Non-Robustness of Gradient Control for 3-D Undirected Formations with Distance Mismatch

Zhiyong Sun, Shaoshuai Mou, Brian D. O. Anderson and A. Stephen Morse

***Abstract*— Gradient control laws can be used for effectively achieving undirected formation shape, by assuming that inter-agent distances between a certain set of joint agent pairs can be accurately specified and measured. This paper examines the formation behavior in a 3-D space context in the case that the neighboring agent pairs have slightly differing views or estimates about the desired interagent distances they are tasked to maintain. It is shown, by using a tetrahedron formation example, that the final formation shape will be slightly distorted as compared to the desired one. Further, in general each agent's motion will be a combination of rotation and translation. Specifically, a helical movement can be observed in the presence of distance mismatch.**

I. INTRODUCTION

Formation control for a group of autonomous mobile agents has gained much attention due to its broad applications in many areas including both civil and military fields. A key problem in this topic that receives particular interest is how to maintain the geometrical shape of formations in a distributed manner. The formation shape can be achieved by controlling a certain set of interagent distances by using relative position measurements, in which rigid graph theory plays an important role in studying this problem [1], [2]. One popular approach to design the controls is from a potential function perspective, and the controls take the form of a gradient based law which can drive the formation to an equilibrium. The stability analysis for rigid formation problems using the gradient law has been studied extensively both in formations modelled using undirected [3], [4] and directed [5], [6] graphs. An important reference is [7], in which Krick et al. have provided a complete study on using the gradient law to achieve an undirected formation shape, proving that the formation shape is locally asymptotically stable if the underlying graph is infinitesimally rigid.

One of the main concerns when implementing any formation shape control method in practice is the robustness issue. Accuracy in measuring some key variables, i.e., the interagent distances, is crucial for achieving the desired formations. In many cases the sensors may produce measurement errors, either due to bias or noise, which may result in discrepancies between the estimates made by each of a pair of agents on the same distance between them. Each agent then has a differing view as to the error in achieving desired inter-agent distance. This problem is also abstractly equivalent to that arising when some agent pairs have differing views of the desired distances that they are tasked to maintain (even though they may consistently measure their actual separation). We use the word *mismatch* to refer to the *inconsistence* of the errors between measured distances and desired distances perceived by two joint agents, whether due to measurement biases or differing views of the desired distance. In a recent paper [8], the robustness issues for controlling undirected formations using gradient control laws with distance mismatch have been discussed in a 2-D space context, which concludes that the rigid formation motion will under a broad set of circumstances converge to a periodic orbit. In this paper we will show that in the 3-D case the distance mismatch will generally drive the agents to move unboundedly. Specifically, the resulting distorted formation will in general experience a motion which is a combination of rotation and translation, and in particular, a helical movement. The main aim in this paper is to identify the agent motions and to explain why this is so by considering the 3-D tetrahedron formation as a starting example.

This paper is organized as follows. Section II presents the problem description from a tetrahedron formation example and then sets up some key equations of agent motions. In Section III we focus on the properties and stability analysis of the error system and z (relative position) dynamical system (the definitions will be clear later). The main result is provided in section IV, which shows some features of agent motions in this 3-D mismatch problem. Finally, some concluding remarks are presented in Section V.

II. PROBLEM DESCRIPTION AND MOTION EQUATIONS

We consider a tetrahedron formation in 3-D space, which consists of four agents labeled as $1, 2, 3, 4$. Let $p_i \in \Re^3, i \in \{1, 2, 3, 4\}$ denotes the position of agent i with $p_i = [p_{ix}, p_{iy}, p_{iz}]^T$. Each agent should maintain the target distances to its three neighbors and each edge is jointly maintained by its two associated agents. For the purpose of writing down an oriented incidence matrix, suppose that the edges are oriented from i to j just when $i < j$. Then we can number the edges in the following order: $12, 23, 34, 13, 24, 14$; see Fig. 1. By doing so, one can obtain the following matrix, which is the transpose of the incidence

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matrix:

$$H = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

For future reference, we shall also define a matrix $\bar{H} = H \otimes I_3$, where the symbol $\otimes$ denotes the Kronecker product.

Let $d_{k_{ij}}$ denote the desired distance between agents i and j, but suppose that the joint agents associated with one edge have different views about the desired distance. Without loss of generality, suppose that when $i < j$, $d_{k_{ij}}$ is the desired distance as perceived by agent i, while the desired distance perceived by agent j may be different. Thus when $i < j$, the actual distances used by agents i and j in formulating their controls are

$$d_{ij}^2 = d_{k_{ij}}^2, d_{ji}^2 = d_{k_{ij}}^2 + \mu_{k_{ij}} \quad (2)$$

where μ_k denotes the distance mismatch corresponding to edge k between its two associated agents. We suppose that all the nominal distances $d_{k_{ij}}$ satisfy the triangular inequalities in each face of the tetrahedron. Furthermore, with small values for μ, the mismatched distances should also satisfy the triangular inequalities and the tetrahedron structure formed by the four agents has positive volume.

We assume each agent's motion is described by a simple kinematic model in the form

$$\dot{p}_i = u_i, i \in \{1, 2, 3, 4\} \quad (3)$$

where u_i is agent i's control input. The controls are derived by a gradient law from a potential function [7], which is defined as

$$V(p_1, p_2, p_3, p_4) = \frac{1}{4} \sum_{1 \le i < j \le 4} [\|p_i - p_j\|^2 - d_{ij}^2]^2 \quad (4)$$

where $\|\cdot\|$ denotes the Euclidean norm. However, when an agent is computing its motion it uses a differing value of the desired distance to any neighbor vertex whose vertex is smaller.

Let $z_{k_{ij}}$ be the relative position vector associated with each edge k_{ij}, which is defined as $z_{k_{ij}} = p_i - p_j$ when $i > j$ and $z_{k_{ij}} = p_j - p_i$ when $i < j$. Also define the error function in the form of

$$e_{k_{ij}}(z) = \|z_{k_{ij}}\|^2 - d_{k_{ij}}^2 \quad (5)$$

Then according to (2), (4) and (5), the equation of agent i's motion can be written as

$$\dot{p}_i = -\sum_{j<i} z_{k_{ij}}(e_{k_{ij}}(z) - \mu_{k_{ij}}) + \sum_{j>i} z_{k_{ij}} e_{k_{ij}}(z) \quad (6)$$

In the following, we shall define some notations to obtain a compact matrix form of the motion equations. Denote $z = [z_1^T, z_2^T, z_3^T, z_4^T, z_5^T, z_6^T]^T$, and $Z = \text{diag}[z_1, z_2, z_3, z_4, z_5, z_6]$. One has $z = \bar{H}p$, and the standard rigidity matrix $R(z)$ is given as $R(z) = Z^T \bar{H}$. (The definition of rigidity matrix can be found in [2]). Also

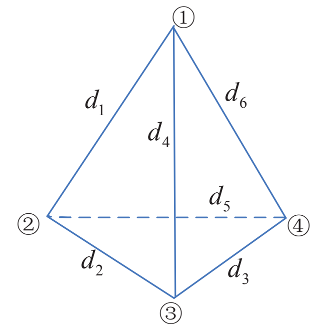

Fig. 1. Undirected tetrahedron formation.

define J and $\bar{J}$ to be the matrices obtained from H and $\bar{H}$ by replacing all -1 entries by zeros, which means that $\bar{J} = J \otimes I_3$. With the definition of $\bar{J}$, we can define a 6×12 matrix $S(z)$ by $S(z) = Z^T \bar{J}$. With $p = [p_1^T, p_2^T, p_3^T, p_4^T]^T$ and obvious definitions of 6-vectors $e(z), \mu$ formed from the $e_{k_{ij}}(z)$ and $\mu_{k_{ij}}$, one can write a compact form for the agent motion equations:

$$\dot{p} = -R^T(z)e(z) + S^T(z)\mu \quad (7)$$

There are two further equations as a consequence of (7) which also play a critical role. First, since $z = \bar{H}p$, multiplying both sides of (7) by $\bar{H}$ yields the following equation for the relative positions:

$$\dot{z} = -\bar{H}R^T(z)e(z) + \bar{H}S^T(z)\mu \quad (8)$$

Second, with the definition of $e_{k_{ij}}$ in (5) and the equation (8) for the relative positions z, it is straightforward to obtain the differential equation for the vector e

$$\dot{e} = -2R(z)R^T(z)e + 2R(z)S^T(z)\mu \quad (9)$$

In the sequel, we shall refer to (7) as *the overall system*, (8) as *the z system*, and (9) as *the error system*.

III. Analysis on the Error System and z System

A. The Error system

One of the aims in this paper is to understand the behavior of the overall system. To begin with, let us first study the property of the error system (9). One nontrivial observation of (9) is that, as for the two-dimensional case [8], the error vector e satisfies a differential equation of the form $\dot{e} = g(e, \mu)$, while z does not appear in the *smooth* function g. In fact, each entry of $R(z)R^T(z)$ is either of the form $\|z_k\|^2$ for some k or of the form $z_i^T z_j$ for some $i \neq j$. Further, the matrix $R(z)S^T(z)$ has entries which are either zero or of the same form as those in the corresponding entries of $R(z)R^T(z)$. By recalling (5), one knows $\|z_k\|^2 = e_k + d_k^2$, and further $z_i^T z_j$ can be expressed using the cosine law as a linear combination of $\|z_k\|^2$. To see the latter point clearly, define $A = -2R(z)R^T(z)$, $B = 2R(z)S^T(z)$, and we write each entry of A in (10) on the next page. As an example, consider the entries in the first row of A. One has $-z_1^T z_2 = \frac{1}{2}(\|z_4\|^2 - \|z_1\|^2 - \|z_2\|^2)$ on the triangular face Δ_{123}. Similarly, the same law applies to $z_1^T z_4$, $-z_1^T z_5$

$$
A = -2R(z)R^T(z)
$$

$$
= -2 \begin{bmatrix} 2\|z_1\|^2 & -z_1^T z_2 & 0 & z_1^T z_4 & -z_1^T z_5 & z_1^T z_6 \\ -z_1^T z_2 & 2\|z_2\|^2 & -z_2^T z_3 & z_2^T z_4 & z_2^T z_5 & 0 \\ 0 & -z_2^T z_3 & 2\|z_3\|^2 & -z_3^T z_4 & z_3^T z_5 & z_3^T z_6 \\ z_1^T z_4 & z_2^T z_4 & -z_3^T z_4 & 2\|z_4\|^2 & 0 & z_4^T z_6 \\ -z_1^T z_5 & z_2^T z_5 & z_3^T z_5 & 0 & 2\|z_5\|^2 & z_5^T z_6 \\ z_1^T z_6 & 0 & z_3^T z_6 & z_4^T z_6 & z_5^T z_6 & 2\|z_6\|^2 \end{bmatrix} \tag{10}
$$

and $z_1^T z_6$, on the triangular faces Δ_{123}, Δ_{124} and Δ_{142}, respectively. In fact, for all the entries $z_i^T z_j$ for $i \neq j$ in matrices A and B, the vectors z_i and z_j are in the same triangular face. Thus, all the inner-product entries in A and also in B can be reexpressed in terms of the $\|z_k\|^2$ and in turn in terms of e_k. Finally, we conclude that the matrices A and B depend smoothly on e and μ but not on z, and in the sequel we will rewrite them as $A(e)$ and $B(e)$.

Based on the above analysis, it is evident that the error system (9) can be rewritten without z in the form as

$$
\dot{e} = g(e, \mu) = A(e)e + B(e)\mu \tag{11}
$$

Note that $A(e)$ is a negative semi-definite symmetric matrix for any e. We are now ready to show the following important result for the error system.

Lemma 1: The unperturbed error system (11) (i.e. $\dot{e} = A(e)e$) has an exponentially stable equilibrium at $e = 0$.

Proof: First note that at the equilibrium of $e = 0$, the formation is rigid. In fact, in Section II we have supposed a realizable tetrahedron formation with positive volume. This leads to the infinitesimal rigidity of the framework, and thus the rigidity matrix $R(z)$ has a maximum rank of 6 at $e = 0$. Accordingly, the matrix $A(0)$ defined in (10) is negative definite for $e = 0$. Since $A(e)$ depends continuously on e and $A(0)$ is negative definite, there will be a positive number ρ sufficiently small so that $A(e)$ is negative definite for all e in the closed bounded set $\mathcal{E} = \{e : \|e\|^2 \leq \rho\}$.

The potential function, which is defined in (4) as a function of p_i, can also be expressed as a simple function of e. That is, $V = \frac{1}{4}\|e\|^2$. Its derivative along the unperturbed error system, i.e. the system (11) with $\mu = 0$, is evidently $\dot{V} = \frac{1}{2}e^T A(e)e$. Since $A(e)$ is negative semi-definite for all e, the function V will be non-increasing. Hence, by assuming that $e(0) \in \mathcal{E}$, it follows that $e(t) \in \mathcal{E}$ for all $t \geq 0$. Let $\lambda(-A(s))$ denote the smallest eigenvalue of $-A(s)$ for $s \in \mathcal{E}$. It is clear that $\lambda > 0$. Further define

$$
\bar{\lambda} = \inf_{s \in \mathcal{E}} \lambda(-A(s)) \tag{12}
$$

One knows that $\bar{\lambda} > 0$ and $\frac{1}{2}e^T A(e)e \leq -\frac{1}{2}\bar{\lambda}\|e\|^2$, which implies that $\dot{V} \leq -2\bar{\lambda}V$. Thus one can conclude that any trajectory starting inside $\mathcal{E}$ must approach $e = 0$ as fast as $e^{-\bar{\lambda}t}$ and the statement in Lemma 1 is proved. ■

The next aim of this section is to show that the error system (11) with small $\|\mu\|$ has an exponentially stable equilibrium close to $e = 0$. This is due to the robust property of exponential stability with respect to small parametric perturbations and also the robust property of the infinitesimal framework with respect to small perturbations on the edge lengths. We summarize this result as a proposition.

Proposition 1: There exists a small $\upsilon > 0$ and a set $\mathcal{B} = \{\mu : \|\mu\| < \upsilon\}$, such that for small mismatch $\mu \in \mathcal{B}$ the perturbed system (11) will approach an exponentially stable equilibrium which is close to $e = 0$. We denote this equilibrium as $\bar{e}(\mu)$, or shortly as $\bar{e}$. Thus, with small μ, the agents will form a slightly distorted formation which is close to the desired one.

The proof for the above result is omitted on the grounds of space limitation.

B. The z system

In the above we have shown the convergence of $e(t)$ to $\bar{e}$ for $\mu \in \mathcal{B}$. Denote $\bar{z}$ as the solution to the z system (8) when $e(t)$ is replaced by the equilibrium state $\bar{e}$. From (5) one has $\|\bar{z}_k\|^2 = \bar{e}_k + d_k^2$, which indicates that the norm of each 3-vector z_k will converge to constant. Consequently, by using the cosine law one can find that $z_i^T z_j$ for $i \neq j$ will also be constant at the equilibrium state $\bar{e}$. To see this clearly, one can examine the entries in the matrices $A(\bar{e})$ and $B(\bar{e})$ by using the same reasoning as stated in the beginning of Section III.(A). Thus in the steady state we can write the matrix A and B as $A(\bar{e})$ and $B(\bar{e})$ by replacing z in each entry with $\bar{z}$. These facts will be quite useful in the later analysis and we restate them in the following proposition:

Proposition 2: Given the convergence of the error system $e(t)$ to the equilibrium state $\bar{e}$, $z_k^T z_k$ for all k and $z_i^T z_j$ for $i \neq j$ will also converge to some constants. Further, the matrices $A(e)$ and $B(e)$ will converge exponentially fast to $A(\bar{e})$ and $B(\bar{e})$, respectively.

We stress here that the convergence of $z_k^T z_k$ for all k and $z_i^T z_j$ for $i \neq j$ does not mean that each z_k itself converges. Also, in general the formation will not actually come to rest when the error system converges to $\bar{e}$. We call the formation motion at the equilibrium state $e(t) = \bar{e}$ an *equilibrium motion*.

In the following, we shall rewrite the z system, which was originally stated in (8), in another compact form to facilitate the stability analysis. Define $E = \text{diag}[e_1, e_2, ..., e_6]$, $\bar{E} = \text{diag}[\bar{e}_1, \bar{e}_2, ..., \bar{e}_6]$, $M = \text{diag}[\mu_1, \mu_2, ..., \mu_6]$ and observe that $Ze = (E \otimes I_3)z$ and likewise for $Z\bar{e}, Z\mu$. One has

$$\begin{aligned}\dot{z} &= -\bar{H}R^T(z)e(z) + \bar{H}S^T(z)\mu \\ &= -\bar{H}\bar{H}^T Ze(z) + \bar{H}\bar{J}^T Z\mu \\ &= -(HH^T E \otimes I_3)z + (HJ^T M \otimes I_3)z \\ &= \big((-HH^T E + HJ^T M) \otimes I_3\big)\, z\end{aligned} \tag{13}$$

Define a matrix F as

$$F = -HH^T E + HJ^T M \tag{14}$$

One should also note that the vectors z_i for $i = \{1, 2, \ldots, 6\}$ are not linearly independent. In fact, the ordering of edges as 12, 23, 34, 13, 24, 14 means that

$$z_4 = z_1 + z_2, z_5 = z_2 + z_3, z_6 = z_1 + z_2 + z_3 \tag{15}$$

Thus one can define a matrix K as

$$K = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

such that $(K \otimes I_3)z = 0$. Notice that K has full row rank and also $KH = 0$. Then the matrix F necessarily has three zero eigenvalues for all M. We would like to examine a reduced-order, self-contained z system. To this end, let us pick three independent z_i, say z_1, z_2, z_3, and eliminate z_4, z_5, z_6 from (13). Furthermore, partition the matrix K in (16) as $K = [K_1 \; I]$, and define the matrix T as

$$T = \begin{pmatrix} I & 0_{3\times3} \\ K_1 & I \end{pmatrix} \tag{17}$$

with its inverse

$$T^{-1} = \begin{pmatrix} I & 0_{3\times3} \\ -K_1 & I \end{pmatrix} \tag{18}$$

According to (15) and (16), it can be easily obtained that $(T \otimes I_3)z = [z_1^T, z_2^T, z_3^T, 0_{1\times3}, 0_{1\times3}, 0_{1\times3}]^T$ and similarly $(T \otimes I_3)\dot{z} = [\dot{z}_1^T, \dot{z}_2^T, \dot{z}_3^T, 0_{1\times3}, 0_{1\times3}, 0_{1\times3}]^T$. Thus, one has

$$\begin{aligned}(T \otimes I_3)\dot{z} &= (T \otimes I_3)(F \otimes I_3)(T^{-1} \otimes I_3)(T \otimes I_3)z \\ &= \big((TFT^{-1}) \otimes I_3\big)\,(T \otimes I_3)z \\ &= \left(\begin{pmatrix} F_1 & F_2 \\ 0_{3\times3} & 0_{3\times3} \end{pmatrix} \otimes I_3\right)(T \otimes I_3)z\end{aligned} \tag{19}$$

where the matrix F_1, which is the upper left 3×3 block of TFT^{-1}, is expressed in (20) on the next page for future reference.

From (19) we have actually arrived at the self-contained z system in the following form:

$$\frac{d}{dt}(z_1^T, z_2^T, z_3^T)^T = (F_1 \otimes I_3)(z_1^T, z_2^T, z_3^T)^T \tag{21}$$

In the above we have shown that three of the six eigenvalues of F are zero. Since the matrix T is of full rank, the other three eigenvalues of F coincide with those of F_1. Now we will focus on the eigenvalues of the matrix F_1.

For ease of notation, we define a 3×3 matrix $\bar{\mathcal{Z}} = [\bar{z}_1, \bar{z}_2, \bar{z}_3]$. Also we can rewrite the system (21) at the steady state in the following form (without the Kronecker product term):

$$\frac{d}{dt}(\bar{z}_1, \bar{z}_2, \bar{z}_3) = (\bar{z}_1, \bar{z}_2, \bar{z}_3)F_1^T(\bar{e}) \tag{22}$$

We shall display a key property of the eigenvalues of the matrix F_1 using a Lyapunov equation method. To this end, define a Gram matrix P as

$$P = \bar{\mathcal{Z}}^T \bar{\mathcal{Z}} = \begin{bmatrix} \bar{z}_1^T \bar{z}_1 & \bar{z}_1^T \bar{z}_2 & \bar{z}_1^T \bar{z}_3 \\ \bar{z}_2^T \bar{z}_1 & \bar{z}_2^T \bar{z}_2 & \bar{z}_2^T \bar{z}_3 \\ \bar{z}_3^T \bar{z}_1 & \bar{z}_3^T \bar{z}_2 & \bar{z}_3^T \bar{z}_3 \end{bmatrix} \tag{23}$$

According to the linear independence of the vectors $\bar{z}_1, \bar{z}_2, \bar{z}_3$, one knows that the matrix P must be positive definite. Furthermore, according to Proposition 2, each entry in the matrix P is constant at the equilibrium motion. Thus we can show that

$$\begin{aligned}\frac{d}{dt}(\bar{\mathcal{Z}}^T \bar{\mathcal{Z}}) &= 0 \\ &= \bar{\mathcal{Z}}^T \dot{\bar{\mathcal{Z}}} + \dot{\bar{\mathcal{Z}}}^T \bar{\mathcal{Z}} \\ &= \bar{\mathcal{Z}}^T \bar{\mathcal{Z}} F_1^T + F_1 \bar{\mathcal{Z}}^T \bar{\mathcal{Z}} \\ &= PF_1^T + F_1 P\end{aligned} \tag{24}$$

Since P is symmetric and positive definite, equation (24) essentially is a Lyapunov equation which immediately gives the following important result concerning the property of the eigenvalues of $F_1(\bar{e})$.

Lemma 2: The eigenvalues of the matrix $F_1(\bar{e})$ have zero real parts.

Evidently, there are two possibilities for the eigenvalues of $F_1(\bar{e})$: either $F_1(\bar{e})$ has two pure imaginary eigenvalues $\pm j\omega$ for some $\omega > 0$ and one zero eigenvalue, or $F_1(\bar{e})$ has three zero eigenvalues. The first case is, as will be seen, generic.

According to Lemma 2, one finds that not only does each $\|z_k\|$ assume a steady state value at the equilibrium motion, but in general the motion of z is a sum of a constant value and a rotation. The detailed nature of the motion will be discussed in next section. To close this section, as a by-product of Lemma 2, we record the following fact which links the steady state error $\bar{e}$ and the sum of the mismatch values.

Proposition 3: In the steady state, the following equality holds

$$\sum_{i=1}^{6} \mu_i = 2 \sum_{i=1}^{6} \bar{e}_i \tag{25}$$

The above equality can be obtained directly by checking the zero trace property of the matrix $F_1(\bar{e})$.

$$F_1 = \begin{bmatrix} -2e_1+\mu_1-e_4-e_6 & e_2-e_4+e_5-e_6 & e_5-e_6 \\ e_1-\mu_1-e_4+\mu_4 & -2e_2+\mu_2-e_4+\mu_4-e_5 & e_3-e_5 \\ e_4-\mu_4-e_6+\mu_6 & e_2-\mu_2+e_4-\mu_4-e_5+\mu_5-e_6+\mu_6 & -2e_3+\mu_3-e_5+\mu_5-e_6+\mu_6 \end{bmatrix} \quad (20)$$

IV. THE OVERALL SYSTEM AND THE HELICAL MOVEMENT

Now we return to the overall system defined in (7). Firstly we can observe the following fact concerning the agent motion.

Proposition 4: The norm of each agent's speed is constant at the equilibrium motion when $e(t) = \bar{e}$.

Proof: To prove Proposition 4, we rewrite (6) by replacing e and z as $\bar{e}$ and $\bar{z}$ at the equilibrium motion:

$$\dot{p}_i = -\sum_{j<i} \bar{z}_{k_{ij}}(\bar{e}_{k_{ij}} - \mu_{k_{ij}}) + \sum_{j>i} \bar{z}_{k_{ij}}\bar{e}_{k_{ij}} \quad (26)$$

One can easily check that $\dot{p}_i^T\dot{p}_i$ contains the terms depending on μ, $\bar{e}$, $\bar{z}_k^T\bar{z}_k$ for some k, $\bar{z}_i^T\bar{z}_j$ for $i \neq j$ and their linear combinations and multiples. According to Propositions 1 and 2, these terms are all constant at the equilibrium motion $e = \bar{e}$. These facts yield the conclusion that $\dot{p}_i^T\dot{p}_i$ is also constant and thus Proposition 4 holds. ■

Now we are ready to present the main result of this paper.

Theorem 1: Assume that the $\bar{z}_i$ moves with a rotational component. Then the motion of each agent, and the motion of the whole rigid formation, will display a helical movement (i.e. a motion involving a rotation in a plane and a simultaneous translation in a direction orthogonal to that plane), due to the presence of mismatched distances.

Proof: By considering (22) and (24), and according to the spectrum of matrix $F_1(\bar{e})$ revealed in Lemma 2, one can write the solution to the $\bar{z}$ system as

$$\bar{z}_i = a_i \cos\omega t + b_i \sin\omega t + c_i \quad (27)$$

where a_i, b_i, c_i are 3-vectors with *constant* entries. Since in the steady state $\bar{e}$ is constant as is μ, then $\dot{p}_i$ has the form of some linear combinations of $\bar{z}_i$ (this can be seen from (26)). Thus one can rewrite the expression of $\dot{p}_i$ in the following form:

$$\dot{p}_i = \alpha_i \cos\omega t + \beta_i \sin\omega t + \delta_i \quad (28)$$

Note that $\alpha_i, \beta_i, \delta_i$ are 3-vectors with *constant* entries which can be obtained by linear combinations of the vectors a_i, b_i, c_i (with weights depending on μ and $\bar{e}$). Further one has

$$\begin{aligned} \dot{p}_i^T\dot{p}_i &= (\alpha_i \cos\omega t + \beta_i \sin\omega t + \delta_i)^T(\alpha_i \cos\omega t + \beta_i \sin\omega t + \delta_i) \\ &= \alpha_i^T\alpha_i\cos^2\omega t + \beta_i^T\beta_i\sin^2\omega t \\ &\quad + 2\alpha_i^T\beta_i\cos\omega t\sin\omega t \\ &\quad + 2\alpha_i^T\delta_i\cos\omega t + 2\beta_i^T\delta_i\sin\omega t \\ &\quad + \delta_i^T\delta_i \end{aligned} \quad (29)$$

Based on the fact that $\dot{p}_i$ has constant norm as shown in Proposition 4, the following must hold

$$\alpha_i^T\delta_i = 0, \alpha_i^T\beta_i = 0, \beta_i^T\delta_i = 0, \alpha_i^T\alpha_i = \beta_i^T\beta_i \quad (30)$$

That is, the three vectors $\alpha_i, \beta_i, \delta_i$ are orthogonal, and also α_i and β_i have identical norm. Equation (30) indicates that the axis about which the agent rotates is parallel to the direction of the translational motion.

From (28) and the definition of $z_{k_{ij}}$, we can obtain another expression for $\dot{z}_{k_{ij}}$ when $i > j$ (the case of $i < j$ is similar):

$$\begin{aligned} \dot{z}_{k_{ij}} &= \dot{p}_i - \dot{p}_j \\ &= (\alpha_i - \alpha_j)\cos\omega t + (\beta_i - \beta_j)\sin\omega t + (\delta_i - \delta_j) \end{aligned} \quad (31)$$

Since $z_{k_{ij}}$ is bounded, one necessarily has

$$\delta_i - \delta_j = 0 \quad (32)$$

That is, all the agents have the same translational velocity (and a single translational direction).

Thus, from (28), (30) and (32), one can show that the 3-D tetrahedron formation with mismatch measurements will generally experience a helical movement for $\omega > 0$. ■

The special case of $\omega = 0$ is discussed in the following subsection (and one can find that it essentially leads to a translation-only movement). Some simulations then follow to support the result.

A. Special case: translation-only movement

We turn our attention to the z system (22). Generally speaking, at the equilibrium $e(t) = \bar{e}$, each $\bar{z}_i$ will not be constant though the norm of $\bar{z}_i$ and the corresponding inner-product terms do converge to constants. In this section we consider a special case of $\dot{\bar{z}}_i = 0$ which leads to constant $\bar{z}_i$ in the steady state. This case essentially corresponds to the special translation-only movement of the formation. The following lemma establishes the nongenericity of this case by stating a necessary and sufficient condition to ensure the translation-only movement for the 3-D tetrahedron formation in the presence of distance mismatch measurements.

Lemma 3: In order to obtain the translation-only movement, the *non-zero* mismatch values should satisfy the following equalities (any three implying the fourth).

$$\begin{aligned} \mu_1 + \mu_2 &= \mu_4 \\ \mu_3 + \mu_4 &= \mu_6 \\ \mu_1 + \mu_5 &= \mu_6 \\ \mu_2 + \mu_3 &= \mu_5 \end{aligned} \quad (33)$$

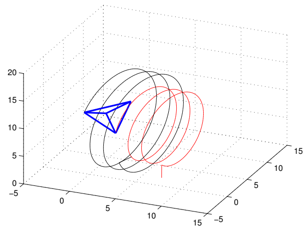

Fig. 2. Helical movement in 3-D tetrahedron formation caused by distance mismatch.

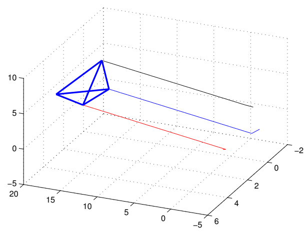

Fig. 3. A translation-only movement in 3-D tetrahedron formation with distance mismatch.

To prove the condition stated above, first notice the linear independence property of the vectors $\bar{z}_i, i \in \{1, 2, 3\}$. Some analysis will show that all the entries in $F_1(\bar{e})$ should be zero such that $\dot{\bar{z}}_i = 0$ can be ensured. The detailed proof is omitted here due to the space limit and will be presented elsewhere.

B. Simulations

In this subsection, we provide some simulations to show the behaviors of 3-D formation motion with mismatches. Suppose a group of four agents wants to form a tetrahedron formation in 3-D space, with the desired distances specified by $d_1 = d_4 = d_6 = 4, d_2 = 5, d_3 = 4.5, d_5 = 5$. However there exist mismatch values in the perceived distances, which are $\mu_1 = \mu_4 = \mu_6 = 0, \mu_2 = 0.13, \mu_3 = 0.12, \mu_5 = 0.13$. Through simulation, a typical helical motion is observed for all the agents, as shown in Fig.2.

We then consider another set of mismatch values. Suppose that the mismatch values are $\mu_1 = 0.05, \mu_2 = 0.05, \mu_3 = 0.1, \mu_4 = 0.1, \mu_5 = 0.15, \mu_6 = 0.2$. One can check that these values satisfy the conditions of Lemma 3. Then a translation-only motion for the tetrahedron formation can be expected. This is depicted in Fig.3.

V. Concluding Remarks

In this paper we have examined the motion behavior in a 3-D tetrahedron formation shape control problem in the presence of distance mismatch by using the gradient control law. The main result shows that in general the formation trajectory is a typical helix. Though we have used the tetrahedron formation throughout this paper to show the analysis, we remark here that such helical movement is not confined to a tetrahedron formation. In fact, the key result for the structure of the eigenvalues of the matrix $F_1(\bar{e})$ as stated in Lemma 2 demonstrates that for rigidity-based, undirected formations in 3-D space, the distance mismatch will generally cause the formation to undergo a helical motion when the gradient control law is employed.

There are some open issues which need to be addressed. On the one hand, we have identified the condition for ensuring translation-only movement but not yet for the rotation-only movement, though simulations have shown the possibility. Further, the angular velocity for the formation motion also needs to be identified, so that with the help of these formulas one can consider steering the formation motion using intentionally introduced distance mismatches. On the other hand, there is motivation for modifying this almost standard gradient control law for formation control, if such non-robust behavior is regarded as undesirable and needs to be suppressed.

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References

[1] R. Olfati-Saber and R. M. Murray, "Graph rigidity and distributed formation stabilization of multi-vehicle systems", in *Proc. of the 41th IEEE Conference on Decision and Control*, 2002, pp. 2965–2971.

[2] B. D. O. Anderson, C. Yu, B. Fidan, and J. M. Hendrickx, "Rigid graph control architectures for autonomous formations", *Control Systems Magazine, IEEE*, vol. 28, no. 6, pp. 48–63, 2008.

[3] R. Olfati-Saber and R. M. Murray, "Distributed cooperative control of multiple vehicle formations using structural potential functions", in *Proc. 15th World Congress of Int. Federation of Automatic Control (IFAC'02)*, 2002, pp. 346–352.

[4] D. V. Dimarogonas and K. H. Johansson, "On the stability of distance-based formation control", in *Proc. of the 47th IEEE Conference on Decision and Control*, 2008, pp. 1200–1205.

[5] M. Cao, C. Yu, A. S. Morse, B. D. O. Anderson, and S. Dasgupta, "Generalized controller for directed triangle formations", in *Proc. 17th World Congress of Int. Federation of Automatic Control (IFAC'08)*, 2008, pp. 6590–6595.

[6] M. Cao, A. S. Morse, C. Yu, B. D. O. Anderson, and S. Dasgupta, "Maintaining a directed, triangular formation of mobile autonomous agents", *Communications in Information and Systems*, vol. 11, no. 1, pp. 1–16, 2011.

[7] L. Krick, M. E. Broucke, and B. A. Francis, "Stabilisation of infinitesimally rigid formations of multi-robot networks", *International Journal of Control*, vol. 82, no. 3, pp. 423–439, 2009.

[8] A. Belabbas, S. Mou, A. S. Morse, and B.D.O. Anderson, "Robustness issues with undirected formations", in *Proc. of the 51th IEEE Conference on Decision and Control*, 2012, pp. 1445–1450.