distance set of the minimally rigid framework to the distance set of its corresponding framework modeled by a complete graph.

*Lemma 2.* Let $r_{\mathcal{G}}(p)$ be the rigidity function for a given infinitesimally minimally rigid framework $(\mathcal{G}, p)$. Further let $r'_{\mathcal{G}'}(p)$ denote the rigidity function for an associated framework $(\mathcal{G}', p)$, in which the vertex set remains the same as $(\mathcal{G}, p)$ but the underlying graph is a complete one (i.e. there exist $n(n-1)/2$ edges which link any vertex pairs). Then there exists a continuously differentiable function $f : r_{\mathcal{G}}(p) \to \mathbb{R}^{n(n-1)/2}$ for which $r'_{\mathcal{G}'}(p) = f(r_{\mathcal{G}}(p))$.

Lemma 2 indicates that all the edge distances in the framework $(\mathcal{G}', p)$ modeled by a complete graph can be expressed in terms of the edge distances of a corresponding minimally infinitesimally framework $(\mathcal{G}, p)$ via some smooth functions. This is almost intuitively obvious. The proof of the above Lemma 2 is omitted here and can be found in Mou et al. (2014).

In this paper, we focus on the formation control problem in which the desired formation shape is feasible [1] and is minimally infinitesimally rigid.

### 2.2 Finite Time Stability

Finite time stability was studied in Bhat and Bernstein (2000), from which we record the following finite time Lyapunov theorem.

*Lemma 3.* (Bhat and Bernstein (2000)) Consider the system $\dot{x} = f(x)$, $f(0) = 0, x \in \mathbb{R}^n$. Suppose there exist a continuous positive definite function $V(x) : U \to R$, real numbers $c > 0$ and $\alpha \in (0, 1)$ and an open neighborhood $U_0 \subset U$ of the origin such that $\dot{V}(x) + c(V(x))^\alpha \leq 0, x \in U_0 \setminus \{0\}$. Then $V(x)$ approaches 0 in finite time. In addition, the finite settling time function $T$ satisfies $T \leq \frac{V(x(0))^{1-\alpha}}{c(1-\alpha)}$.

In the later part of this paper, we will also employ the following inequality in deriving an upper bound for the finite settling time.

*Lemma 4.* (Hardy et al. (1952)) Let $x_1, x_2, \cdots, x_n \geq 0$. Given $\alpha \in (0, 1]$, then

$$\left(\sum_{i=1}^{n} x_i\right)^\alpha \leq \sum_{i=1}^{n} x_i^\alpha \tag{7}$$

## 3. FORMATION STABILIZATION CONTROL FOR SINGLE INTEGRATOR AGENTS

In this section, we will devise a new control law to stabilize a minimally and infinitesimally rigid formation in finite time when each agent is modeled by a single integrator. The problem considered in this section is formally formulated as follows:

[1] A feasible formation shape is one that can actually be realized; for a triangle, for example, prescribed distances must satisfy the triangle inequality.

*Problem 1.* Consider a network of $n$ agents in $d$-dimensional space with associated minimally rigid graph and in which

$$\dot{p}_i = u_i, \quad i = 1, 2, \cdots, n \tag{8}$$

Design the control $u_i$ for each agent $i$ in terms of $p_i - p_j$, $j \in \mathcal{N}_i$ such that $||p_i - p_j||$ converges to the desired distance $d^*_{ij}$ in *finite time.*

Let $e_{k_{ij}} = \|p_i - p_j\|^2 - (d^*_{ij})^2$ denote the squared distance error. One popular distance-based formation control is the following gradient control

$$u_i = \sum_{j \in \mathcal{N}_i} e_{k_{ij}}(p_j - p_i)$$

which was first proposed by Krick et al. (2009) and then developed by Dimarogonas and Johansson (2009), Dorfler and Francis (2010), Cao et al. (2011) and Oh and Ahn (2013). However, all of these results establish asymptotic convergence, i.e. the convergence of the formation shape can only be achieved in infinite time. To solve Problem 1, we propose the following control for each agent $i$:

$$u_i = \sum_{j \in \mathcal{N}_i} sig(e_{k_{ij}})^\alpha (p_j - p_i) \tag{9}$$

Using (9), we shall establish the main result of this section.

*Theorem 1.* The modified controller (9) achieves the finite time convergence of the formation shape.

### 3.1 Obtaining the Overall System

To prove Theorem 1, we let $p = \left[\, p_1^T,\, p_2^T,\, \cdots,\, p_n^T \,\right]^T$ and $e = \left[\, e_1^T,\, e_2^T,\, \cdots,\, e_m^T \,\right]^T$. One obtains the following *overall system*:

$$\dot{p} = -R^T(z) sig(e)^\alpha \tag{10}$$

The above compact form of the overall system can be derived by using the expression of the rigidity matrix as shown in (4) and (5). Note from the overall system one has immediately the following lemma, the proof of which will be found elsewhere.

*Lemma 5.* The system defined by (8) with the designed finite time controller (9) have the following properties:

- **(i)** The controller is decentralized in that each agent requires only relative position measurements of its neighboring agents.
- **(ii)** The center of the mass of the formation is stationary.
- **(iii)** The controller for each agent is independent of any global coordinates. That is, each agent can use its own coordinate system to measure relative positions and implement the control.

### 3.2 Stability Analysis of the Error System

We firstly define the equilibrium set for the overall system (10). For a given realization $p^* = [p_1^{*T}, \cdots, p_n^{*T}] \in \mathbb{R}^{dn}$ with the desired distances $d^*_{ij}$, the set $\mathcal{P}_S$ of the target formation shape which is congruent to $p^*$ is defined as:

$$\mathcal{P}_S = \{p \in \mathbb{R}^{dn} : \|p_j - p_i\| = \|p^*_j - p^*_i\| = d^*_{ij}, \quad \forall\, i, j \in \mathcal{V} \text{ and } i \neq j\} \tag{11}$$

Note that $\mathcal{P}_S$ is generally not compact, which complicates the stability analysis when the overall system is discussed.

For this reason, we will analyze instead the error system, which describes the evolution of $e$. Note that

$$\dot{e} = \frac{\partial e}{\partial p}\dot{p} = 2\frac{\partial r_{\mathcal{G}}(p)}{\partial p}\dot{p} = 2Z^T\bar{H}\dot{p} = 2R(z)\dot{p} \tag{12}$$

Using this and (10) one has the *error system* as follows:

$$\dot{e} = -2R(z)R^T(z)sig(e)^\alpha \tag{13}$$

Observe that the error system (13) involves the matrix product $M(z) := R(z)R^T(z)$, the entries of which have the following property.

*Lemma 6.* When the formation is close to the desired one, the entries of the matrix $M(z) = R(z)R^T(z)$ are continuously differentiable functions of $e$.

Note that the above lemma holds locally when $e$ is small and the formation is close to the desired one. A proof for triangular formations can be found at Belabbas et al. (2012), and the rather more difficult proof for minimally rigid formations with four or more vertices can be found at Mou et al. (2014). Also note that it indicates that the error system (13) is self-contained. Hence we could rewrite it as $\dot{e} = -2M(e)sig(e)^\alpha$, in which the matrix $M(z)$ is rewritten as $M(e)$ in order to reflect the result from Lemma 6.

### *3.3 Proof of the Main Result*

Now we are ready to give the proof of Theorem 1 as follows:

**Proof of Theorem 1:** Define a compact level set $\mathcal{H}(\rho) = \{e : V(e) \leq \rho\}$ for some small $\rho$ ($\rho > 0$), such that for all the points in the set $\mathcal{H}(\rho)$ the formation is infinitesimally and minimally rigid. The set also defines the initial formation shape which is close to the target one. Note that the set does not need to be arbitrarily small since the infinitesimal rigidity of the formation is a generic property for a dense set.

Consider the following Lyapunov function candidate:

$$V = \frac{1}{\alpha+1}\sum_{i=1}^{m}|e_i|^{1+\alpha} \tag{14}$$

Obviously $V$ is locally Lipschitz and continuously everywhere, and $V > 0$ for $e \neq 0$. Its derivative along the error system (13) is

$$\begin{aligned}\dot{V} &= \frac{1}{\alpha+1}\sum_{i=1}^{m}\frac{\partial|e_i|^{1+\alpha}}{\partial e_i}\frac{\mathrm{d}e_i}{\mathrm{d}t}\\ &= \sum_{i=1}^{m}sig(e_i)^\alpha\dot{e}_i\\ &= \left(sig(e)^\alpha\right)^T\left(-2M(e)\right)sig(e)^\alpha\end{aligned} \tag{15}$$

The second equality of the above equation has used the following fact:

$$\frac{\partial}{\partial e_i}|e_i|^{\alpha+1} = (\alpha+1)sig(e_i)^\alpha \tag{16}$$

Note that in the set $\mathcal{H}(\rho)$ the rigidity matrix is of full row rank, and further $M(e)$ is positive definite according to *Lemma 1.* Further let $\lambda$ denote the smallest eigenvalue of $RR^T$ when $e$ is in the set $\mathcal{H}$ (i.e. $\lambda = \min_{e\in\mathcal{H}} eig(M(e)) > 0$). The reason that $\lambda$ exists is because that the set $\mathcal{H}(\rho)$ is a compact set and the eigenvalues of a matrix are continuous functions of the matrix elements. Then the following holds:

$$\dot{V} \leq -2\lambda\left(sig(e)^\alpha\right)^T sig(e)^\alpha \tag{17}$$

Since $\alpha \in (0,1)$, then $2\alpha/(\alpha+1) \in (0,1)$. According to (14), one has

$$\begin{aligned}V^{\frac{2\alpha}{\alpha+1}} &= \left(\frac{1}{\alpha+1}\right)^{\frac{2\alpha}{\alpha+1}}\left(\sum_{i=1}^{m}|e_i|^{1+\alpha}\right)^{\frac{2\alpha}{\alpha+1}}\\ &\leq \left(\frac{1}{\alpha+1}\right)^{\frac{2\alpha}{\alpha+1}}\sum_{i=1}^{m}|e_i|^{2\alpha}\\ &= \left(\frac{1}{\alpha+1}\right)^{\frac{2\alpha}{\alpha+1}}\left(sig(e)^\alpha\right)^T sig(e)^\alpha\end{aligned} \tag{18}$$

For all $e \in \mathcal{H}(\rho)\setminus\{0\}$, $V^{\frac{2\alpha}{\alpha+1}}(t) > 0$. Thus

$$\begin{aligned}\dot{V} &\leq -2\lambda\left(sig(e)^\alpha\right)^T sig(e)^\alpha\\ &= -2\lambda\frac{\left(sig(e)^\alpha\right)^T sig(e)^\alpha}{V^{\frac{2\alpha}{\alpha+1}}}V^{\frac{2\alpha}{\alpha+1}}\\ &\leq -2\lambda\frac{\left(sig(e)^\alpha\right)^T sig(e)^\alpha}{\left(\frac{1}{\alpha+1}\right)^{\frac{2\alpha}{\alpha+1}}\left(sig(e)^\alpha\right)^T sig(e)^\alpha}V^{\frac{2\alpha}{\alpha+1}}\\ &= -2\lambda(\alpha+1)^{\frac{2\alpha}{\alpha+1}}V^{\frac{2\alpha}{\alpha+1}}\end{aligned} \tag{19}$$

By choosing $K = 2\lambda(\alpha+1)^{\frac{2\alpha}{\alpha+1}}$, the finite time stability of the error system is thus proved according to *Lemma 3.* According to the definition of rigidity (Definition 1), the finite time convergence of the error system is equivalent to the finite time convergence of the set $\mathcal{P}_{\mathcal{S}}$. Furthermore, the settling time satisfies $T \leq \frac{(1+\alpha)V(0)^{\frac{1-\alpha}{1+\alpha}}}{K(1-\alpha)}$. ■

Note that the convergence of the inter-agent distances will not directly guarantee the convergence of the agents' positions to some fixed points. However, we observe that a sufficient condition for the position $p_i$ of agent $i$ to converge to a fixed point is that $\int_0^\infty |u_i(t)|\mathrm{d}t < \infty$, which is true since all $|u_i(t)|$ are upper bounded and converge to the origin in finite time. Thus $p_i(t)_{t>T} = p_i^*$ which is constant in $\mathcal{P}_S$. To sum up, one has the following Corollary:

*Corollary 1.* For $t > T$, the control law achieves the finite time convergence of $p$ to a fixed point in $\mathcal{P}_S$.

*Remark 1.* The modified controller in (9) introduces the *sig* function, which indicates that the right side of (9) is continuous everywhere and locally Lipschitz everywhere except the origin. Thus the uniqueness of the solution with initial condition in $\mathbb{R}^{dn}\setminus\{0\}$ is guaranteed in forward time on a sufficiently small time interval. We note that there are some recent papers which focus on sliding mode control (involving discontinuous control) in distributed controller design; see e.g. Cao et al. (2010), Meng et al. (2013), Jafarian and De Persis (2013). Since discontinuous/non-smooth terms are involved in these works, non-smooth analysis comes into play and one should also be cautious about the often undesired chattering phenomena. In this paper, since the function $sig(\cdot)^\alpha$ with $\alpha \in (0,1)$ guarantees the continuity of the controllers, there can be no chattering.

*Remark 2.* The target formation shape set $\mathcal{P}_S$ is not a compact set, which thus complicates the stability analysis. In Krick et al. (2009), the center manifold theorem is employed for the stability analysis. In this section we have focused first on the error system for which the equilibrium set is compact.

*Remark 3.* The multi-agent finite time control problem (but not shape control problem) with more realistic agent models are discussed in several recent papers. For example, in Meng et al. (2010) the authors discussed the finite time attitude containment control for agents modeled by rigid body dynamics, while in Li and Wang (2013) the problem of position consensus for multi-AUV systems with collision avoidance was addressed. How to design finite time formation shape controller by considering these more realistic models could be an interesting future direction.

## 4. FORMATION AND FLOCKING CONTROL FOR DOUBLE INTEGRATOR AGENTS

In this section, we are interested in solving the following problem:

*Problem 2.* Consider a network of $n$ agents in $d$-dimensional space with associated minimally rigid graph and in which

$$\begin{aligned} \dot{p}_i &= v_i \\ \dot{v}_i &= u_i, \quad i = 1, 2, \cdots, n \end{aligned} \tag{20}$$

where $v_i \in \mathbb{R}^d$ is the velocity of agent $i$. Design the control $u_i$ for each agent $i$ in terms of $p_i - p_j$, $v_i - v_j$ where $j \in \mathcal{N}_i$ such that $||p_i - p_j||$ converges to the desired distance $d^*_{ij}$ and all $v_i$ converge to the same $v^*$ in *finite time.*

Anderson et al. (2012) employed a combination of the distance-based formation shape control and flocking control to achieve the exponential convergence. In this paper, we borrow the idea of previous works Bhat and Bernstein (1998) and Hong et al. (2002) on finite time stability of scalar double-integrator systems to obtain a modified control:

$$\begin{aligned} \dot{p}_i &= v_i \\ \dot{v}_i &= \sum_{j \in \mathcal{N}_i} sig(v_j - v_i)^{\alpha} + \sum_{j \in \mathcal{N}_i} sig(e_{ij})^{\frac{\alpha}{2-\alpha}}(p_j - p_i) \end{aligned} \tag{21}$$

which will be shown to solve Problem 2 as stated in the following theorem.

*Theorem 2.* For minimally and infinitesimally rigid formation shapes, the controller in (21) will guarantee the finite time convergence for the flocking of desired formation shape, with convergence of agents' velocities to $v^* = \frac{1}{n}\sum_{i=1}^n v_i(0)$.

### *4.1 Analysis*

For a given realization $p^* = [p_1^{*T}, \cdots, p_n^{*T}] \in \mathbb{R}^{dn}$ with the desired distances $d^*_{ij}$ and the desired final velocity $v^*$, we define the following set $\mathcal{P}_D$

$$\begin{aligned} \mathcal{P}_D = \{[p^T, v^T]^T \in \mathbb{R}^{2dn} : \|p_j - p_i\| = \|p_j^* - p_i^*\| = d^*_{ij}, \\ v_i = v_j = v^*, \forall i, j \in \mathcal{V} \text{ and } i \neq j\} \end{aligned} \tag{22}$$

By using a similar strategy as in Section 3 and realizing that $\dot{e} = 2R\dot{p}$ (see (12)), we derive the following equations which involve the distance error $e$ and the velocity term $v$:

$$\begin{aligned} \dot{e} &= 2Rv \\ \dot{v}_i &= \sum_{j \in \mathcal{N}_i} sig(v_j - v_i)^{\alpha} + \sum_{j \in \mathcal{N}_i} sig(e_{ij})^{\frac{\alpha}{2-\alpha}}(p_j - p_i) \end{aligned} \tag{23}$$

Further let $\bar{\delta} \in \mathbb{R}^d$ denote the average velocity of all the agents, i.e. $\bar{\delta}(t) = \frac{1}{n}\sum_{i=1}^n v_i(t)$. Observe that $\bar{\delta}(t)$ is time invariant, since a simple calculation using (23) shows that $\dot{\bar{\delta}} = 0$. We introduce the velocity disagreement vector $\delta = [\delta_1^T, \delta_2^T, \cdots, \delta_n^T]^T$, where $\delta_i$ is defined as $\delta_i = v_i - \bar{\delta}$. Thus, one has $\dot{\delta}_i = \dot{v}_i$. Notice that $v_i - v_j = \delta_i - \delta_j$. Therefore, there hold $Rv = Z^T\bar{H}v = Z^T\bar{H}\delta = R\delta$. Hence, one can transform (23) into the following equation:

$$\begin{aligned} \dot{e} &= 2R\delta \\ \dot{\delta}_i &= \sum_{j \in \mathcal{N}_i} sig(\delta_j - \delta_i)^{\alpha} + \sum_{j \in \mathcal{N}_i} sig(e_{ij})^{\frac{\alpha}{2-\alpha}}(p_j - p_i) \end{aligned} \tag{24}$$

To prove Theorem 2, one also needs the following lemma, the proof of which is omitted here.

*Lemma 7.* The set of velocity disagreements satisfies the following equality in the $n$-agent network:

$$\sum_{i=1}^{n}\sum_{j \in \mathcal{N}_i} \delta_i^T sig(\delta_j - \delta_i)^{\alpha} = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j \in \mathcal{N}_i} |\delta_j - \delta_i|^{1+\alpha} \tag{25}$$

### *4.2 Proof of the Main Result*

Now we are ready to give the proof of Theorem 2:

**Proof of Theorem 2:** For the same reason as mentioned in Theorem 1, the equilibrium set defined in $\mathcal{P}_D$ is not compact. Hence we will focus on (24) which involves the distance error $e$ and disagreement velocity $\delta$. Note that (24) is not a self-contained system, since the relative position term $z_i$ appears in the right-hand sides of (24). Nevertheless, we can still proceed with the stability analysis by employing a Lyapunov-like function.

Consider the following Lyapunov-like function candidate

$$V(e, \delta) = \frac{1}{2}\sum_{i=1}^{n}\delta_i^T\delta_i + \frac{2-\alpha}{4}\sum_{i=1}^{m}|e_i|^{\frac{2}{2-\alpha}} \tag{26}$$

It is obvious that $V$ is continuously differentiable everywhere, positive definite and radially unbounded. Differentiating $V(e, \delta)$ along system (24), one has

$$
\begin{aligned}
\dot{V}(e,\delta) &= \sum_{i=1}^{n} \delta_i^T \dot{\delta}_i + \frac{1}{2}\sum_{i=1}^{m} sig(e_i)^{\frac{\alpha}{2-\alpha}} \dot{e}_i \\
&= \sum_{i=1}^{n} \delta_i^T \sum_{j\in\mathcal{N}_i} sig(\delta_j - \delta_i)^{\alpha} \\
&\quad -\delta^T R^T sig(e)^{\frac{\alpha}{2-\alpha}} + \left(sig(e)^{\frac{\alpha}{2-\alpha}}\right)^T R\delta \\
&= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} |\delta_i - \delta_j|^{1+\alpha} \qquad \underline{\text{due to (25)}} \\
&\leq -\frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} (\delta_i - \delta_j)^2\right)^{(1+\alpha)/2} \qquad \underline{\text{due to (7)}} \\
&= -\frac{1}{2}\left(2\delta^T \bar{L}\delta\right)^{(1+\alpha)/2} = -\frac{2^{(1+\alpha)/2}}{2}|\bar{H}\delta|^{1+\alpha}
\end{aligned}
$$

The level sets of $V(e,\delta)$ are compact w.r.t. the distance error $e$ and velocity disagreement term $\delta$. Since $\dot{V}(e,\delta) \leq 0$, one has $V(e(t),\delta(t)) \leq V(e(0),\delta(0))$, which indicates that $e(t)$ and $\delta(t)$ are bounded. Due to the fact that (24) is not a self-contained system, it is not straightforward to apply LaSalle's Invariance Principle here. Let us examine the term $\dot{V}(e,\delta)$ and its derivative:

$$
\ddot{V}(e,\delta) = -\frac{1+\alpha}{2}\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i}\left(sig(\delta_i - \delta_j)^{\alpha}\right)^T\left(\dot{\delta}_i - \dot{\delta}_i\right) \quad (27)
$$

Since $\ddot{V}(e,\delta)$ is continuous and bounded, $\dot{V}$ is uniformly continuous. By invoking Barbalat's Lemma Slotine and Li (1991), one can conclude that all the agents will converge to a configuration in which $|\bar{H}\delta|^{1+\alpha} = 0$, or equivalently, $\bar{H}\delta = 0$. Note that $\bar{H} = H \otimes I_d$ and $ker(H) = span\{\mathbf{1}_n\}$. This implies $\delta_1 = \delta_2 = \cdots = \delta_n$. Further note that $\sum_{i=1}^{n}\delta_i = \mathbf{0}$, which indicates that $\delta_1 = \delta_2 = \cdots = \delta_n = \mathbf{0}$ at the steady state. Thus, all the components of $v$ in each direction will be the same, i.e., the velocity alignment is achieved.

Note that at the steady state $\delta_i = 0$ also implies that $\dot{\delta}_i = \sum_{j\in\mathcal{N}_i} sig(e_{ij})^{\frac{\alpha}{2-\alpha}}(p_j - p_i) = 0$, which can be written in a compact form as $\dot{\delta} = -R^T sig(e)^{\frac{\alpha}{2-\alpha}} = 0$. Since $R^T$ is of full column rank due to the minimal and infinitesimal rigidity of the target framework, one obtains $sig(e)^{\frac{\alpha}{2-\alpha}} = 0$, or equivalently, $e = 0$. In conclusion, the trajectories will converge to the largest invariant set $\Omega = \{(e,v) : v = v^* \otimes \mathbf{1}_n \text{ and } e = 0\}$. Note that the set $\Omega$ is the same as the objective formation set $\mathcal{P}_D$ according to the definition of the framework rigidity. Thus the control goal is achieved, at least asymptotically.

In the following, we will prove that the convergence of the velocity alignment, as well as the convergence of formation shape stabilization, can be achieved in finite time. The proof is inspired by Bhat and Bernstein (1998). It can be verified that, for $k > 0$,

$$
\begin{aligned}
V(k^{2-\alpha}e, k\delta) &= k^2 V(e,\delta) \qquad (28)\\
\dot{V}(k^{2-\alpha}e, k\delta) &= k^{1+\alpha}\dot{V}(e,\delta)
\end{aligned}
$$

By letting $k = (V(e,\delta))^{-\frac{1}{2}}$, one can obtain

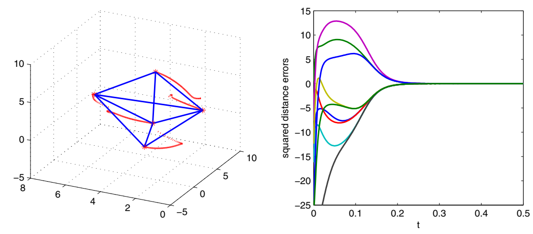


Fig. 1. Finite time stabilization of a double tetrahedron formation. Left: the trajectories of five agents and the final formation shape. Right: Time evolutions of the squared distance errors.

$$
\begin{aligned}
&V\left((V(e,\delta))^{-\frac{2-\alpha}{2}}e, (V(e,\delta))^{-\frac{1}{2}}\delta\right) \\
&\quad = \left((V(e,\delta))^{-\frac{1}{2}}\right)^2 V(e,\delta) = 1
\end{aligned}
$$

Define the set $\mathcal{S} = \{(e^T,\delta^T)^T \in \mathbb{R}^{m+dn} : V(e,\delta) = 1\}$. Since $V$ is radially unbounded, the set $\mathcal{S}$ is compact. Furthermore, since $\dot{V}$ is continuous, $\dot{V}$ can achieve its maximum on the compact set $\mathcal{S}$. Thus, one obtains

$$
\begin{aligned}
\frac{\dot{V}(e,\delta)}{(V(e,\delta))^{\frac{1+\alpha}{2}}} &= \dot{V}\left((V(e,\delta))^{-\frac{2-\alpha}{2}}e, (V(e,\delta))^{-\frac{1}{2}}\delta\right) \\
&\leq \max_{(e,\delta)\in\mathcal{S}} \dot{V}(e,\delta) = -c \qquad (29)
\end{aligned}
$$

where $c > 0$. This is due to the fact that $-\dot{V}$ is positive definite on the set $\mathcal{S}$. The following result then follows

$$
\dot{V}(e,\delta) \leq -c(V(e,\delta))^{\frac{1+\alpha}{2}} \qquad (30)
$$

Note that since $\alpha \in (0,1)$, one has $\frac{1+\alpha}{2} \in (0,1)$. From *Lemma 3*, one obtains the origin of the system is a finite time stable equilibrium with the settling time satisfying $T \leq \frac{2}{c(1-\alpha)}\left(V(e(0),\delta(0))\right)^{\frac{1-\alpha}{2}}$. This completes the proof. ■

## 5. SIMULATIONS

We first perform simulations of controlling a five-agent minimally rigid formation in 3D, in which each agent is modeled by a single integrator. The underlying graph is with a double tetrahedron of nine edges. The desired distance for each edge is set to be 6. The initial positions are chosen such that the initial formation is infinitesimally rigid and is close to the target formation shape. By employing the finite time controller (9) with $\alpha = 0.5$, the five agents finally achieve the desired formation shape in finite time as shown in Fig. 1. Furthermore, it is worth mentioning that there is no chattering occurring in the formation stabilization process.

Then we let each agent be modeled by a double integrator. The control goal is not only to achieve the desired double tetrahedron shape but also to drive all agents' velocities to be the same in finite time. The simulation settings are the same as those in the single integrator case, while the initial velocities for each agent are chosen randomly. The simulation results are depicted in Fig. 2, which shows that the five-agent group achieves the desired formation shape in finite time and then moves together with the common velocity and with the desired formation shape.

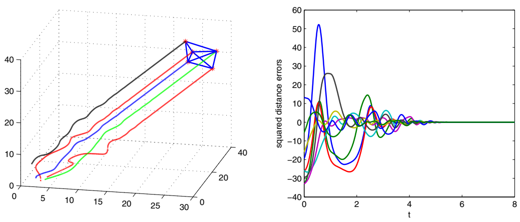

Fig. 2. Flocking control of five agents with double tetrahedron formation. Left: the flocking trajectories of five agents. Right: Time evolutions of the squared distance errors.

## 6. CONCLUSION

In this paper we have studied the finite time distance-based formation control problem. A modified gradient control law was proposed to stabilize a predefined formation shape for a group of agents modelled by single integrators. Several interesting properties of the finite time formation controller were discussed. It should be noted that the equilibrium set of the overall system is not compact, which complicates the stability analysis. To deal with this issue, we have proved that the error system is a self-contained one with a compact equilibrium set and then the finite time stability was established. Furthermore, we have considered double integrator agents and proposed a finite time flocking algorithm so that the agent group can achieve desired flocking motions with finite time velocity alignments and finite time convergence of the formation shape. The main results were proved by finite time Lyapunov theorem and Barbalat's Lemma. Typical simulations with a five-agent group in 3-D space were provided to verify the performances of the developed finite time controllers.

## REFERENCES

Anderson, B.D.O., Lin, Z., and Deghat, M. (2012). Combining distance-based formation shape control with formation translation. In L. Qiu, J. Chen, T. Iwasaki, and H. Fujioka (eds.), *Developments in Control Theory Towards Glocal Control*, 121–130. IET.

Anderson, B.D.O., Yu, C., Dasgupta, S., and Morse, A.S. (2007). Control of a three coleader persistent formation in the plane. *Systems and Control Letters*, 573–578.

Asimow, L. and Roth, B. (1979). The rigidity of graphs, ii. *Journal of Mathematical Analysis and Applications*, 68(1), 171–190.

Belabbas, A., Mou, S., Morse, A.S., and Anderson, B.D.O. (2012). Robustness issues with undirected formations. In *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, 1445–1450. IEEE.

Bhat, S.P. and Bernstein, D.S. (1998). Continuous finite-time stabilization of the translational and rotational double integrators. *Automatic Control, IEEE Transactions on*, 43(5), 678–682.

Bhat, S.P. and Bernstein, D.S. (2000). Finite-time stability of continuous autonomous systems. *SIAM Journal on Control and Optimization*, 38(3), 751–766.

Cao, M., Morse, A.S., Yu, C., Anderson, B.D.O., and Dasgupta, S. (2011). Maintaining a directed, triangular formation of mobile autonomous agents. *Communications in Information and Systems*, 11(1), 1–16.

Cao, Y., Ren, W., and Meng, Z. (2010). Decentralized finite-time sliding mode estimators and their applications in decentralized finite-time formation tracking. *Systems & Control Letters*, 59(9), 522–529.

Dimarogonas, D.V. and Johansson, K.H. (2009). Further results on the stability of distance-based multi-robot formations. In *American Control Conference, 2009. ACC'09.*, 2972–2977. IEEE.

Dorfler, F. and Francis, B. (2010). Geometric analysis of the formation problem for autonomous robots. *Automatic Control, IEEE Transactions on*, 55(10), 2379–2384.

Hardy, G.H., Littlewood, J.E., and Polya, G. (1952). *Inequalities*. Cambridge university press.

Hendrickson, B. (1992). Conditions for unique graph realizations. *SIAM Journal on Computing*, 21(1), 65–84.

Hong, Y., Xu, Y., and Huang, J. (2002). Finite-time control for robot manipulators. *Systems & control letters*, 46(4), 243–253.

Hui, Q., Haddad, W.M., and Bhat, S.P. (2008). Finite-time semistability and consensus for nonlinear dynamical networks. *Automatic Control, IEEE Transactions on*, 53(8), 1887–1900.

Jafarian, M. and De Persis, C. (2013). Formation control with binary information. *arXiv preprint arXiv:1304.4893*.

Krick, L., Broucke, M.E., and Francis, B.A. (2009). Stabilisation of infinitesimally rigid formations of multi-robot networks. *International Journal of Control*, 82(3), 423–439.

Li, S. and Wang, X. (2013). Finite-time consensus and collision avoidance control algorithms for multiple auvs. *Automatica*, 49(11), 3359–3367.

Meng, Z., Lin, Z., and Ren, W. (2013). Robust cooperative tracking for multiple non-identical second-order nonlinear systems. *Automatica*, 49(8), 2363–2372.

Meng, Z., Ren, W., and You, Z. (2010). Distributed finite-time attitude containment control for multiple rigid bodies. *Automatica*, 46(12), 2092–2099.

Mou, S., Belabbas, M.A., Morse, A.S., and Anderson, B.D.O. (2014). Non-robustness issues of gradient control on undirected formations. *IEEE Transactions on Automatic Control, In Preparation*.

Mou, S., Cao, M., and Morse, A.S. (2011). A distributed control law for acyclic formations. *Proc. of the 18th IFAC Congress*, 7818–7823.

Oh, K.K. and Ahn, H.S. (2013). Distance-based undirected formations of single-integrator and double-integrator modeled agents in n-dimensional space. *International Journal of Robust and Nonlinear Control*, In press.

Oh, K.K., Park, M.C., and Ahn, H.S. (2012). A survey of multi-agent formation control: Position-, displacement- and distance-based approaches. *avaiable online: http://dcas.gist.ac.kr/pubs/TR201202.pdf*.

Slotine, J.J.E. and Li, W. (1991). *Applied nonlinear control*, volume 199. Prentice hall New Jersey.

Xiao, F., Wang, L., Chen, J., and Gao, Y. (2009). Finite-time formation control for multi-agent systems. *Automatica*, 45(11), 2605–2611.

Yu, C., Anderson, B.D.O., Dasgupta, S., and Fidan, B. (2009). Control of minimally persistent formations in the plane. *SIAM Jounal on Control and Optimization*, (1), 206–233.