Equivariant Morse Theory and Formation Control

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***Abstract*— In this paper we study the critical points of potential functions for distance-based formation shape of a finite number of point agents in Euclidean space $\mathbb{R}^d$ with $d \leq 3$. The analysis of critical formations proceeds using equivariant Morse theory for equivariant Morse functions on manifolds of configuration spaces. We establish lower bounds for the number of critical formations. For $d = 2$ these bounds agree with the bounds announced in [3], while for $d = 3$ we obtain new bounds. We also propose a control law of the form of a decentralized gradient flow that evolves on a configuration manifold for agents in $\mathbb{R}^d$ such that collisions among the agents do not occur. By computing the equivariant cohomology of the configurations spaces we establish new lower bounds for the number of critical collision-free formations in the configuration space. Our work parallels earlier research in geometric mechanics by Pacella [19] and McCord [18] on enumerating central configurations for the N-body problem.**

I. Introduction

Formation shape control for a finite number of point agents in Euclidean space $\mathbb{R}^d$ is concerned with devising decentralized control laws which ensure that the formation will converge to critical formations that realize prescribed inter-agent distances. Communication between the agents is defined by a finite, connected and undirected graph $\Gamma = (\mathcal{V}, \mathcal{E})$ whose set of vertices $\mathcal{V} = \{1, \ldots, N\}$ labels the N agents while the set of edges $ij \in \mathcal{E}$ corresponds to the unordered pairs of agents $x_i, x_j \in \mathbb{R}^d$ among which prescribed distances $d^*_{ij} > 0$ are to be realized. Such control laws are often derived as gradient flows to minimize a suitable potential function on the formation space which is invariant under translations and rotations. In this paper we study, following [3], [14], the critical point structure of the potential function defined by

$$V(x_1, \ldots, x_N) = \frac{1}{4} \sum_{ij \in \mathcal{E}} \left(\|x_i - x_j\|^2 - d^{*2}_{ij} \right)^2. \qquad (1)$$

The associated steepest descent gradient flow for $i = 1, \ldots, N$ is

$$\dot{x}_i = - \sum_{j: ij \in \mathcal{E}} \left(\|x_i - x_j\|^2 - d^{*2}_{ij} \right) (x_i - x_j). \qquad (2)$$

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Critical points of V then correspond bijectively to equilibrium points of (2), i.e. to *critical formations*. Due to the symmetry properties of the potential function, the critical formations are entire orbits of the Euclidean group and therefore are always nonisolated. Classical Morse theory on shape manifolds has been proposed in [3],[4] to establish lower bounds for the number of critical formations in $\mathbb{R}$ and $\mathbb{R}^2$. The analysis in [3] depends on the non-trivial fact that the shape space of all planar formations that are not collocated at one point forms a smooth manifold. This is wrong for formations in $\mathbb{R}^d$ with $d \geq 3$. In this paper we offer an alternative approach based on equivariant Morse theory that allows us to establish lower bounds for the numbers of critical orbits also for $d = 3$. The results by Anderson for $d = 2$ in [3] are recovered. We prove that rigidity of the graph Γ is a necessary condition for (1) to be an equivariant Morse function. For $d = 1$, it is shown in [4] that rigidity is a sufficient condition for (1) to define an equivariant Morse function, for generic choices of distances d^*_{ij}. We leave the problem to decide whether or not for $d \geq 2$ the function (1) is generically an equivariant Morse function for any rigid graph as an open research problem. Proceeding in a different direction we note that in many formation control algorithms proposed so far, see, e.g., [1], [2], [9], [16], collisions among the agents in the steepest descent flows are possible and cannot be ruled out. The phase space for collision-free agent formations is the configuration manifold $\mathbb{F}_N(\mathbb{R}^d)$ of N pairwise distinct points in $\mathbb{R}^d$. A natural potential function for collison-free formation control is, e.g.,

$$W(x_1, \ldots, x_N) = \frac{1}{2} \sum_{ij \in \mathcal{E}} \frac{\left(\|x_i - x_j\|^2 - d^{*2}_{ij} \right)^2}{\|x_i - x_j\|^2}. \qquad (3)$$

whose steepest descent gradient flow for $i = 1, \ldots, N$ is

$$\dot{x}_i = - \sum_{j: ij \in \mathcal{E}} \left(1 - \frac{d^{*4}_{ij}}{\|x_i - x_j\|^4} \right) (x_i - x_j). \qquad (4)$$

This defines a decentralized steepest descent algorithm on the configuration manifold $\mathbb{F}_N(\mathbb{R}^d)$ that evolves collision free, even asymptotically for $t \to \infty$. We apply equivariant Morse theory to establish new lower bounds for the number of critical formations of (4) in the configuration space.

We proceed as follows. Section II formulates the problem as an optimization task on shape spaces. After a brief summary of equivariant Morse theory we apply these techniques in Section IV to estimate the number of critical orbits for the cost function (1) in dimensions $d = 2, 3$. Section V contains a corresponding analysis for the numbers of collision-free critical points of (3). We show that there are at least $N!$ critical orbits of (3), provided the critical orbits are all nondegenerate. Section VI discusses the genericity issue of the equivariant Morse-property for the two potential functions V and W and outlines an alternative approach using stratified Morse theory on Cayley-Menger varieties.

II. Formations and Shape Spaces

Formations of N mobile agents in $\mathbb{R}^d$ are often best described in terms of graph theory. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ denote any finite, connected and undirected graph with vertex set $\mathcal{V} = \{1, \ldots, N\}$ and $M = |\mathcal{E}|$ edges, denoted by ij and ordered lexicographically. Since Γ is connected, we have $M \geq N-1$. Given any vertex element $i \in \mathcal{V}$ of the graph we associate to it a point x_i in Euclidean space $\mathbb{R}^d$. The matrix

$$X = [x_1, \ldots, x_N] \in \mathbb{R}^{d\times N} \tag{5}$$

then describes a **formation** of N points, the columns being labelled by the set of vertices of Γ. Similarly, for any edge ij we associate the Euclidean distance $\|x_i - x_j\|$. We assume that a given set of positive numbers $d^*_{ij} > 0$ are specified for any edge $ij \in \mathcal{E}$. These numbers represent the desired distances that we want to achieve by a suitable formation, at least to some degree of approximation. The invariance properties of potential functions are formulated in terms of the Euclidean group action on formations. Let $O(d)$ denote the compact matrix Lie group of all real orthogonal $d \times d$-matrices; let $SO(d)$ denote the compact, connected subgroup of real orthogonal matrices $S \in \mathbb{R}^{d\times d}$ of determinant $\det S = 1$. The Euclidean group $SE(d)$ then parameterizes all Euclidean group transformations of the form $x \mapsto Sx + v$, where $S \in SO(d)$ is an orientation preserving rotation matrix and $v \in \mathbb{R}^d$ denotes an arbitrary translation vector. Thus $SE(n)$ is a connected Lie group of dimension $\frac{1}{2}d(d+1)$, which is in fact a semidirect product of $SO(d)$ and $\mathbb{R}^d$.

In order to describe the *shape space* of formations it is useful to first eliminate the effects of translations. This is done here simply by translating the last agent x_N to the origin, i.e. by passing to the reduced shape space as

$$M = \{X \in \mathbb{R}^{d\times N} | \quad x_N = 0\}. \tag{6}$$

Of course, M is diffeomorphic to $\mathbb{R}^{d\times(N-1)}$. The origin in M then corresponds to the formations that are all collocated at the origin. To eliminate the effects of rotations one passes to a suitable moduli space for the Lie group action $X \mapsto SX$ of $SO(d)$ on M. These moduli spaces are called **shape spaces**. A natural choice of such a shape space of formations is the **orbit space** $M/SO(d)$, which defines a semialgebraic set of dimension $d(N-1) - \frac{1}{2}d(d-1)$. However, unless $d = 1$, it is never a smooth manifold. Similarly, by eliminating the effect of translations in the collision-free case, we are led to the reduced **configuration space**

$$\mathbb{F}_{N,d} = \{(x_1, \ldots, x_{N-1}) \in (\mathbb{R}^d \setminus \{0\})^{N-1} \mid x_i \neq x_j \forall i \neq j\}$$

which is an open subset of M. Again, the Lie group $SO(d)$ of rotations acts on $\mathbb{F}_{N,d}$ using orthogonal group transformations $(x_1, \ldots, x_{N-1}) \mapsto (Sx_1, \ldots, Sx_{N-1}), S \in SO(d)$. The associated shape space then is the orbit space $\mathbb{F}_{N,d}/SO(d)$, i.e. an open dense subset of the semialgebraic set $M/SO(d)$. The space $\mathbb{F}_{N,d}/SO(d)$ is a smooth manifold for $d = 2$, but it is not a manifold for $d \geq 3$. The latter fact is important to us as it prevents one to apply classical techniques from Morse theory to the problem.

To analyze the critical point structure of the potential functions (1) and (3) we use the property that both functions are invariant under arbitrary Euclidean group transformations. Thus, by eliminating translational ambiguity, we consider the **reduced potential functions** $V_N : M \longrightarrow \mathbb{R}$ and $W_N : \mathbb{F}_{N,d} \longrightarrow \mathbb{R}$ with $x_N = 0$ as

$$V_N(x_1, \ldots, x_{N-1}) = \frac{1}{4}\sum_{ij\in\mathcal{E}} \left(\|x_i - x_j\|^2 - d^{*2}_{ij}\right)^2 \tag{7}$$

and

$$W_N(x_1, \ldots, x_{N-1}) = \frac{1}{4}\sum_{ij\in\mathcal{E}} \frac{\left(\|x_i - x_j\|^2 - d^{*2}_{ij}\right)^2}{\|x_i - x_j\|^2} \tag{8}$$

respectively. From now on we focus on analyzing (7), (8). As far as the critical point structure is concerned there is no loss of generality in passing from (1) and (3) to (7) and (8), respectively. Of course, the invariance property of (1), (3) under Euclidean transformations then corresponds to the invariance of the functions (7), (8) under orthogonal group transformations $(x_1, \ldots, x_{N-1}) \mapsto (Sx_1, \ldots, Sx_{N-1}), S \in SO(d)$. We conclude that V_N, W_N are smooth $SO(d)$-invariant functions that have the same critical point structure as the unrestricted functions (1), (3).

III. Equivariant Morse Theory

We next introduce the main technical tool for analyzing the critical point structures of (7) and (8), i.e. equivariant Morse theory. The purpose of this generalization of classical Morse theory is to capture the possibility that there are symmetries in the construction of an objective function f; a situation that often occurs in practise. Such symmetries imply that f does not have isolated critical points, but rather has critical points constituting orbits of a Lie group action. For example, in our case, the orbits will be associated with the fact that formation shapes are invariant with respect to rotations. Classical Morse theory establishes a connection between the numbers of isolated nondegenerate critical points of a smooth function $f : M \longrightarrow \mathbb{R}$ on a manifold M and topological invariants such as the Betti numbers of M. Dealing with the presence of orbits of critical points demands an extension of classical Morse theory, such as equivariant Morse theory, in order to be able to cope with the existence of continua of equilibrium points. This notion is captured in the following way. We refer to [17] for an introductory text on classical Morse theory, to [7] for an elegant introduction to equivariant Morse theory and to [8] background in topology.

Consider any Lie group action $G \times M \longrightarrow M, (g,x) \mapsto g \cdot x$ by a compact Lie group G on a connected manifold M; see [11] for a textbook reference on Lie groups. A smooth function $f : M \longrightarrow \mathbb{R}$ is called *G-invariant*, if $f(g \cdot x) = f(x)$ holds for all $g \in G$ and $x \in M$. The critical points of a smooth G-invariant function form orbits under the Lie group action. Recall, that the *G-orbit* of $x \in M$ is defined as the set $G \cdot x := \{g \cdot x | g \in G\}$ and is a compact submanifold of M. The stabilizer subgroup of a point x is the compact Lie subgroup $G_x := \{g \in G | g \cdot x = x\}$. If G is connected then, by the surjectivity of the exponential map for G, it follows that the stabilizer subgroups G_x are connected, too. An action is called free, if all stabilizer subgroups $G_x = \{e\}$ are trivial; here e denotes the unit element of G. Given any such Lie group action, the orbit space M/G is defined as the set $\{G \cdot x | x \in M\}$ of orbits. This space is endowed with a natural topology, the quotient topology, which turns M/G into a connected Hausdorff space. Any G-invariant smooth function $f : M \longrightarrow \mathbb{R}$ induces a continuous function $\overline{f} : M/G \longrightarrow \mathbb{R}$, which is smooth if the orbit space M/G is a smooth manifold. Note that M/G is however not a manifold, unless e.g. the group action is free. This creates considerable technical difficulties when trying to apply standard Morse theory to functions with symmetries. Coping with these difficulties proceeds as follows. Let EG denote the infinite-dimensional contractible space on which G acts freely and let $BG = EG/G$ denote the classifying space. We will not review here the well-known topological construction of these spaces and refer to e.g. [7], [8] for further details. Suffice here to say that for the real orthogonal group $G = O(d)$ of rotation matrices the space EG can be identified with the infinite dimensional Hilbert manifold of all real $d \times \infty$-matrices with orthonormal and square-summable row vectors. The classifying space $BO(d)$ then is identified with the Grassmann manifold of d-dimensional linear subspaces in the Hilbert space ℓ^2 of square-summable sequences. Since G acts freely on EG, it induces a free diagonal action $g \cdot (x,p) \mapsto (g \cdot x, gp)$ on the product space $M \times EG$. Let

$$M_G = (M \times EG)/G \tag{9}$$

denote the associated orbit space. Note that M_G is an infinite-dimensional space and thus may have nonzero Betti numbers in all dimensions. Similarly, the *equivariant cohomology* groups of M are defined as

$$H^*_G(M) = H^*(M_G; \mathbb{R}); \tag{10}$$

these are infinite-dimensional $\mathbb{R}$-vector spaces. The *equivariant Poincaré-series* then is defined as the formal power series

$$P^G_t(M) := \sum_{q=0}^{\infty} \dim H^q(M_G; \mathbb{R}) t^q \tag{11}$$

Now assume that $f : M \longrightarrow \mathbb{R}$ denotes a smooth G-invariant function with compact sublevel sets. Fix a G-invariant Riemannian metric on M. A G-orbit $Z = G \cdot x$ is called a *nondegenerate critical orbit* of f, if (i) x is a critical point of f and (ii) the Hessian of f is nonsingular in the normal direction to Z. Now assume that all critical orbits of f are nondegenerate and there are only finitely many. Then f is called an **equivariant Morse function**. Let $Z_j = G \cdot x_j, j = 1, \cdots, r$, denote the critical orbits of f. Let $G_j \subset G$ denote the stabilizer subgroups of $x_j, j = 1, \cdots, r$. We assume that the G_j are connected. This condition is always satisfied if, e.g., $G = SO(d)$. Let $BG_j = EG_j/G_j$ denote the classifying space and let λ_j denote the number of positive eigenvalues of the Hessian $H_f(x_j)$ in the normal direction to Z_j. The *equivariant Morse inequalities* then are

$$\sum_{j=1}^{r} t^{\lambda_j} P_t(BG_j) = P^G_t(M) + (1+t) q(t), \tag{12}$$

where $q(t)$ is a power series in t with nonnegative coefficients. More specifically, by comparing coefficients of t^j in the identity (12) of power series we obtain linear inequalities relating the number of critical orbits of Morse index λ_j with the $j - th$ equivariant Betti number of M. If $G = \{e\}$ is the trivial group then $P^G_t(M) = P_t(M)$ coincides with the ordinary Poincaré series of M, λ_j is just the Morse-index of x_j and $P_t(BG_j) = P_t(B\{e\}) = 1$. Thus, in this case, the equivariant Morse inequalities are equivalent to the classical Morse inequalities. Note that the coefficients in the Poincaré polynomial $P_t(M)$ of a manifold M are the Betti numbers. Therefore the Euler characteristic of M, which is the alternating sum of all Betti numbers of M, coincides with evaluating the Poincaré polynomial at $t = -1$. Similarly, by evaluating at $t = 1$, we obtain the sum of all Betti numbers.

IV. Lower Bounds for Critical Formations

We now specialize the above abstract results to the setting of formation control. We consider the smooth invariant function (7) defined on Euclidean space $\mathbb{R}^{d \times (N-1)}$ which has the important property that $V_N(X)$ is invariant under left multiplication $(g, X) \mapsto gX$ by arbitrary orthogonal transformations $g \in SO(d)$. Since the zero matrix is a fixed point, this action is not free and one can check for $d \geq 2$ that the orbit space is not a smooth manifold. Therefore we cannot apply classical Morse theory on the orbit space $\mathbb{R}^{d(N-1)}/SO(d)$. However, we can apply equivariant Morse theory and to this end we have to compute the equivariant Poincaré series. Here it is.

Proposition 1: Let $SO(d)$ act linearly from the left on matrix space $\mathbb{R}^{d \times (N-1)}, N > d$. The equivariant Poincaré series then coincides with the ordinary Poincaré series of the classifying space $BSO(d)$, i.e.

$$P^{SO(d)}_t(\mathbb{R}^{d \times N}) = P_t(BSO(d)). \tag{13}$$

Proof: Consider the fiber bundle, whose fibres are topological cells, as

$$\mathbb{R}^{d\times(N-1)} \longrightarrow \mathbb{R}^{d\times(N-1)}_{SO(d)} \longrightarrow BSO(d). \tag{14}$$

Since the fibres are contractible spaces, their cohomology is trivial. Thus there exist a finite number of cohomology classes in $H^q(\mathbb{R}^{d\times(N-1)}_{SO(d)};\mathbb{R})$ that restrict on each fibre to a basis of $H^q(\mathbb{R}^{d\times(N-1)};\mathbb{R})$. By the Leray-Hirsch theorem, we obtain an isomorphism of cohomology

$$H^*_{SO(d)}(\mathbb{R}^{d\times(N-1)}) \simeq H^*(BSO(d)) \otimes H^*(\mathbb{R}^{d\times(N-1)}). \tag{15}$$

The result follows. ■

As examples for equivariant Poincaré series we mention that $P_t(BSO(1)) = 1$, $P_t(BSO(2)) = \frac{1}{1-t^2}$ and $P_t(BSO(3)) = \frac{1}{1-t^4}$. Thus, for $N \geq n$ we conclude

$$\begin{aligned} &P_t^{SO(1)}(\mathbb{R}^{1\times N}) = 1, \quad P_t^{SO(2)}(\mathbb{R}^{2\times N}) = \frac{1}{1-t^2} \\ &P_t^{SO(3)}(\mathbb{R}^{3\times N}) = \frac{1}{1-t^4} \end{aligned} \tag{16}$$

We now analyze the situation in the special cases where $N > d$ and $d = 2, d = 3$. The subsequent computations apply to any smooth $SO(d)$-invariant function $f : \mathbb{R}^{d\times(N-1)} \longrightarrow \mathbb{R}$ with compact sublevel sets and which has only finitely many nondegenerate critical orbits $Z_1 = gX_1, \ldots, Z_r = gX_r$.

A. The 2D-case

Here $N > d = 2$. For any $2 \times (N-1)$-matrix X, the stabilizer subgroup G_X of X in $SO(2)$ is always connected and therefore either $G_X = \{I_2\}$ is the trivial group or $G_X = SO(2)$. The first case happens if and only if $X \neq 0$, while the second case happens if and only if $X = 0$. Let $m_i, i = 0, \ldots, 2N-3$ and $M_i, i = 0, \ldots, 2N-2$ denote the numbers of critical orbits of Morse index $0 \leq i \leq 2N-2$, with stabilizer subgroup $\{I_2\}$ and $SO(2)$, respectively. Since $P_t(B\{I_2\}) = 1$ and $P_t(BSO(2)) = \frac{1}{1-t^2}$, the equivariant Morse inequalities are

$$\sum_{i=0}^{2N-3} m_i t^i + \frac{\sum_{j=0}^{2N-2} M_j t^j}{1-t^2} = \frac{1}{1-t^2} + (1+t)q(t) \tag{17}$$

with $q(t)$ a power series with only nonnegative coefficients. Actually, from the Morse inequalities one sees that $q(t)$ is rational with poles at most at $t = \pm 1$. Note that $X = 0$ is always a critical point, due to the invariance of f. Moreover, it is the unique critical orbit with stabilizer group $SO(2)$. Thus exactly one of the M_j is nonzero. We rederive a result of [3].

Theorem 1: Let $d = 2$. Suppose that (7) is an equivariant Morse function, i.e. V_N has only finitely many nondegenerate critical $SO(2)$-orbits. Then V_N has at least one critical orbit of Morse index $2i$ for each $i = 0, \ldots, N-2$. If m_i denote the number of critical $SO(2)$-orbits with Morse index i, then

$$\sum_{k=0}^{2N-3} (-1)^k m_k = N - 1. \tag{18}$$

Proof: The potential function (7) is a smooth $SO(d)$ invariant function with compact sublevel sets and is easily verified to have a local maximum at $X = 0$. Thus $\sum_{j=0}^{2N-2} M_j t^j = t^{2N-2}$ and (17) is equivalent to

$$\sum_{i=0}^{2N-3} m_i t^i = \frac{1-t^{2N-2}}{1-t^2} + (1+t)q(t) = \sum_{j=0}^{N-2} t^{2j} + (1+t)q(t).$$

This implies that $q(t)$ is a polynomial and $m_{2i} \geq 1$ holds for any $i = 0, \ldots, N-2$. By substituting $t = -1$ we obtain

$$\sum_{i=0}^{2N-3} (-1)^i m_i = N - 1 \tag{19}$$

while $t = 1$ yields

$$\sum_{i=0}^{2N-3} m_i \geq N - 1. \tag{20}$$

■

Note that the analysis in [3] is quite different to ours and depends on identifying the smooth shape space $\mathbb{R}^{2\times(N-1)} \setminus \{0\}/SO(2)$ with the direct product manifold $\mathbb{P}^{N-2}(\mathbb{C}) \times (0,\infty)$. In this way one can apply ordinary Morse theory on the complex projective space $\mathbb{P}^{N-2}(\mathbb{C})$ without passing to equivariant cohomology. Note however that the analysis in [3] contains a gap as the cost function on $\mathbb{P}^{N-2}(\mathbb{C}) \times (0,\infty)$ does not have compact sublevel sets; see [4] for a resolution of this difficulty. In our approach, using equivariant Morse theory, we do not have to exclude singular points from the shape space and therefore can apply the methods from equivaraint Morse theory directly to the function (7) on the entire Euclidean space, where it has compact sublevel sets.

B. The 3D-case

The situation is considerably more complicated for $d = 3$. Here we have three possible stabilizer subgroups $\{I_3\}, SO(2)$ and $SO(3)$. An N-tuple X of points has full stabilizer $SO(3)$ if and only if $X = 0$. Similarly, the stabilizer of X is isomorphic to $SO(2)$ if and only if the column vectors of X are all collinear, i.e. iff $\operatorname{rk} X = 1$. The critical orbits with trivial stabilizer subgroup of X are characterized by $\operatorname{rk} X \geq 2$. Let $m_i, i = 0, \cdots, 3N-6$ denote the numbers of critical orbits X of index i with $\operatorname{rk} X \geq 2$. Let $M_i, i = 0, \cdots, 3N-5$ denote the number of critical orbits of index i with $\operatorname{rk} X = 1$. The point $X = 0$ is always a local maximum for (7).

Theorem 2: Let $d = 3$. Suppose that (7) is an equivariant Morse function, i.e. V_N has only finitely many nondegenerate critical $SO(3)$-orbits. Then the numbers m_i, M_i of critical orbits with Morse indices i satisfy the equivariant Morse inequalities

$$\sum_{i=0}^{3N-6} m_i t^i + \frac{\sum_{i=0}^{3N-5} M_i t^i}{1-t^2} = \frac{1-t^{3N-3}}{1-t^4} + (1+t)q(t) \quad (21)$$

for a power series $q(t) = \sum_{k=0}^{\infty} q_k t^k$ with nonnegative coefficients. We easily deduce

$$\sum_{j=0}^{3N-5} (-1)^j M_j = \frac{1-(-1)^{3N-3}}{2}. \quad (22)$$

Moreover, if $N = 4\ell + 1$ then for $k = 0, \ldots, 3\ell - 1$

$$m_{4k} + \sum_{i=0}^{2k} M_{2i} \geq 1. \quad (23)$$

Proof: The Morse inequalities (21) follow from (12), using the explicit formulas for the equivariant Poincaré series in (16). For $N = 4\ell + 1$ we have

$$\frac{1-t^{3N-3}}{1-t^4} = \sum_{j=0}^{3\ell-1} t^{4j}$$

Thus (23) follows by comparing coefficients of t^{4k} and using the property that the coefficients of $q(t)$ are nonnegative. To prove (23) one might consider the idea of multiplying both sides by $1-t^4$ and then substituting $t = -1$. Note that the power series $q(t)$ is rational with poles at most at the roots of unity $t^4 = 1$. If $q(t)$ were known to be analytic at $t = -1$ then the claim would immediately follow. Unfortunately, we do not know this and thus prove (23) by an independent argument which we now outline. Recall that the Euler characteristic $\chi(Q)$ of a manifold satisfies

$$\chi(Q) = \sum_j (-1)^{\operatorname{codim} Q_j} \chi(Q_j)$$

for any finite filtration into submanifolds M_j. In our case $Q = \mathbb{R}^{3(N-1)}$ and the Q_j are the positive eigenspace bundle of critical orbits. Now consider any critical orbit Z of a formation X with trivial stabilizer group $\{I_3\}$. The positive eigenspace bundle of Z then is a bundle over the orbit $SO(3)$ and thus has Euler characteristic zero. Thus the critical orbits of X with $\operatorname{rk} X \geq 2$ do not contribute to the Euler characteristic $\chi(\mathbb{R}^{3(N-1)}) = 1$. The critical orbits of collinear formations are diffeomorphic to the two-sphere $S^2 = SO(3)/SO(2)$, which has Euler characteristic 2. Moreover, the positive eigenspace bundle Q_j of any such orbit with Morse index $j = 0, \ldots, 3N-5$ has the fibre $\mathbb{R}^{3N-5-j}$. Thus the total space Q_j of this bundle over S^2 has codimension j. Finally, the maximum $X = 0$ has codimension $3N-3$. Applying the above decomposition formula yields

$$1 = \chi(\mathbb{R}^{3(N-1)}) = 2 \sum_{j=0}^{3N-5} (-1)^j M_j + (-1)^{3N-3}.$$

This completes the proof. ∎

It should be obvious that one would wish to extract more information from (21) than we have done here. Unfortunately we have not been able to do so. In the next section we show how the more complicated geometry of configuration spaces allows us to obtain more explicit bounds.

V. Counting Collision-free critical formations

Here we turn to an analysis of the potential function (8), defined on the configuration manifold

$$\mathbb{F}_{N,d} = \{(x_1, \ldots, x_{N-1}) \in (\mathbb{R}^d \setminus \{0\})^{N-1} \mid x_i \neq x_j \forall i \neq j\}$$

We note that the smooth function W_N has compact sublevel sets on $\mathbb{F}_{N,d}$ and is $SO(d)$-invariant. Let us assume that the critical orbits are nondegenerate, i.e. that W_N defines an equivariant Morse function. We refer to section VI for a brief discussion how reasonable this assumption is. Thus the equivariant Morse inequalities (12) apply. The calculation of the equivariant Poincaré series for the configuration space is however much more complicated than for the Euclidean case discussed in the preceding section. This part of our work has been strongly influenced by earlier work in celestial mechanics on characterizing central configurations in the N-body problem; see [18], [19].

A. The 2D-case

Consider first the case where $d = 2$, i.e. the task of collision-free formation control in the plane. In this case the rotation group $SO(2)$ acts freely on $\mathbb{F}_{N,2}$ and we can use the action to normalize the agent x_{N-1} to $x_{N-1} = \lambda(1,0)$ for some real positive number $\lambda > 0$. By identifying $\mathbb{R}^2$ with $\mathbb{C}$ this implies that the orbit space $\mathbb{F}_{N,2}/SO(2)$ is a smooth manifold of dimension $2N-3$ which is diffeomorphic to

$$\mathbb{F}_{N,d}/SO(2) \simeq \mathbb{F}_{N-2}(\mathbb{C} \setminus \{0,1\}) \times (0,\infty)$$

where $\mathbb{F}_{N-2}(\mathbb{C} \setminus \{0,1\}) =$

$$\{(x_1, \ldots, x_{N-2}) \in (\mathbb{C} \setminus \{0,1\})^{N-2} \mid x_i \neq x_j \forall i \neq j\}.$$

Note that $\mathbb{F}_{N-2}(\mathbb{C} \setminus \{0,1\})$ is just the configuration space of $N-2$ distinct points in $\mathbb{C}$ that avoid two nominated points. The Betti numbers of configuration spaces of points in Euclidean space have been investigated in topology over the least 40 years. For example, Arnold [5] has computed the cohomology of the configuration space $\mathbb{F}_N(\mathbb{C})$ of N-tuples of pairwise distinct complex numbers. We have

Proposition 2 ([12]): The $\bmod 2$ Poincaré polynomial of $\mathbb{F}_{N,2}/SO(2)$ is

$$P_t\left(\mathbb{F}_{N,2}/SO(2)\right)) = (1+2t)(1+3t)\cdots(1+(N-1)t).$$

Evaluating the right hand side at $t = -1$ yields that the Euler characteristic of $\mathbb{F}_{N-2}(\mathbb{C} \setminus \{0,1\})$ is equal to $(-1)^N (N-2)!$. By evaluating at $t = 1$ yields that the sum of all $\bmod 2$ Betti numbers is

$$\sum_{k=0}^{N-2} \dim H_k\left(\mathbb{F}_{N-2}(\mathbb{C} \setminus \{0,1\})\right) = \frac{N!}{2}.$$

Since $SO(2)$ acts freely on $\mathbb{F}_{N,2}$ we can apply ordinary Morse theory to the induced function $\overline{W}_N$ on the smooth quotient manifold $\mathbb{F}_{N,2}/SO(2)$. Then the critical $SO(2)$-orbits of W_N are in one-to-one correspondence with the critical points of $\overline{W}_N$. Thus it suffices to estimate the number of critical points of $\overline{W}_N$. From the preceding topological results we conclude

Theorem 3: Let $d = 2$. Assume that W_N induces a Morse function $\overline{W}_N$ on $\mathbb{F}_{N,2}/SO(2)$. Then $\overline{W}_N$ has at least $\frac{N!}{2}$ critical points on $\mathbb{F}_{N,2}/SO(2)$ and the numbers m_i of critical orbits of Morse index i satisfies

$$\sum_{k=0}^{2N-3} (-1)^k m_k = (-1)^N (N-2)!. \tag{24}$$

Moreover, if $\overline{W}_N$ is not necessarily a Morse function, then $\overline{W}_N$ has at least $N-2$ critical points.

Proof: The first part follows from Proposition 2, by applying the Morse inequalities to $\overline{W}_N$. The second part depends on a more refined topological argument. McCord [18] has shown that the cup length of the $\bmod 2$ cohomology ring $H^*(\mathbb{F}_{N,2}/SO(2)); \mathbb{Z}_2)$ is at least $N-3$. By the theorem of Ljusternik-Schirelman [13] the number of critical points is at least 1 plus the cup length, i.e. is $\geq N-2$. ■

B. The 3D-case

In contrast to the $2D$-case, the critical point analysis of $3D$-formations cannot be simply reduced to ordinary Morse theory, as the relevant shape space $\mathbb{F}_{N,3}/SO(3)$ of dimension $3N-6$ has singularities. Thus we apply equivariant Morse theory to the smooth $SO(3)$-invariant function W_N with compact sublevel sets. Using the same notation as in Theorem 2, our main result is as follows.

Theorem 4: Let $d = 3$ and assume that all the critical $SO(3)$- orbits of W_N are nondegenerate. There exists a power series $q(t) = \sum_{k=0}^{\infty} q_k t^k$ with nonnegative coefficients q_k such that the equivariant Morse inequalities

$$\begin{aligned}\sum_{i=0}^{3N-6} m_i t^i + \frac{\sum_{i=0}^{3N-5} M_i t^i}{1-t^2} &= \\ \frac{(1+2t^2)\cdots(1+(N-1)t^2)}{1-t^2} &+ (1+t)q(t)\end{aligned} \tag{25}$$

hold. Furthermore

$$\sum_{j=0}^{3N-5} (-1)^j M_j = \frac{N!}{2} \tag{26}$$

and defining

$$\gamma_{2k} = \sum_{2 \leq i_1 < \cdots < i_k \leq N-1} i_1 \cdots i_k$$

there holds for $k = 0, \ldots, N-2$

$$m_{2k} + \sum_{j=0}^{k} M_{2j} \geq \sum_{j=0}^{k} \gamma_{2j}. \tag{27}$$

There are at least $N!$ critical orbits of W_N.

Proof: Note that $X = 0$ does not belong to $\mathbb{F}_{N,3}$. Therefore the equivariant Morse inequalities are

$$\sum_{i=0}^{3N-6} m_i t^i + \frac{\sum_{i=0}^{3N-5} M_i t^i}{1-t^2} = P_t^{SO(3)}(\mathbb{F}_{N,3}) + (1+t)q(t),$$

for a power series $q(t) = \sum_{k=0}^{\infty} q_k t^k$ with nonnegative coefficients. By [19], the equivariant Poincaré series of $\mathbb{F}_{N,3}$ is given as

$$P_t^{SO(3)}(\mathbb{F}_{N,3}) = \frac{(1+t^2)(1+2t^2)\cdots(1+(N-1)t^2)}{1-t^4}.$$

Moreover,

$$P_t(\mathbb{F}_{N,3}) = (1+t^2)(1+2t^2)\cdots(1+(N-1)t^2).$$

This implies (25). The topological argument for the Euler characteristic in the proof of Theorem 2 remains in force and yields

$$N! = \chi(\mathbb{F}_{N,3}) = 2\sum_{j=0}^{3N-5} (-1)^j M_j.$$

By clearing denominators we obtain

$$\begin{aligned}(1-t^2)\sum_{j=0}^{3N-6} m_j t^j + \sum_{j=0}^{3N-5} M_j t^j = \sum_{j=0}^{N-2} \gamma_{2j} t^{2j} + \\ + (1+t)(1-t^2)\sum_{k=0}^{\infty} q_k t^k\end{aligned}$$

with $q_k \geq 0$. Comparing coefficients of t^{2j} for $j \geq 0$ leads to

$$m_{2j} - m_{2(j-1)} + M_{2j} = \gamma_{2j} + q_{2j} + q_{2j-1} - q_{2j-2} - q_{2j-3}.$$

Thus by adding up these equations for $0 \leq j \leq k$ we obtain

$$m_{2k} + \sum_{j=0}^{k} M_{2j} = \sum_{j=0}^{k} \gamma_{2j} + q_{2k} + q_{2k-1} \geq \sum_{j=0}^{k} \gamma_{2j}.$$

We have $(1+2t^2)\cdots(1+(N-1)t^2) = \sum_{k=0}^{N-2} \gamma_{2k} t^{2k}$. Thus for $k = N-2$ we obtain

$$m_{2(N-2)} + \sum_{j=0}^{N-2} M_{2j} \geq \sum_{j=0}^{N-2} \gamma_{2j} = \frac{N!}{2}.$$

This completes the proof. ■

VI. Genericity of Morse functions

In the preceding sections we have applied topological methods to establish lower bounds on the number of critical orbits for the the two potential functions V_N and W_N. Of course these results rest upon the assumption that V_N and W_N are in fact equivariant Morse functions. One might optimistically assume that this condition is always satisfied, at least for generically defined desired distances d_{ij}^*. Unfortunately life is not that easy and there is in fact no guarantee that this Morse property is satisfied at all. In this section we therefore address the question of when one can establish genericity of the Morse property for the formation control fucntions. This depends on some properties of the underlying graph that we next begin to review.

A. Graph Rigidity and Cayley-Menger Varieties

Given any undirected connected graph $\Gamma = (\mathcal{V}, \mathcal{E})$ on N vertcies and M edges, we consider the smooth real algebraic distance map ($x_N = 0$)

$$\mathcal{D} : \mathbb{R}^{d\times(N-1)} \longrightarrow \mathbb{R}^M, \mathcal{D}(X) = (\frac{1}{2}\|\|x_i - x_j\|\|^2)_{ij\in\mathcal{E}}.$$

The image set

$$CM_d(\Gamma) = \{\mathcal{D}(X) \mid X \in \mathbb{R}^{d\times(N-1)}\}.$$

is called the **Cayley-Menger variety** and forms a closed semialgebraic subset of $\mathbb{R}^M$. Its dimension is easily calculated for rigid graphs, as we will now show. The **rigidity matrix** of a formation is defined as the $M \times d(N-1)$-Jacobi matrix $R(X) = Jac_{\mathcal{D}}(X)$ whose $ij-$th row ($ij \in \mathcal{E}$) is

$$R(X)_{ij} = (e_i - e_j)^\top \otimes (x_i - x_j)^\top.$$

A formation X is called **regular** whenever

$$\operatorname{rk} R(X) = \max_{Z\in\mathbb{R}^{d\times(N-1)}} \operatorname{rk} R(Z)$$

holds. The regular formations form an open and dense subset in the space of all formations. Since $\mathcal{D}$ is invariant under orthogonal rotations, i.e. $\mathcal{D}(SX) = \mathcal{D}(X)$ holds for all $S \in SO(d)$, the tangent space to such a group orbit is always contained in the kernel of the rigidity matrix $R(X)$. A formation is called **infinitesimally rigid** if the kernel of the rigidity matrix concides with the tangent space $T_X(SO(d)\cdot X)$. Equivalently, infinitesimal rigidity holds if and if the following rank condition is satisfied

$$\operatorname{rk} R(X) = d(N-1) - \frac{1}{2}\operatorname{rk} X(2d - \operatorname{rk} X - 1).$$

Note that for $\operatorname{rk} X = 1$ and any d we have $\operatorname{rk} R(X) \le N-1$. For $\operatorname{rk} X = 2$ and any d with $M \ge 2N-3$ one shows $\operatorname{rk} R(X) \le 2N-3$. In particular, a formation of $N \ge 3$ points in the plane $\mathbb{R}^2$ is infinitesimally rigid if and only if $\operatorname{rk} X = 2$ and the rank of $R(X)$ is equal to $2N-3$. Similarly, a formation of $N \ge 4$ points in $\mathbb{R}^3$ is infinitesimally rigid if and only if $\operatorname{rk} X \ge 2$ and the rank of $R(X)$ is equal to $3N-6$. A formation X is called **rigid** whenever the orbit $SO(d)\cdot X$ is isolated in the fibre $\mathcal{D}^{-1}(\mathcal{D}(X))$. Any infinitesimally rigid formation is rigid, but the converse does not hold. It is a result of Asimow and Roth [6] that regular formations are infinitesimally rigid if and only if they are rigid. A **rigid graph** in $\mathbb{R}^d$ is one for which almost every $X \in \mathbb{R}^d$ is infinitesimally rigid. Thus Γ is rigid in $\mathbb{R}^d$ if and only if the rigidity matrix $R(X)$ has generic rank equal to $dN - \frac{d(d+1)}{2}$. Rigid graphs in $\mathbb{R}^2$ are characterized in a combinatorial manner by the so-called Laman condition [15]. An explicit combinatorial characterization of rigid graphs in $\mathbb{R}^3$ is unknown.

B. Genericity results

We now address the question of under which conditions there exist parameters $d^*_{ij} > 0$ such that the potential functions V_N or W_N are equivariant Morse functions. It is easily seen that this in turn is equivalent to the question of whether the equivariant Morse property holds for a generic set of desired distances. Clearly this question depends only upon the graph Γ and the dimension d. We first state a geometric result for the Cayley-Menger variety; we omit the straightforward proof.

Theorem 5: Let Γ be a rigid graph in $\mathbb{R}^d$. Then the Cayley-Menger variety $CM_d(\Gamma)$ is a semialgebraic set of dimension $d(N-1) - \frac{1}{2}d(d-1)$. Let $\mathbb{R}^{d\times(N-1)}_{\text{reg}}$ denote the dense open subset of infinitesimally rigid formations X with $\operatorname{rk} R(X) = d(N-1) - \frac{1}{2}d(d-1)$. Then $CM_d(\Gamma)_{\text{reg}} := \mathcal{D}(\mathbb{R}^{d\times(N-1)}_{\text{reg}})$ consists of smooth points of $CM_d(\Gamma)$ and

$$\mathcal{D} : \mathbb{R}^{d\times(N-1)}_{\text{reg}} \longrightarrow CM_d(\Gamma)_{\text{reg}}$$

defines a smooth $O(d)$ fibre bundle.

It is instructive to consider the special case where $\Gamma = K_N$ is the complete graph on N vertices. Then any formation of points $x_1, \ldots, x_{N-1}$ with $x_N = 0$ is uniquely determined by the mutual distances $\|x_i - x_j\|, 1 \le i, j \le N$. Let $S(N)$ denote the vector space of real symmetric $N \times N$ matrices. The Laplacian map then is defined as the linear map $\mathcal{L} : S(N) \longrightarrow S(N), M \mapsto L(M)$, such that the ij-th entry of $L(M)$ is

$$L_{ij} := \begin{cases} m_{ij} - \frac{1}{2}(m_{ii} + m_{jj}) & \text{if} \quad i \neq j \\ -\sum_{j=1}^N (m_{ij} - \frac{1}{2}(m_{ii} + m_{jj})) & \text{if} \quad i = j \end{cases}$$

If X is $n \times N$ and $M := X^\top X$, then $L(M)$ is positive semidefinite and coincides with the usual Laplacian distance matrix $L(M)$ having entries

$$L_{ij} = \frac{1}{2}\begin{cases} -\|x_i - x_j\|^2 & \text{if } i \neq j \\ \sum_{j=1}^N \|x_i - x_j\|^2 & \text{if } i = j \end{cases}$$

It is easily seen that $\mathcal{L}$ is injective on the subspace $S_0(N)$ of symmetric matrices whose last columns and rows are zero. For any $0 \le r \le d$ let $\mathcal{P}(r, N)$ denote the subset of $S_0(N)$ consisting of all positive semidefinite symmetric matrices of rank equal to r. It is well-known that $\mathcal{P}(r, N)$ is a smooth manifold of dimension $rN - \frac{1}{2}r(r+1)$. We conclude that $\mathcal{L}(\mathcal{P}(r, N))$ is a smooth submanifold of $S_0(N)$, which is diffeomorphic to $\mathcal{P}(r, N)$. It forms a finite stratification by submanifolds of the Cayley-Menger variety $CM_d(K_N) \simeq \mathcal{L}(S_0(N))$.

Returning back to the general situation, let $d^* \in \mathbb{R}^M_+$ denote an arbitrary vector of desired distances. Then the potential function $V_N : \mathbb{R}^{d(N-1)} \longrightarrow \mathbb{R}$ is the composition $V_N = g_* \circ \mathcal{D}$ of the distance map $\mathcal{D}$ with the least squares distance function $g_* : CM_d(\Gamma) \longrightarrow \mathbb{R}$

$$g_*(P) = \frac{1}{4}\|P - d^*\|^2$$

on the Cayley-Menger variety. Since $CM_d(\Gamma)$ is semialgebraic it admits a finite decomposition into disjoint submanifolds $\mathcal{S}_1, \ldots, \mathcal{S}_q$, called strata of $CM_d(\Gamma)$. By Theorem 5, the highest dimensional stratum S_q contains $CM_d(\Gamma)_{\rm reg}$ as an open dense subset. The function g_* is called a **stratified Morse function** on $CM_d(\Gamma)$ provided the restriction $g_*|\mathcal{S}_j$ on each stratum $\mathcal{S}_j$ has only nondegenerate critical points.

Theorem 6: There exists an open and dense subset of $d^* \in \mathbb{R}_+^M$ such that g_* is a stratified Morse function with a finite number $\nu(g^*)$ of critical points. In particular, for such choices of d^* and Γ a rigid graph, the potential function V_N defines an equivariant Morse function on the open and dense subset of infinitesimally rigid formations $\mathbb{R}_{\rm reg}^{d\times(N-1)}$. The function V_N has at least $\nu(g^*)$ many $O(d)$-orbits of critical formations.

Proof: The first part follows from a more general result by Pignoni [20]. The second part then is implied by Theorem 5. Let $P = \mathcal{D}(X)$ be any critical point of g^*. Then X is a critical point of $V_N = g^* \circ \mathcal{D}$. ■

Unfortunately, the critical orbits of V_N will not always be infinitesimally rigid. Thus the preceding result does not imply that V_N is generically an equivariant Morse function on the **full space** of formations $\mathbb{R}^{d(N-1)}$. It follows that the optimization problem of the distance function on the Cayley-Menger variety seems to be a better conditioned problem than that of optimizing the potential function V_N. In fact, recent work by Draisma et al. [10] yields promising new results in this direction.

Theorem 7: Let $d \geq 1$. A necessary condition that V_N is an equivariant Morse function for generic $d_{ij}^* > 0$ is that the graph is rigid. For $d = 1$ this condition is also sufficient.

Proof: If V_N is an equivariant Morse function then the critical orbits are isolated. Suppose that Γ is not rigid. Then for almost all X the rank of the rigidity matrix is less than $d(N-1) - \frac{1}{2}d(d-1)$. Let S_q denote the smooth stratum of $CM_d(\Gamma)$. Choose a critical point $P_* = \mathcal{D}(X_*) \in S_q$ of q^*. Then X_* is a critical point of V_N, as well is every point in the fibre $(g^*)^{-1}(P_*)$. Since Γ is not rigid, the dimension of $(g^*)^{-1}(P_*)$ is strictly greater than the dimension of $SO(d)$. This contradicts the equivariant Morse property and completes the proof of the first part. For a proof of the second part we refer to [4]. ■

VII. Conclusions

We have presented topological methods based on equivariant Morse theory to establish lower bounds for the number of critical formations in $\mathbb{R}^2$ and $\mathbb{R}^3$, thus extending earlier work in [3], [4]. While such methods are certainly powerful enough to yield nontrivial bounds on critical formations, they apply only under the so far unqualified assumption that the potential functions of interest are in fact equivariant Morse functions, at least generically. An open problem is thus to give sufficient conditions on a graph such that the potential functions V_N, W_N are generically equivariant Morse functions. At this point it is not even known whether the genericity assumption is satisfied for complete graphs. To tackle this problem it might be useful to consider a reformulation of the optimization problem as a least squares distance problem on Cayley-Menger varieties. This suggests one might apply techniques from stratified Morse theory but this would require a more detailed investigation of the geometry of Cayley-Menger varieties. Another problem that would be worthwhile to consider is to study potential functions where the distances d_{ij}^* are defined to be the lenghts of generic formations of N agents $x_i^* \in \mathbb{R}^{d'}, d' \geq d$, when the number of edges exceeds the generic rank of the rigidity matrix.

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