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Path Planning for Minimizing Detection

B. Jiang * **A. N. Bishop** ** **B. D. O. Anderson** **
S. P. Drake ***

* *Research School of Engineering, Australian National University, Canberra, Australia (e-mail: u5225976@anu.edu.au).*
** *Research School of Engineering, Australian National University and National ICT Australia (NICTA), Canberra, Australia (e-mail: adrian.bishop,brian.anderson@anu.edu.au).*
*** *Defence Science and Technology Organisation, Edinburgh, Australia and School of Chemistry and Physics, The University of Adelaide, Adelaide, Australia (e-mail: sam.drake@dsto.defence.gov.au).*

Abstract: For a flying military vehicle, avoiding detection can be a key objective. To achieve this, flying the least-probability-of-detection path from A to B through a field of detectors is a fundamental strategy. While most of the previous optimization models aim to minimize the cumulative radar exposure, this paper derives a model that can directly minimize the probability of being detected. Furthermore, a variational dynamic programming method is applied to this model which allows one finding a precise local optimal path with low computational complexity. In addition, a homotopy method is derived to adjust the optimal path with exceptionally low computational complexity when the detection rate function changes due to the removal of detectors, the addition of detectors or the changes of understanding of detectors.

1. INTRODUCTION

For flying military vehicles, avoiding detection is often a key objective. There has been much prior research leading to many different sophisticated strategies to avoid detection. Maithripala et al. (2008) explained how to deceive a ground radar network into seeing a spurious phantom track in its radar space by intercepting the pulse, introducing a time delay and re-transmitting the radars pulses. Hai (1999) treats the use of radar stealth material to absorb radar pulses.

For a detection minimizing problem, which is the focus of this paper, it is an obvious strategy to fly the least-probability-of-detection path from the source to the destination through a field of detectors; this strategy has been studied with increasing intensity as vehicles have become increasingly intelligent. Much research emphasises the modelling aspect of this problem. Examples of such work are Zabarankin et al. (2006); Kim and Hespanha (2003); Murphey and Pardalos (2002); Murphey et al. (2003), in which risk functions are employed to represent the cumulative radar exposure when flying through the field of detectors. The objective of these models is to minimize the cumulative radar exposure. Other research focuses on the algorithms to solve this problem. Examples of such research are Zabarankin et al. (2006); Royset et al. (2009); Murphey and Pardalos (2002), in which the problem is abstracted to be a weight constrained shortest path problem (WCSPP) and then solved.

In the above papers, the overall objective is to minimise the cumulative radar exposure instead of the probability of being detected. These two objectives may intuitively seem to be the same but they are not precisely equivalent. For example, if a mobile vehicle enters the non-escape zone of a radar, the model may still try to minimize the cumulative radar exposure which will ultimately make no difference in the probability of being detected (which always equals 1). Again, if a mobile vehicle is very far away from the detector and the signal to noise ratio is very low, the optimization of radar exposure will also make no difference.

As an example of the model whose objective is to minimise the probability of being detected, the paper Royset et al. (2009) modelled radars using threat circles and solved the problem using Lagrangian relaxation plus (near-shortest path) enumeration (LRE). However, due to the idealized detection model and the high complexity of the algorithm, the method can only provide a very rough optimal path.

This paper considers the problem where the objective is to minimize the probability of being detected rather than cumulative radar exposure. While the majority of radars work based on a pulse-by-pulse mechanism, we assume that the radar refresh frequency is high enough so that the detection event can be considered as a continuous event. The derived cost function has the identical form but different underlying meaning as those in cumulative exposure models. Consequently, most of the numerical algorithms are still applicable with a partial adjustment on the typical detection rate function.

Considering the high computational complexity of previous numerical algorithms, this paper utilizes the variational dynamic programming approach which can obtain accurate local optima. Furthermore, a homotopy method is derived to adjust the optimal path when the detection rate function changes due to the removal of detectors, the addition of detectors, the changes of understanding of detectors, and so on. The underlying principle of this homotopy method is to linearize the change of optimal path using the first order calculus of variations. The linearized model can be found with several matrix operations; further because the major coefficient matrix is a tridiagonal matrix, the computational complexity of this approach is exceptionally low. As a result, even miniature unmanned aerial vehicles (UAVs) such as quadrotors can deploy this method using on-board calculation.

The novel contributions of the paper are as follows (a) adjusting detection rate function analogous to a Poisson Process which is closer to reality, (b) applying variational dynamic programming on this problem, which can obtain local optima with low complexity, and (c) deriving a homotopy method which can adjust the optimal path with extremely low computational complexity when the detection rate function changes.

2. PROBLEM STATEMENT USING PROBABILITY OF DETECTION MEASURE

Consider a closed bounded region $\mathcal{C}$ in $\mathbb{R}^2$. Within this region, it is understood that there is a moving agent or target, and a set of detectors. The target wishes to remain undetected, and the detectors are not perfect. The target is assumed to be moving with constant (unit) speed between two fixed points in $\mathcal{C}$ along a simple (i.e. non-self-intersecting), piecewise smooth curve.

As the detectors are imperfect, associated with every point $(x, y) \in \mathcal{C}$ is a nonnegative value $p(x, y)$, with the interpretation that the probability that a target located at the point (x, y) would be detected in a time interval of length Δt is $p(x, y)\Delta t + \mathcal{O}(\Delta t)$. The function p is assumed to be smooth. Detection events in non-overlapping time intervals are independent. For a target stationary at (x, y), the assumptions made so far mean that the probability of first detection in a time interval of length T is $1 - \exp[-p(x, y)T]$.

The problem of interest is then to determine, given two fixed points in $\mathcal{C}$ and some function $p(x, y)$, an optimal trajectory for the target between these two points such that the detection probability is minimised.

2.1 Computing the Probability of Detection for a Path

Consider a parameterised path $(x(s), y(s))$ of length L from (x_0, y_0) to (x_T, y_T). For a small positive Δ, determine the points $(x_i, y_i) = (x(i\Delta), y(i\Delta))$ such that the unit speed assumption implies a sequence of Δ spaced points (x_i, y_i) along the path.

The probability of non-detection for the target is the probability that there is no detection in each interval with end points (x_i, y_i) and (x_{i+1}, y_{i+1}), $i = 0, 1, \ldots, (L-1)/\Delta$. The probability that there is a detection in the i-th interval is $p(x_i, y_i)\Delta + o(\Delta)$ and so the probability that there is no detection is simply $[1 - p(x_i, y_i)\Delta + o(\Delta)]$. Since detection events are independent through time, the probability that there is no detection along the whole path is

$$P_{no}(x(\cdot), y(\cdot)) = \Pi_{i=1}^{(L-1)/\Delta}[1 - p(x_i, y_i)\Delta + o(\Delta)] \quad (1)$$

Evidently,

$$\ln P_{no}(x(\cdot), y(\cdot)) = \sum_{i=1}^{(L-1)/\Delta} \ln[1 - p(x_i, y_i)\Delta + o(\Delta)] \quad (2)$$

Letting Δ go to zero, we see that the probability of detection, call it P_{det}, obeys

$$\begin{aligned} \ln(1 - P_{det}) &= \ln P_{no}(x(\cdot), y(\cdot)) \quad (3) \\ &= -\lim_{\Delta\to 0} \sum_{i=1}^{(L-1)/\Delta} p(x_i, y_i)\Delta \\ &= -\int_Q p(x(s), y(s))ds \end{aligned}$$

where Q is a path starting from the point (x_0, y_0) and ends at the point (x_T, y_T). Note the integration is effectively a parameterised path integral. Equivalently, we thus have

$$P_{det} = 1 - \exp[-\int_Q p(x(s), y(s))ds] \quad (4)$$

It is straightforward to compute the probability of detection for a prescribed trajectory and knowledge of $p(x, y)$.

The problem of interest requires the computation of a trajectory that minimises the detection probability given two fixed points (x_0, y_0), (x_T, y_T) and knowledge of the detection probability function $p(x, y)$. This is a calculus of variations problem which is equivalent to finding a given trajectory that minimises the path integral

$$\int_Q p(x(s), y(s))ds \quad (5)$$

If we view (5) in isolation then the problem formulation resembles the widely considered risk function approach to path planning where $p(x, y)$ could be regarded as a detection probability per unit time; see Zabarankin et al. (2006); Kim and Hespanha (2003); Murphey and Pardalos (2002); Murphey et al. (2003). However, the interpretation as presented here is based on a derivation starting from an infinitesimal detection rate $p(x, y)$ where the probability of detecting a target at (x, y) for one second is *not* $p(x, y)$ but rather $1 - \exp[-p(x, y)]$. We believe the idea of defining an infinitesimal detection rate in this sense is more natural. Indeed, the function $p(x, y)$ can be thought of as an intensity function for a Poisson process defined at each point (x, y) in the sense that for a stationary target at (x, y) the time of first detection could be thought of as a nonnegative random variable with distribution function $1 - \exp[-p(x, y)T]$ over T. In addition to detection events being independent across non-overlapping time intervals, it also follows that detection events for fixed targets at distinct points in $\mathbb{R}^2$ are also independent. Along any constant velocity path the intensity $p(x, y)$ as a function of the path (and consequently time) essentially defines a non-homogeneous Poisson process (Franceschetti and Meester, 2007, p. 9). The typical detection rate function is given in Appendix and that model will be used in the rest of this paper unless otherwise noted.

Note that when deriving (5), the constant speed constraint (throughout taken to be unity) has been taken into account. A more convenient form of (5) for our purposes is given by

$$\int_0^1 p(x(\tau), y(\tau))\sqrt{{x'_\tau}^2 + {y'_\tau}^2}d\tau \quad (6)$$

where $\tau \in [0,1]$ is a given parameterization obeying $(x(0), y(0)) = (x_0, y_0)$ and $(x(1), y(1)) = (x_T, y_T)$ and $'$ denotes $\frac{d}{d\tau}$. The objective is to find the function $(x(\cdot), y(\cdot))$ defined on $[0,1]$ that minimizes (6).

Throughout we assume that the target trajectory is piecewise smooth as requiring total smoothness rules out paths with an isolated corner, and such paths may be optimal when compared to a totally smooth path.

2.2 The Detection Probability with Multiple Detectors

Consider a collection $p_1(x,y), p_2(x,y), \ldots, p_i(x,y)$ of detection rate functions associated with, for example, multiple detectors. Then the relevant value of $p(x,y)$ is simply the sum so long as the individual $p_i(x,y)$ allow one to define the probability of detection on an infinitesimal interval δt and each detector operates independently (i.e. detection events occur independently).

More formally, the probability that detector i does not detect the target is $1 - p_i\delta t$ and the probability that all detectors fail to detect the target is $\prod(1 - p_i\delta t)$ owing to the independence assumption. Then the probability of detection is $1 - \prod(1 - p_i\delta t) = (\sum p_i)\delta t + o(\delta t)$.

3. A VARIATIONAL DYNAMIC PROGRAMMING ALGORITHM

Consider a continuously parameterised target trajectory $(x(s), y(s))$ from (x_0, y_0) to (x_T, y_T) in some compact set $\mathcal{C} \subset \mathbb{R}^2$. For a positive Δ, determine the points $(x_i, y_i) = (x(i\Delta), y(i\Delta))$ such that the unit speed assumption implies a sequence of Δ spaced points (x_i, y_i) along $(x(s), y(s))$. Suppose Δ is chosen such that there are $N+1$ such points starting at (x_0, y_0) and ending at $(x_N, y_N) = (x_T, y_T)$. If we approximate $(x(s), y(s))$ by a straight line on each interval between (x_i, y_i) and (x_{i+1}, y_{i+1}) then the target path will be a piecewise smooth polygonal curve. The optimal trajectory that minimises the stated probability of detection is given by finding a trajectory that minimises

$$\int_Q p(x(s), y(s))ds \approx \sum_{i=0}^{N} \Delta p(x_i, y_i)\sqrt{{x_i'}^2 + {y_i'}^2} \qquad (7)$$

or more specifically, the optimal trajectory is determined by a set of waypoints (x_i, y_i), $i = 1, \ldots, N-1$ that minimise the sum. This simple approximation transforms a calculus of variations problem into a classical optimisation problem over the parameters (x_i, y_i). The approximation of a line integral by a Riemann sum over straight line segments and the convergence of this sum as $N \to \infty$ justifies this approximation.

Denote the cost to move from (x_a, y_a) to (x_b, y_b) by $\mathcal{J}_{a,b}$. Then, under the discretisation assumptions thus far

$$\int_Q p(x(s), y(s))ds \approx \sum_{i=1}^{N} \mathcal{J}_{i-1,i} \qquad (8)$$

and if we relax the approximation that each interval between (x_i, y_i) and (x_{i+1}, y_{i+1}) is a straight line then the incremental cost to move from (x_{n-1}, y_{n-1}) to (x_n, y_n) is

$$\mathcal{J}_{n-1,n} = \int_{\frac{n-1}{N}}^{\frac{n}{N}} p(x,y)\sqrt{{x_\tau'}^2 + {y_\tau'}^2}d\tau \qquad (9)$$

Going forward, the assumptions thus far allow us to define $\mathcal{C}_n$ by a finite subset of $\mathbb{R}^2$ so $(x_i, y_i) \in \mathcal{C}_n$. Suppose then that $\mathcal{J}^*_{0,n}$ is the opimal cost when the target moves from (x_0, y_0) to (x_n, y_n). Since the value of (x_0, y_0) is prescribed it follows that $\mathcal{J}^*_{0,n}$ is only a function of $(x_n, y_n) \in \mathcal{C}_n$. Then

$$\mathcal{J}^*_{0,n} = min\{\mathcal{J}^*_{0,n-1} + \mathcal{J}_{n-1,n}\}, \quad 1 \le n \le N \qquad (10)$$

Now we have enough equations to state a variable dynamic programming solution to the given optimisation problem. The broad idea of the algorithm is to guess a path, then consider variations to it based on local perturbations and when a reduction in the cost function is achieved, select the adjusted path. Let $x_1^*, x_2^*, \ldots, x_{N-1}^*$ and $y_1^*, y_2^*, \ldots, y_{N-1}^*$ denote the optimal waypoints. Then the algorithm is given by Algorithm 1.

Algorithm 1. Variational Dynamic Programming

Select a check point number N
Select a recursive loop number M
Generate an initial path as near as possible to the optimal path represented by check points (x_n, y_n) where $0 \le n \le N$
for $i = 1; i \le M; i = i+1$ **do**
  for $n = 1, n \le N, n = n+1$ **do**
    Generate a finite set of points $\mathcal{C}_n$ whose elements include (x_n, y_n) and a so-called set of neighbour points surrounding (x_n, y_n).
  end for
  for $n = 2, n \le N, n = n+1$ **do**
    For each $(x_n, y_n) \in \mathcal{C}_n$, choose $(x_{n-1}, y_{n-1}) \in \mathcal{C}_{n-1}$ to minimize the function $\mathcal{J}^*_{0,n-1} + \mathcal{J}_{n-1,n}$. Then every (x_n, y_n) is mapped with one or more (x_{n-1}, y_{n-1}) and $\mathcal{J}^*_{0,n}$ can be expressed as a function of (x_n, y_n) only.
  end for
  Use the destination coordinates (x_N, y_N) to find points $(x^*_{N-1}, y^*_{N-1}), (x^*_{N-2}, y^*_{N-2}), \ldots, (x_1^*, y_1^*)$.
  Assign the value of $(x_1^*, y_1^*), (x_2^*, y_2^*), \ldots, (x_N^*, y_N^*)$ to ordered pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$ accordingly.
end for

In the above algorithm, when generating the set $\mathcal{C}_n$, a set of neighbour points was introduced. This set of neighbour points can be any arbitrary finite set of points within a local neighbourhood of the point (x_n, y_n). A typical example of a set of neighbour points, one which is used in our own simulations later, is a set of points lying on a rectangular grid centred on (x_n, y_n); e.g. the points $\{(x_n, y_n \pm \delta), (x_n \pm \delta, y_n), (x_n \pm \delta, y_n \pm \delta), \ldots\}$.

Suppose the total cost after $m \in [0, M]$ iterations is ω_m and the total cost of the global optimal path is ω_{min}. Because the sequence $\omega_0, \omega_1, \omega_2 \ldots$ is monotonically decreasing and

$$\omega_m \ge \omega_{min}, \quad \forall m \in [0, M]$$

we can conclude that the sequence $\omega_0, \omega_1, \omega_2 \ldots$ converges.

Now suppose the total cost converge to a value Ω. The statement $\forall \epsilon > 0, \exists N$ *and* Δ *such that* $\Omega - \omega_{min} \le \epsilon$ may not hold because the method can only ensure a local optima is achieved. As a result, the generation of the original path (i.e. the initial guess) has a critical impact on the final result. This could be even more critical when there is more than one detector.

The above Variational Dynamic Programming algorithm can be extended to high dimensional spaces, where its advantage on computational complexity in comparison to Weighted Shortest Path is more significant.

3.1 Illustrative Example

Given only one detector, the WSP algorithm proposed in Murphey et al. (2003) provides a rough approximate solution to (5) for a given $p(x, y)$ within a reasonable amount of time and the probability of being detected cannot be improved upon much by the variational method proposed in this section. When the number of detectors increases however this is not necessarily true.

We consider a simple illustration of the variational method in Figure 1. The circles represent radar arrays, the dash line is the optimal path generated by the WSP algorithm from Murphey et al. (2003), the solid line is the path modified from this using the variational algorithm proposed in this section. The probability of being detected along the blue path is 31% while the probability of being detected along the green path is 27%. Furthermore, as real vehicles cannot have instantaneous changes in velocity, the new path is more feasible.

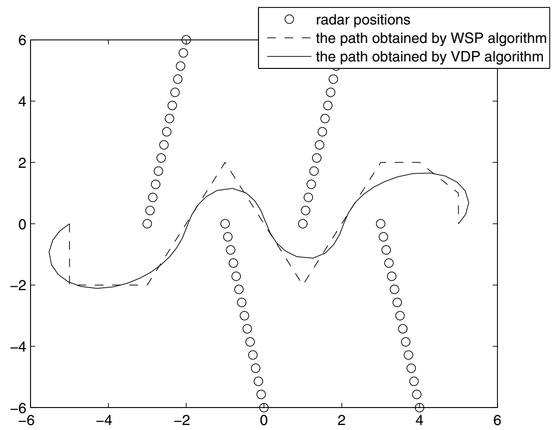

Fig. 1. Simulation result using Variational Dynamic Programming

4. A HOMOTOPY METHOD FOR FAST TRAJECTORY COMPUTATION

The main contribution of this paper is a computationally fast homotopy method for deriving a modified optimal path given a change in the total detection probability rate $p(x, y)$ and the previously derived optimal path. This method is practically important as the environment in which the target moves may be dynamic; e.g. the number of detectors may change as new detectors are added/discovered, detectors may be removed or switched off or the detector function at particular detectors may evolve, etc. in all such cases it would be very time consuming to re-run the optimal trajectory planning algorithm and obtain the complete path from the beginning.

More formally, we suppose one already has an optimal solution to (6) (or (5)), for some $p(x, y)$, expressed by $x(\tau)$ and $y(\tau)$. Given the addition/removal of detectors or an otherwise change in detection rate at some detectors, we suppose the detection rate is changed to $p(x, y) + q(x, y)$. The homotopy method introduced in this section allows one to modify the optimal path to the updated detection rate with rapid modification if the change is small. The computation of the updated path can be accomplished by iteratively performing certain matrix operations and the computational complexity is considerably less than variational dynamic programming algorithm introduced previously and even the rough approximation of the WSP algorithm proposed in Murphey et al. (2003).

4.1 The Homotopy Approach: A Special Case

A special case will be considered first due to its analytic solvability. Furthermore, it can also serve as introduction to more complicated general cases. The special case is when probability rate of detection is constant on any circle whose centre is the detector and there is only one such detector. Now set up a polar coordinates whose origin on the detector. Suppose ρ is the radial coordinate and θ is the angle. The probability of detection is only a function of ρ.

If the desired initial and terminal points for the trajectory are on the same radial line then the optimal path is just a straight line. Otherwise, suppose θ_0 and θ_T define the polar angles of the desired initial and terminal points. Then (5) becomes

$$\int_{\theta_0}^{\theta_T} p(\rho(\theta))\sqrt{\rho^2(\theta) + (\frac{d\rho(\theta)}{d\theta})^2}d\theta \tag{11}$$

from which, using the Beltrami identity by Fox (1950), we find

$$\frac{p(\rho(\theta))\rho^2(\theta)}{\sqrt{\rho^2(\theta) + (\frac{d\rho(\theta)}{d\theta})^2}} = C \tag{12}$$

where C is some constant.

Now if the first detector is changed but remains unmoved, one can use homotopy method to fast modify its path. Suppose for a certain detection rate $p(\rho)$ we know the optimal path $\rho(\theta)$, $\theta \in [\theta_0, \theta_T]$. Then consider a small variation and let the detection rate function be

$$p(\rho) + \lambda q(\rho)$$

where $q(\rho)$ is a bounded function and λ is a sufficiently small number. Correspondingly, a small variation to the optimal path is expected and we model this as

$$\rho(\theta) + \lambda\sigma(\theta)$$

where $\sigma(\theta)$ is also bounded. Furthermore, the constant on the right hand side of the Beltrami identity will become $C + \lambda\gamma$ where γ is a finite number.

Now we have

$$\frac{[p(\rho) + \lambda q(\rho)] \cdot [\rho(\theta) + \lambda\sigma(\theta)]^2}{\sqrt{[\rho(\theta) + \lambda\sigma(\theta)]^2 + [\frac{d\rho(\theta)}{d\theta} + \lambda\frac{d\sigma(\theta)}{d\theta}]^2}} = C + \lambda\gamma \tag{13}$$

Therefore, on subtracting (12) from (13) and eliminating the terms in higher order of λ we have

$$2p\rho\sigma + \rho^2 q = C\frac{\rho\sigma + (\frac{d\rho}{d\theta})(\frac{d\sigma}{d\theta})}{\sqrt{\rho^2 + (\frac{d\rho}{d\theta})^2}} + \gamma\sqrt{\rho^2 + (\frac{d\rho}{d\theta})^2} \tag{14}$$

where $p(\rho)$, $q(\rho)$ and $\rho(\theta)$ are known functions and C is a known constant which can be obtained from (12). Now we have a linear differential equation in $\sigma(\theta)$

$$a_1(\theta)\sigma(\theta) + a_2(\theta)\sigma'(\theta) + a_3(\theta)\gamma + a_4(\theta) = 0 \quad (15)$$

where $a_1 = \rho p(2 - \frac{\rho^2}{\rho^2+\rho'^2})$, $a_2(\theta) = -\frac{p\rho^2\rho'}{\rho^2+\rho'^2}$, $a_3(\theta) = -\sqrt{\rho^2+\rho'^2}$ and $a_3(\theta) = \rho^2 q$. The solution to this first order linear differential equation is

$$\begin{aligned}\sigma(\theta) =& exp(\int_{\theta_0}^{\theta} \frac{\rho^2 + 2\rho'^2}{\rho\rho'} d\theta) \\ &\cdot [\int_{\theta_0}^{\theta} exp(\int_{\theta_0}^{\theta} -\frac{\rho^2 + 2\rho'^2}{\rho\rho'} d\theta) g(\theta) d\theta]\end{aligned} \quad (16)$$

where

$$g(\theta) = -\frac{(\gamma\sqrt{\rho^2+\rho'^2} - \rho^2 q)(\rho^2+\rho'^2)}{p\rho^2\rho'} \quad (17)$$

where $'$ denotes $\frac{d}{d\theta}$. The initial condition is $\sigma(\theta_0) = 0$ while $\sigma(\theta_T) = 0$ from which the constant γ can be obtained.

It is not clear a priori whether (16) is bounded. Simulation result shows that (16) is bounded for many choices of $q(\theta)$ but unbounded for others. Often (16) is unbounded when the detection rate falls off so fast that the optimal path is at infinity.

To this point we made use of a linearization approximation after (13). As a consequence, (13) will typically not hold exactly with the computed $\sigma(\theta)$. In this case, one can solve for an error function of $\sigma(\theta)$ to make further adjustments.

4.2 The Homotopy Method: The General Case

Recall the equation

$$\int_0^1 p(x(\tau), y(\tau))\sqrt{{x'_\tau}^2 + {y'_\tau}^2} d\tau \quad (18)$$

from (6), in which $'$ denotes $\frac{d}{d\tau}$. Recall we are seeking that path $(x(\tau), y(\tau))$ such that (18) is minimised given an initial and terminal point for the trajectory and $p(x(\tau), y(\tau))$.

The Euler Lagrange equations applied to (18) are

$$p(x,y){x''}_\tau - \frac{\partial p}{\partial x}{y'_\tau}^2 + \frac{\partial p}{\partial y}x'_\tau y'_\tau = 0 \quad (19)$$

and

$$p(x,y){y''}_\tau - \frac{\partial p}{\partial y}{x'_\tau}^2 + \frac{\partial p}{\partial x}x'_\tau y'_\tau = 0 \quad (20)$$

where $'$ denotes $\frac{d}{d\tau}$. Now suppose for a certain detection rate $p(x, y)$, we already know the optimal path expressed by $x(\tau)$ and $y(\tau)$ (e.g. from the variational method derived previously), and consider a small variation to the detection rate function given by

$$p(x, y) + \lambda q(x, y)$$

where $q(x, y)$ is a bounded function and λ is a sufficiently small number. Correspondingly, a small variation to the optimal path is expected

$$x(\tau) + \lambda\sigma_x(\tau)$$

and

$$y(\tau) + \lambda\sigma_y(\tau)$$

where $\sigma_x(\tau)$ and $\sigma_y(\tau)$ are also bounded.

By following the same operations as with the special case we obtain the linear differential equations regarding $\sigma_x(\tau)$ and $\sigma_y(\tau)$ which are

$$\begin{aligned}&p{\sigma_x}'' + qx'' - 2\frac{\partial p}{\partial x}y'{\sigma_y}' \\ &\quad - \frac{\partial q}{\partial x}{y'}^2 + \frac{\partial q}{\partial y}x'y' + \frac{\partial p}{\partial y}x'{\sigma_y}' + \frac{\partial p}{\partial y}{\sigma_x}'y' = 0\end{aligned} \quad (21)$$

and

$$\begin{aligned}&p{\sigma_y}'' + qy'' - 2\frac{\partial p}{\partial y}x'{\sigma_x}' \\ &\quad - \frac{\partial q}{\partial y}{x'}^2 + \frac{\partial q}{\partial x}x'y' + \frac{\partial p}{\partial x}x'{\sigma_y}' + \frac{\partial p}{\partial x}{\sigma_x}'y' = 0\end{aligned} \quad (22)$$

in which $'$ denotes $\frac{d}{d\tau}$. In the above equations, $p(x, y)$, $q(x, y)$, $x(\tau)$ and $y(\tau)$ are known functions. Consequently, we have a system of second order linear differential equations with two unknown functions

$$\begin{aligned}a_1(\tau){\sigma_x}''(\tau) + b_{11}(\tau){\sigma_x}'(\tau) + b_{12}(\tau){\sigma_y}'(\tau) + c_1(\tau) = 0 \\ a_2(\tau){\sigma_y}''(\tau) + b_{21}(\tau){\sigma_x}'(\tau) + b_{22}(\tau){\sigma_y}'(\tau) + c_2(\tau) = 0\end{aligned} \quad (23)$$

Because the above equations are second order differential equations, there are two constants of integration in the solution for $\sigma_x(\tau)$ and $\sigma_y(\tau)$. Correspondingly, $\sigma_x(\tau)$ has two boundary conditions $\sigma_x(0) = \sigma_x(1) = 0$ while $\sigma_y(\tau)$ also has two boundary conditions $\sigma_y(0) = \sigma_y(1) = 0$. Now the numerical solutions of $\sigma_x(\tau)$ and $\sigma_y(\tau)$ can be obtained. The detail will be discussed in Section 4.4.

As with the special case we have applied a linearization approximation and thus an error could occur in the form

$$\begin{aligned}&(p + \lambda q)(x'' + \sigma''_x) - \frac{\partial(p+\lambda q)}{\partial x}(y' + \sigma'_y)^2 \\ &\quad + \frac{\partial(p+\lambda q)}{\partial y}(x' + \sigma'_x)(y' + \sigma'_y) = \epsilon_1(\tau)\end{aligned} \quad (24)$$

and

$$\begin{aligned}&(p + \lambda q)(y'' + \sigma''_y) - \frac{\partial(p+\lambda q)}{\partial y}(x' + \sigma'_x)^2 \\ &\quad + \frac{\partial(p+\lambda q)}{\partial x}(x' + \sigma'_x)(y' + \sigma'_y) = \epsilon_2(\tau)\end{aligned} \quad (25)$$

from which one can calculate the error $\epsilon_1(\tau)$ and $\epsilon_2(\tau)$. In order to adjust the obtained σ_x and σ_y and diminish $\epsilon_1(\tau)$ and $\epsilon_2(\tau)$, let

$$x(\tau) + \lambda\sigma_x(\tau) \to x(\tau) + \lambda\sigma_x(\tau) + \xi_x(\tau)$$

and

$$y(\tau) + \lambda\sigma_y(\tau) \to y(\tau) + \lambda\sigma_y(\tau) + \xi_y(\tau)$$

Again following the same line of reasoning as with the special case we can derive now two differential equations regarding $\xi_x(\tau)$ and $\xi_y(\tau)$

$$\begin{aligned}-\epsilon_1(\tau) =& (p + \lambda q)\xi''_x - 2\frac{\partial(p+\lambda q)}{\partial x}(y' + \sigma'_y)\xi'_y \\ &+ \frac{\partial(p+\lambda q)}{\partial y}(x' + \sigma'_x)\xi'_y + \frac{\partial(p+\lambda q)}{\partial y}(y' + \sigma'_y)\xi'_x\end{aligned} \quad (26)$$

and

$$-\epsilon_2(\tau) = (p+\lambda q)\xi_y'' - 2\frac{\partial(p+\lambda q)}{\partial y}(x' + \sigma_x')\xi_x' + \frac{\partial(p+\lambda q)}{\partial x}(yy' + \sigma_y')\xi_x' + \frac{\partial(p+\lambda q)}{\partial x}(x' + \sigma_x')\xi_y' \quad (27)$$

These equations have a similar form to (21) and (22) and one can then obtain $\xi_x(\tau)$ and $\xi_y(\tau)$ following the same operations used to find $\sigma_x(\tau)$ and $\sigma_y(\tau)$.

4.3 An Iterative Algorithm for Trajectory Updating

The discussion thus far in this section allows us to compute the variation of the optimal path $\lambda\sigma_x(\tau)$ and $\lambda\sigma_y(\tau)$ corresponding to a small variation of the detection rate function $\lambda q(x, y)$. Suppose now there is a bounded variation $q(x, y)$ occurring on the detection rate function $p(x, y)$ and we want to find the optimal path when detection rate function is $p(x, y) + q(x, y)$. We construct a parameterised function

$$H(x, y, \Lambda) = p(x, y) + \Lambda q(x, y) \quad (28)$$

where $\Lambda \in [0, 1]$. Note the optimal path corresponding to $H(x, y, 0)$ is known to us while the optimal path corresponding to $H(x, y, 1)$ is desired. We then discretize the interval according to $0 = \Lambda_0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_n = 1$ where each increment $\Lambda_{i+1} - \Lambda_i$ $(i = 0, 1, 2, \cdots, n-1)$ is a sufficient small number that can be thought of as corresponding to λ in the previous subsections. Consequently, the original problem is equivalent to solving this sequence of differential equations just previously derived. During each iteration, the objective is to find the optimal path with the detection rate $H(x, y, \Lambda_{i+1})$ and knowledge of the optimal path corresponding to the case $H(x, y, \Lambda_i)$.

Remark 1. Suppose a vehicle has already obtained curve $\widehat{AB}$ as the optimal path from A to B, and then after the vehicle has travelled from A to an intermediate point C, it finds an additional detector. Now it can use the homotopy method to modify the optimal path. Note that it should modify $\widehat{CB}$ rather than $\widehat{AB}$ as a whole because the vehicle is not at A now; C should be considered as its new origin.

4.4 Numeric Solution

During each iteration the problem boils down to solving a system of second order linear differential equations with two unknown functions. When analytic solutions are not available, numeric methods can be used. The work in Keller (1992) shows a general approach for numerically solving second order ordinary linear differential equations with time complexity $O(N)$ where N is the dimension of the unknown vector. In our particular case, N is the number of waypoints.

4.5 Caution with Discontinuities

The change in the optimal path is unfortunately not always continuous in Λ which means a small variation on the detection rate may result in an abruptly large change to the optimal path. In such cases the homotopy method will fail to provide a good result.

Example Suppose there are two detectors in $\mathbb{R}^2$ located symmetrically with respect to the line segment between the desired initial and terminal points for the trajectory; e.g. as shown by Figure 2. At first, the target knows $p(x, y)$ corresponding just to *Detector 1* and computes an optimal path shown by *Line 1*. Then *Detector 2* with detection rate $q(x, y)$ is introduced. Further suppose that the detectors are of the same type but that *Detector 2* has a much higher sensitivity than *Detector 1*.

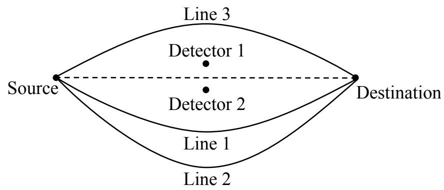

Fig. 2. Potential discontinuity in the homotopy method

Now the optimal path corresponding to $H(x, y, 0)$ is *Line 1*. When Λ increases from zero, the optimal path will be pushed down as shown by *Line 2*. However, because $q(x, y)$ will ultimately dominate $H(x, y, 1)$, the optimal path corresponding to $H(x, y, 1)$ should be *Line 3*. This analysis leads to us to the conclusion that the optimal path will have to jump at some value of Λ.

4.6 Illustrative Examples and Computational Comparison

We suppose there is a target in $\mathbb{R}^2$ that needs to travel from $(-5, 5)$ to $(5, 5)$ and there is a radar placed at the origin. The flying vehicle has knowledge about that detector and has previously computed an optimal path accordingly. The probability rate function in this section is as Appendix A.

Now a new active radar is added to the origin and the target then intends to quickly adjust its optimal path. As shown in Figure 3, the blue line is the original optimal path with only one detector; the red line is the path found after adjusting the original path using the homotopy method and the green line is the optimal path with both detectors found using variational dynamic programming.

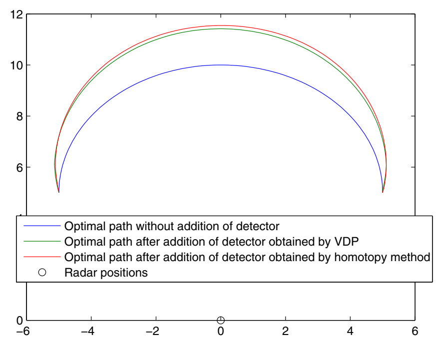

Fig. 3. Simulation result of Section 4.2

Now a new passive sensor is added to the point $(6, 6)$ and the vehicle intends to quickly adjust its optimal path. As shown in Figure 4, the blue line is the optimal path before adjustment; the red line is the path after adjustment using the homotopy method and the green line is the optimal path generated using variational dynamic programming.

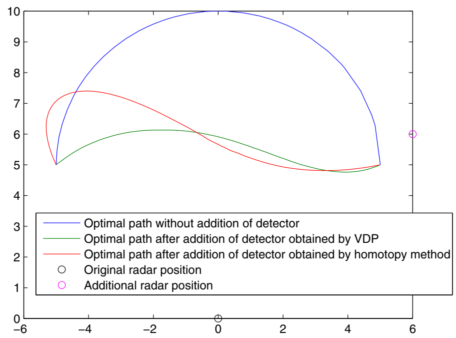

Fig. 4. Simulation result of Section 4.2

In the above simulation, no adjustment using (26) and (27) is made in any steps of the homotopy method. Figure 5 shows the result when adjustments are made.

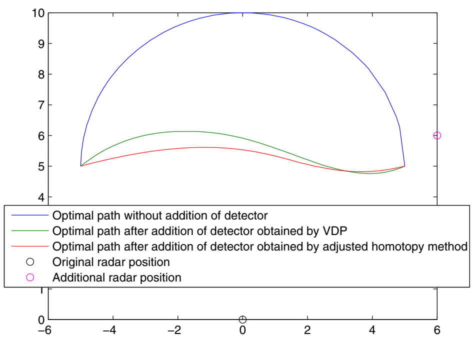

Fig. 5. Simulation result of Section 4.2

Suppose k_n is the element numbers in $\mathcal{C}_n$ in Variational Dynamic programming (VDP), N and M are the check-point numbers and iteration numbers respectively in the VDP and homotopy methods. The operation count in variational dynamic programming is $O(M \cdot \sum_0^N k_n)$ and the operation count in homotopy method is $O(M \cdot N)$. Furthermore, in each operation, VDP has to evaluate the numerical integral of the cost function while the homotopy method only requires simple arithmetic. Thus the time consumed in each step of VDP is generally 10^3 times that for the homotopy method without adjustment; this number can vary depend on how accurate one needs to evaluate the numerical integral.

5. CONCLUSION

In this paper, we proposed an optimization model where the objective is to minimize the probability of being detected rather than cumulative radar exposure. Furthermore, the variational dynamic programming approach is utilized to obtain accurate local optima of a dedicated original path. In addition, the homotopy method is proposed to adjust the optimal path when the detection rate function changes.

In further research, we are studying the use of convex optimization for selection of detectors when a target vehicle can plan its path to avoid detection. In addition, the detection rate function for radars with Doppler capability will also be discussed.

Appendix A. A TYPICAL DETECTION RATE

We briefly outline the relationship of our infinitesimal detection rate $p(x, y)$ to a common signal detection probability function used in radar design. Target detection in practice is based on thresholding and depends naturally on the signal-to-noise ratio of the receiver.

For a pulsed radar, let f_r denote the pulse frequency (alternatively $1/f_r$ could correspond to some time period in a continuous-wave radar). Then the probability of detection during one pulse is $1 - exp[-p(x, y)/f_r]$. Suppose SNR is the signal-to-noise ratio. The probability of detection in one pulse can then be approximated by the Marcum q-function

$$1 - exp[-p(x, y)/f_r] = Q[\sqrt{2SNR}, \sqrt{-2\ln P_{fa}}]$$

, which appears often in such detection problems; see Simon (1998); Mahafza (2013).

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