

On the Properties of Linear Multirate Systems with Coprime Output Rates

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Abstract—This paper studies discrete-time linear systems with multirate outputs, assuming that two measured output streams are available at coprime rates. In the literature this type of system, which can be considered as periodic time-varying, is commonly studied in its blocked version, since the well-known techniques of analysis developed for linear time-invariant systems can be used. In particular, we focus on some structural properties of the blocked systems and we prove that, under a generic setting i.e. for a generic choice of parameter matrices, the blocked systems are minimal when the underlying multirate system is defined using a minimal dimension system. Moreover, we focus on zeros of tall blocked systems i.e. blocked systems with more outputs than inputs. In particular, we study those cases where the associated system matrix attains full-column rank. We exhibit situations where they generically have no finite nonzero zeros.

I. INTRODUCTION

Multirate linear systems have been studied in different disciplines such as systems and control [4], signal processing [12] and econometric modeling [5] for some decades. In particular, with recent theoretical advances in the field of econometric modeling see e.g. [8], multirate systems analysis has found more potential applications in multirate analysis. The authors of this paper have also become interested in this area while studying generalized dynamic factor models in the field of econometric modeling.

This is because in econometric modeling, it is common to have some data which are collected monthly, while other data may be obtained quarterly or even annually [11], [5]. It is also the case that some data is aggregated bimonthly and other data obtained quarterly. The mentioned scenarios can be studied under multirate systems analysis. In systems and control, the former scenario is associated with multirate systems whose measured outputs have two parts, one available at all times (*fast outputs*) and the other one is available every N time instants (*slow outputs*). In this paper, we refer to such systems as **multirate systems type-1**; in partial contrast, the second scenario corresponds to multirate systems whose measured outputs still have two parts, fast outputs and slow outputs, but, unlike the former case the rates of availability

of the fast outputs and the slow outputs are coprime integers. We use the term **multirate systems type-2** to refer to these systems.

In modeling of high dimensional time-series using generalized dynamic factor models, one has very frequently to deal systems which have a larger number of outputs than inputs i.e. tall systems [10], [7]. In this context, the latent variables i.e. the noiseless part of the outputs, or the part remaining after removal of contaminating additive measurement noise, are modeled by systems with unobserved white noise inputs. In a single-rate setting i.e. monthly data only, the authors in [7] have shown when the generalized dynamic factor model is zero-free then the latent variables can be modeled as a singular autoregressive process whose parameters can be easily identified using Yule-Walker equations. These results were subsequently extended to multirate systems type-1 [1] and [13]. The authors in [13] have demonstrated that the blocked system associated with multirate systems type-1 has no finite nonzero zeros in a generic setting i.e. when the parameter matrices in the underlying state variable realization assume generic values. The authors in [1] demonstrate, under the reasonable postulate that there exists an autoregressive model operating at the highest sampling frequency, that the parameters of this model are generically identifiable from those population second order moments of the multirate system which can be observed in principle¹. Later, the results of [1] are extended to autoregressive models with the noise input having singular covariance matrix [14]. A corresponding demonstration is still lacking for the multirate systems type-2. This paper is a step in that direction. Here, we do not focus on the applications problem, but rather on the system theoretical issues involved with multirate systems type-2 with tall structure.

In particular, we focus on zero-freeness of multirate systems of type-2. In our work [15], we have studied the similar problem for multirate systems type-1. In the rest of this paper, the term multirate systems is used to refer to multirate systems type-2 unless otherwise mentioned. *We explore those cases where the associated blocked system matrix has full-column normal rank. Under this condition we explain when multirate systems with generic parameter matrices have no finite nonzero zeros.*

This paper is structured as follows. Section II introduces the problem under study. Then in Section III, we introduce the idea of two-step blocking and based on that explore

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¹Second order moments which are observed in "principle" are those obtainable from sample statistics of measured variables when the number of samples goes to infinity.

the controllability and observability of the blocked system associated with a multirate system described in Section II. In Section IV the dynamic properties and particularly zeros of tall blocked systems are investigated. Finally, Section V provides concluding remarks.

II. PROBLEM FORMULATION

In this section, first the formulation of the problem under study is given. The dynamics of an underlying system are defined by

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t, \\ y_t &= Cx_t + Du_t, \end{aligned} \quad (1)$$

where $x_t \in \mathbb{R}^n$ is the state, $y_t \in \mathbb{R}^p$ the output, and $u_t \in \mathbb{R}^m$ the input. For this system, y_t exists for all t , and can in principle be measured at every time t . However, we are specifically interested in the situation where y_t exists for all t , but not all of its entries are measured at every time instant. In particular, we consider the case where y_t has components that are observed at different rates. Indeed, we suppose there are two output streams, one available every t_f time instants and the other every t_s time instants, with t_f and t_s coprime integers. Without loss of generality we assume $t_f < t_s$. The noncoprime case is not treated here, since we believe that extensions of the ideas of this paper or of [15] can tackle such a problem.

Without loss of generality we decompose y_t as $y_t = [y_t^f \ y_t^s]'$ where $y_t^f \in \mathbb{R}^{p_f}$ is observed at $t = 0, t_f, 2t_f, \dots$, the fast part, and $y_t^s \in \mathbb{R}^{p_s}$ is observed at $t = 0, t_s, 2t_s, \dots$, the slow part; also $p_f > 0$, $p_s > 0$ and $p_f + p_s = p$. Correspondingly, we decompose C and D as $C = [C^f \ C^s]'$, $D = [D^f \ D^s]'$.

Thus, the multirate linear system (which we denote by Σ) corresponding to what is measured has the following dynamics:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \quad t = 0, 1, 2, \dots \\ y_t^f &= C^f x_t + D^f u_t \quad t = 0, t_f, 2t_f, \dots \\ y_t^s &= C^s x_t + D^s u_t \quad t = 0, t_s, 2t_s, \dots \end{aligned} \quad (2)$$

Since the rates of availability of fast outputs and slow outputs are assumed to be coprime, in order to obtain a blocked linear time-invariant system associated with the system (2) one has to block the system by the rate $T \triangleq t_f t_s$. Actually there exist T distinct ways to block the above multirate system. All T resultant blocked systems share common poles properties but their zeros might not be identical. The distinction is made apparent immediately below.

For the tag point $\tau \in \{1, 2, \dots, T\}$, define

$$\begin{aligned} U_t^\tau &\triangleq [u'_{t+\tau} \ u'_{t+\tau+1} \ \dots \ u'_{t+\tau+T-1}]' \\ Y_t^\tau &\triangleq [Y_t^{f,\tau} \ Y_t^{s,\tau}]' \\ Y_t^{f,\tau} &= [y_{t+\tau+\theta_f}^{f'} \ \dots \ y_{t+\tau+\theta_f+(t_s-1)t_f}^{f'}] \\ Y_t^{s,\tau} &= [y_{t+\tau+\theta_s}^{s'} \ y_{t+\tau+\theta_s+t_s}^{s'} \ \dots \ y_{t+\tau+\theta_s+(t_f-1)t_s}^{s'}] \end{aligned} \quad (3)$$

where $t = 0, T, 2T, \dots$, $\theta_f \triangleq (T - \tau) \bmod t_f$ and $\theta_s \triangleq (T - \tau) \bmod t_s$. Finally $x_t^\tau \triangleq x_{t+\tau}$.

The integer θ_f (θ_s) admits a physical interpretation as the delay between the tag point τ and the first among the following time instants in which a sample of y^f (y^s) is available. Figure 1 shows an example for $t_f = 2$, $t_s = 3$ and $\tau = 1$, which results in $\theta_f = 1$, $\theta_s = 2$.

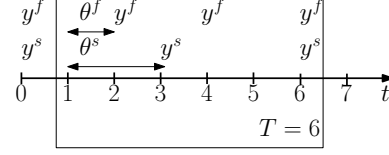


Fig. 1. Collection of observations in a blocked system. The input signal u is assumed to be available at every time instant.

The blocked system Σ_τ is defined by

$$\begin{aligned} x_{t+T}^\tau &= A_\tau x_t^\tau + B_\tau U_t^\tau, \\ Y_t^\tau &= C_\tau x_t^\tau + D_\tau U_t^\tau, \end{aligned} \quad (4)$$

where,

$$\begin{aligned} A_\tau &\triangleq A^T, \quad B_\tau \triangleq [A^{T-1}B \ A^{T-2}B \ \dots \ AB \ B], \\ C_\tau &\triangleq [C_\tau^{f'} \ C_\tau^{s'}]', \quad D_\tau \triangleq [D_\tau^{f'} \ D_\tau^{s'}]'. \\ C_\tau^f &\triangleq [(C^f A^{\theta_f})' (C^f A^{\theta_f+t_f})' \ \dots \ (C^f A^{\theta_f+T-t_f})']', \end{aligned} \quad (5)$$

Furthermore, $D_{\tau[i,j]}^f \triangleq Q^f(\theta_f + (i-1)t_f + (j-1))$, with $D_{\tau[i,j]}^f$ denoting the $p_f \times m$ block of matrix D_τ^f in block position i, j and

$$Q^f(\nu) = \begin{cases} 0 & \nu < 0, \\ D^f & \nu = 0, \\ C^f A^{\nu-1} B & \nu > 0. \end{cases} \quad (6)$$

The matrices D_τ^s and C_τ^s can be defined exactly like D_τ^f and C_τ^f by replacing $C^f, D^f, \theta_f, t_f, p_f$ with their associated slow counterparts, i.e. $C^s, D^s, \theta_s, t_s, p_s$.

The following example suggests a possible structure of the blocked system Σ_τ .

Example 1: Consider the system

$$\begin{aligned} x_{t+1} &= ax_t + bu_t \quad t = 0, 1, 2, \dots \\ y_t^f &= c^f x_t + d^f u_t \quad t = 0, t_f, 2t_f, \dots \\ y_t^s &= c^s x_t + d^s u_t \quad t = 0, t_s, 2t_s, \dots \end{aligned} \quad (7)$$

where the $a, b, c^f, d^f, c^s, d^s \in \mathbb{R}$. We assume the same values for t_f and t_s as in Fig. 1 i.e. $t_f = 2$ and $t_s = 3$, so that the blocking rate is $T = 6$. Then, for $\tau = 1$ the blocked system Σ_τ is described by the matrices

$$\begin{aligned} A_\tau &= a^6, \quad B_\tau = [a^5b \ a^4b \ a^3b \ a^2b \ ab \ b], \\ C_\tau^f &= [c^f a \ c^f a^3 \ c^f a^5]', \quad C_\tau^s = [c^s a^2 \ c^s a^5]', \\ D_\tau^f &= \begin{bmatrix} c^f b & d^f & 0 & 0 & 0 & 0 \\ c^f a^2 b & c^f a b & c^f b & d^f & 0 & 0 \\ c^f a^4 b & c^f a^3 b & c^f a^2 b & c^f a b & c^f b & d^f \end{bmatrix}, \\ D_\tau^s &= \begin{bmatrix} c^s a b & c^s b & d^s & 0 & 0 & 0 \\ c^s a^4 b & c^s a^3 b & c^s a^2 b & c^s a b & c^s b & d^s \end{bmatrix}. \end{aligned} \quad (8)$$

III. STRUCTURAL PROPERTIES

A. An interpretation in terms of two-step blocking

In the previous section the structure of the blocked system Σ_τ has been provided. The presence of two measured output streams available at coprime rates seems to induce a complex structure for Σ_τ . In this section, we first demonstrate that the system Σ_τ admits a natural interpretation in terms of a two-step blocking once the fast and slow rate outputs are considered separately. The main idea is to split Σ into the systems $\Sigma^f \triangleq (A, B, C^f, D^f)$ and $\Sigma^s \triangleq (A, B, C^s, D^s)$ and then to separately block these systems (according to a certain procedure which will be discussed in more details later). The two resultant blocked systems apparently share the same input-state dynamics which is identical to the input-state dynamics of the system Σ_τ . Furthermore, one can construct the output dynamics of the system Σ_τ by simply stacking the output dynamics of the blocked system obtained only by considering the fast rate outputs over those obtained by only considering the slow rate outputs. To illustrate the approach in more details, we first focus on Σ^f only; we show that the system $\tilde{\Sigma}_\tau^f \triangleq (A_\tau, B_\tau, C_\tau^f, D_\tau^f)$, which is the part of the blocked final system Σ_τ associated with the fast rate output only, can be obtained by the following operations:

- 1) Block the system Σ^f with $\tau_f = t_f - \theta_f$ replacing the tag point τ and the input block size of t_f (**step 1**).
- 2) Block the system resulting from the previous step, call it Σ_{τ_f} , with the input and the output block size of t_s and the tag point equal to zero (**step 2**).

It is worthwhile remarking that one can easily obtain the system $\tilde{\Sigma}_\tau^s \triangleq (A_\tau, B_\tau, C_\tau^s, D_\tau^s)$ by blocking Σ^s with the tag point equal to $\tau_s = t_s - \theta_s$ and input block size of t_s . Then blocking the system obtained from the previous step, say Σ_{τ_s} , by the input and output block size of t_f and the tag point equal to zero. Finally, the system Σ_τ is obtained by considering A_τ and B_τ and simply stacking the matrices C_τ^f over C_τ^s and D_τ^f over D_τ^s , respectively. For the sake of complete explanation, we now show how A_τ, B_τ, C_τ^f and D_τ^f can be obtained through the two-step blocking approach.

1) *Step 1*: The result of step 1 of the blocking procedure is the system Σ_{τ_f} , which is defined by a quadruple $(A_{\tau_f}, B_{\tau_f}, C_{\tau_f}, D_{\tau_f})$, where

$$\begin{aligned} A_{\tau_f} &\triangleq A^{t_f}, & B_{\tau_f} &\triangleq [A^{(t_f-1)}B \quad \dots \quad AB \quad B], \\ C_{\tau_f} &\triangleq C^f A^{\theta_f}, \\ D_{\tau_f} &\triangleq [C^f A^{\theta_f-1}B \quad C^f A^{\theta_f-2}B \quad \dots \quad D^f \quad 0_{p_f \times m(t_f-\theta_f-1)}]. \end{aligned} \quad (9)$$

2) *Step 2*: In this step one performs blocking on the system Σ_{τ_f} . It is shown in the following that the resulting system is exactly the system $\tilde{\Sigma}_\tau^f$. One can easily observe that when the system Σ_{τ_f} is blocked with input block size of t_s , the state matrix of the resultant system will be in the form $A_{\tau_f}^{t_s} = A^{t_s t_f} = A_\tau$. Furthermore, the input matrix of the resultant system has the following structure

$$\begin{bmatrix} A_{\tau_f}^{(t_s-1)} B_{\tau_f} & A_{\tau_f}^{(t_s-2)} B_{\tau_f} & \dots & A_{\tau_f} B_{\tau_f} & B_{\tau_f} \end{bmatrix}. \quad (10)$$

Using simple algebra computations one can see that

$$\begin{aligned} A_{\tau_f}^{(t_s-i)} B_{\tau_f} &= A^{t_f(t_s-i)} [A^{(t_f-1)}B \quad A^{(t_f-2)}B \quad \dots \quad B] \\ &= [A^{(T-(i-1)t_f-1)}B \quad A^{(T-(i-1)t_f-2)}B \quad \dots \\ &\quad \dots \quad A^{(T-it_f+1)}B \quad A^{(T-it_f)}B], \end{aligned} \quad (11)$$

and it is straightforward to see that the resultant matrix is B_τ . Moreover, the output matrix of the resulting blocked system is

$$[C_{\tau_f}' \quad (C_{\tau_f} A_{\tau_f})' \quad (C_{\tau_f} A_{\tau_f}^2)' \quad \dots \quad (C_{\tau_f} A_{\tau_f}^{t_s-1})']'. \quad (12)$$

Again, one can see that $C_{\tau_f} A_{\tau_f}^i = C^f A^{\theta_f + i t_f}$, hence the resulting matrix is C_τ^f . Finally, the direct feedthrough matrix of the blocked system is

$$\begin{bmatrix} D_{\tau_f} & 0 & \dots & 0 \\ C_{\tau_f} B_{\tau_f} & D_{\tau_f} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ C_{\tau_f} A_{\tau_f}^{t_s-2} B_{\tau_f} & C_{\tau_f} A_{\tau_f}^{t_s-3} B_{\tau_f} & \dots & D_{\tau_f} \end{bmatrix} \quad (13)$$

and by substituting parameters from (9) one can easily verify that the matrix in (13) is D_τ^f .

Hence, the dynamics of the fast rate outputs of the blocked system Σ_τ can be obtained from the two-step blocking process described above. One can easily observe that the slow rate outputs of the blocked system Σ_τ can be obtained in a similar way.

B. Minimality of the blocked system

In this section, we consider the controllability and observability properties of the blocked system Σ_τ given corresponding properties for the unblocked multirate system. To this end, we need to recall the following result from [6].

Lemma 1: [6] The unblocked multirate system (2) is controllable (observable) at time τ if and only if the system Σ_τ is controllable (observable).

With the help of the above lemma the following result proves the minimality of the system Σ_τ for a set of generic parameter matrices A, B , etc.

Theorem 1: Consider the system Σ_τ defined by the quadruple $\{A_\tau, B_\tau, C_\tau, D_\tau\}$, where $A_\tau, B_\tau, C_\tau, D_\tau$ are specified by (5). Then, for a generic choice of the matrices A, B, C^f, D^f, C^s and D^s the system Σ_τ is controllable and observable (and so minimal).

Proof: The proof is straightforward. Since the parameter matrices A, B, C^f, C^s, D^f and D^s assume generic values the associated multirate dynamics is both controllable and observable at time τ . Then using the result of Lemma 1 the same should hold for Σ_τ . ■

Remark 1: It is immediate that the systems Σ_{τ_f} and Σ_{τ_s} are generically controllable and observable.

IV. ZEROS OF THE BLOCKED SYSTEM

In this section we analyze the zero properties of Σ_τ . While the poles properties are well known [6], [3], [2], the problem of verifying the possible presence of zeros is

nontrivial and needs a deep analysis. In particular, motivated by applications in econometrics modeling and systems and control, we consider tall systems. The general condition for the system Σ_τ to be tall is $t_s p_f + t_f p_s > Tm$. Such a relation can be interpreted as a subset of the first orthant of the plane (p_f, p_s) ; for example it can be partitioned as

- 1) $p_f > t_f m$;
- 2) $p_s > t_s m$;
- 3) $\frac{Tm - t_f p_s}{t_s} < p_f \leq t_f m, p_s \leq t_s m$.

These partitions are depicted in Fig. 2. In rest of the this

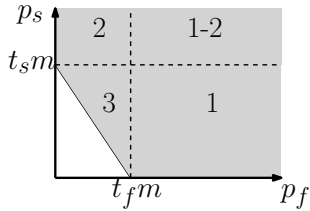


Fig. 2. Partitioning of the condition for tallness.

paper we will focus on the finite zeros. Infinite zeros and zeros at the origin certainly require separate techniques for the examination in the case $t_f = 1$ treated in [13], and we presume that to be the case here too. Thus, infinite zeros and zeros at the origin will be considered in future. Further, we restrict our attention to regions one and two in Fig. 2. It is worthwhile remarking that the region three is associated with tall blocked systems with fast and slow parts neither of which alone is tall. The analysis of zeros for this region appear to be harder compared to the other regions. In particular, we show that for a generic choice of the parameter matrices A, B , etc., the blocked system Σ_τ has no finite nonzero zeros provided that their state space dimension satisfies certain inequalities. The object of our interest is the so-called system matrix Σ_τ , defined as the $(t_s p_f + t_f p_s + n) \times (n + Tm)$ matrix

$$M_\tau(Z) \triangleq \begin{bmatrix} ZI - A_\tau & -B_\tau \\ C_\tau & D_\tau \end{bmatrix}, \quad Z \in \mathbb{C} \quad (14)$$

since such a matrix is of paramount importance for the study of the zeros of a system, as can be seen in the following definition, which can be considered as a standard in the literature of linear systems theory [9].

Definition 1: The finite zeros of the system Σ_τ are defined as those values of Z at which the rank of (14) fall below its normal rank. Moreover, the system Σ_τ is said to have an infinite zero when $n + \text{rank}(D_\tau)$ is less than the normal rank of $M_\tau(Z)$ or equivalently the rank of D_τ is less than the normal rank of $W_\tau(Z) \triangleq C_\tau(ZI - A_\tau)^{-1}B_\tau + D_\tau$.

Remark 2: Note that when the quadruple $\{A_\tau, B_\tau, C_\tau, D_\tau\}$ is a minimal realization of the matrix transfer function $W_\tau(Z)$, the zeros are just determined by the transfer function. Furthermore, they will coincide with those obtained from other well-known starting points for consideration of zeros e.g. Smith-McMillan form and coprime polynomial matrix fraction description. It is also worthwhile mentioning that when the realization is nonminimal, any uncontrollable and unobservable mode is also a zero.

We now use the concept of two-step blocking introduced earlier in the paper, to deal with zeros of the system Σ_τ when $p_f > t_f m$ and generic parameter matrices A, B, C^f, D^f, C^s, D^s . The case $p_s > t_s m$ is similar, and is considered below.

Similar to Definition 1, we define zeros for the system Σ_{τ_f} . We first introduce the symbol ζ , denoting both a complex variable and the t_f -steps forward shift operator such that $\zeta y_t = y_{t+t_f}$.

Definition 2: The finite zeros of the system Σ_{τ_f} are defined as those finite values of ζ at which the rank of the system matrix

$$S_{\tau_f}(\zeta) \triangleq \begin{bmatrix} \zeta I - A_{\tau_f} & -B_{\tau_f} \\ C_{\tau_f} & D_{\tau_f} \end{bmatrix}, \quad \zeta \in \mathbb{C}$$

falls below its normal rank.

In the following we examine the zeros of Σ_{τ_f} . We start our analysis by considering the value $\tau_f = 1$, which corresponds to $\theta_f = t_f - 1$. First, we need to define a square submatrix of $S_1(\zeta)$, call it $N_1(\zeta)$, such that $\text{normal rank}(N_1(\zeta)) = \text{normal rank}(S_1(\zeta))$. Then

$$S_1(\zeta) = \begin{bmatrix} N_1(\zeta) \\ C_2 \quad D_2 \end{bmatrix} \quad (15)$$

where C_2 and D_2 capture those rows of C_{τ_f} and D_{τ_f} that are not included in $N_1(\zeta)$.

Proposition 1: Let the matrix $N_1(\zeta)$ be a submatrix of $S_1(\zeta)$ formed via the procedure described. Then for generic values of the matrices A, B , etc. with $p_f > t_f m$, for any finite ζ_0 for which the matrix $N_1(\zeta_0)$ has less rank than the normal rank of $N_1(\zeta)$, its rank is one less than its normal rank.

Proof: The proof can be done similarly to the proof of Proposition 2.8 in [15] and thus is skipped. ■

The next theorem shows that if the two-step blocking procedure is used to construct Σ_τ , then after the first step, there is freedom from finite nonzero zeros generically.

Theorem 2: For a generic choice of the matrices $\{A, B, C^f, D^f\}$ with $p_f > t_f m$, the system matrix $S_{\tau_f}(\zeta)$, has rank equal to its normal rank for all finite nonzero values or equivalently the system Σ_{τ_f} has no finite nonzero zeros.

Proof: We first focus on the case where $\tau_f = 1$. This proof runs along the lines of the proof of Theorem 2.9 in [15]. Apart from the last $p_f - t_f m$ rows of the matrices C_{τ_f} and D_{τ_f} , which do not enter the matrix $N_1(\zeta)$, choose generic values for the defining matrices, so that the conclusions of the preceding proposition are valid.

Let ζ_a, ζ_b, \dots be the finite set of ζ for which $N_1(\zeta)$ has less rank than its normal rank (the set may have less than n elements, but never has more), and let w_a, w_b, \dots be vectors which are in the corresponding kernels and orthogonal to the subspace in the kernel obtained from the limit of the kernel of $N_1(\zeta)$ as $\zeta \rightarrow \zeta_a, \zeta_b, \dots$ etc.

Now to obtain a contradiction, let us suppose that the system matrix $S_1(\zeta)$ is such that for $\zeta_0 \neq 0$, $S_1(\zeta_0)$ has rank less than its normal rank; in fact, we can assume that its rank is precisely one less than the normal rank because

any existing vector in kernel of $S_1(\zeta)$ is in the kernel of $N_1(\zeta)$ and there is only one independent vector in $N_1(\zeta_0)$ that is not the limit of vectors obtained as $\zeta \rightarrow \zeta_0$. Then there is necessarily a ζ_0 and associated nonzero w_0 in the kernel of $S_1(\zeta_0)$ and which is orthogonal to the limit of the kernel of $S_1(\zeta)$ as $\zeta \rightarrow \zeta_0$. Then w_0 is necessarily in the kernel of $N_1(\zeta_0)$, orthogonal to the limit as $\zeta \rightarrow \zeta_0$ of the kernel of $N_1(\zeta)$ and thus w_0 in fact must coincide to within a nonzero multiplier with one of the vectors w_a, w_b, \dots . Write this w_0 as $w_0 = [x'_1 u'_1 u'_2 \dots u'_{t_f}]'$ and suppose the input sequence u_i is applied for $i = 1, \dots, t_f$ to the original system, starting in initial state x_1 at time 1.

Let $y_{t_f}^f$ denote the fast output at time t_f . Furthermore, let \mathcal{C}_1 and \mathcal{D}_1 represent those rows of C_{τ_f} and D_{τ_f} entering in the matrix $N_1(\zeta)$. We now break up $y_{t_f}^f$ into two subvectors $y_{t_f}^{f1}, y_{t_f}^{f2}$, where $y_{t_f}^{f1}$ are those entries of $y_{t_f}^f$ associated with $\mathcal{C}_1, \mathcal{D}_1$, while $y_{t_f}^{f2}$ captures the remainder entries of $y_{t_f}^f$. We have $N_1(\zeta_0)w_0 =$

$$\begin{bmatrix} \zeta I - A_{\tau_f} & B_{\tau_f} \\ \mathcal{C}_1 & \mathcal{D}_1 \end{bmatrix} w_0 = \begin{bmatrix} \zeta_0 x_1 - x_{t_f+1} \\ y_{t_f}^{f1} \end{bmatrix} = 0.$$

Now it must be true that $x_1 \neq 0$. For otherwise, we would have $N_1(\zeta)w_1 = 0$ for all ζ , which would violate assumptions. Since also $\zeta_0 \neq 0$, there must hold $x_{t_f+1} \neq 0$. Hence there cannot hold both $x_{t_f} = 0$ and $u_{t_f} = 0$. Consequently, we can always find $\mathcal{C}_2, \mathcal{D}_2$ such that $y_{t_f}^{f2} = \mathcal{C}_2 x_{t_f} + \mathcal{D}_2 u_{t_f} \neq 0$, i.e. part of the fast output at the instant t_f is necessarily nonzero, no matter whether $w_0 = w_a, w_b$ etc. Hence, $S_1(\zeta)$ generically has rank equal to its normal rank for all finite nonzero ζ . So far we have shown the conclusion of theorem for a particular choice of τ_f . Now, we show that the latter property holds for all $S_{\tau_f}(\zeta)$. First, note that since the parameter matrices A, B, C^f, D^f assume generic values, the system Σ_{τ_f} is minimal (see Remark 1). Now, consider $\zeta_0 \in \mathbb{C} - \{0, \infty\}$; if it does not coincide with an eigenvalue of A_{τ} then the following equality holds

$$\text{rank}(S_{\tau_f}(\zeta_0)) = n + \text{rank}(V_{\tau_f}(\zeta_0)), \quad (16)$$

where $V_{\tau_f}(\zeta)$ is the matrix transfer function of the system Σ_{τ_f} . Further, observe that

$$V_{\tau_f+1}(\zeta) = \zeta I_{p_f} V_{\tau_f}(\zeta) \begin{bmatrix} 0 & \zeta^{-1} I_m \\ I_{m(t_f-1)} & 0 \end{bmatrix}. \quad (17)$$

With the help of the above equalities i.e. (16) and (17), one can conclude that $S_{\tau_f}(\zeta_0)$ has the same rank for all possible values of τ_f . Moreover, all $S_{\tau_f}(\zeta)$ have the same normal rank. If ζ_0 coincides with one or more eigenvalues of A_{τ} , one can find a state feedback to move the eigenvalues while the zeros defined using $S_{\tau_f}(\zeta)$ remain unchanged (note that the pair (A_{τ_f}, B_{τ_f}) is controllable). Hence, we can conclude that $S_{\tau_f}(\zeta)$ has rank equal to its normal rank for all finite nonzero ζ , or that the system defined by the quadruple $\{A, B, C^f, D^f\}$ generically has no finite nonzero zeros. ■

We now recall the following result from one of our previous works [16].

Lemma 2: [16] The system $\tilde{\Sigma}_{\tau}^f$ has a zero if and only if the system Σ_{τ_f} has a zero.

Now with the help of Lemma 2 and Theorem 2, the following corollary becomes immediate.

Corollary 1: For a generic choice of a quadruple $\{A, B, C^f, D^f\}$, with $p_f > t_f m$, the blocked system Σ_{τ}^f has no finite nonzero zeros.

Then one can easily observe that the following result also holds.

Corollary 2: For a generic choice of a quadruple $\{A, B, C^s, D^s\}$, with $p_s > t_s m$, the blocked system $\tilde{\Sigma}_{\tau}^s$ has no finite nonzero zeros.

Our ultimate interest is to study the zero-freeness of Σ_{τ} . Note that according to Definition 1 the normal rank of system matrix $M_{\tau}(Z)$ plays an important role in the zero properties of Σ_{τ} . The proof of zero-freeness for the system Σ_{τ} is much simpler when the associated system matrix has full-column normal rank compared to the case where it has less than full-column normal rank. Here, we deal with the simpler problem before tackling the harder one in the future. In the following we show sufficient conditions for $M_{\tau}(Z)$ to have full-column normal rank. To achieve this, we first focus on the normal rank of $S_{\tau_f}(\zeta)$.

The following proposition even though restricted to a particular choice of τ_f , will help us later to provide sufficient conditions for all $S_{\tau_f}(\zeta)$ to have full-column normal rank.

Proposition 2: Consider a generic choice of the matrices $\{A, B, C^f, D^f\}$. Assume that $n \geq (t_f - 1)m$, $p_f > t_f m$; then the matrix D_{τ_f} , for $\tau_f = 1$, has full-column rank.

Proof: First, observe that when $\tau_f = 1$, we have $\theta_f = t_f - 1$. Furthermore, the matrix D_{τ_f} admits the factorization, $[C^f \dots C^f D^f] \Gamma$, and $\Gamma := \text{diag}\{A^{\theta_f-1}B, A^{\theta_f-2}B, \dots, B, I_m\}$. Since $n \geq m$, Γ is not full-row rank. Note that the matrix A is generic and thus is nonsingular; hence, one can select $n - m$ rows from each block $A^{\theta_f-1}B, A^{\theta_f-2}B, \dots, B, I_m$ and discard them in order to obtain a matrix, say $\tilde{\Gamma}$, which becomes full-row rank, namely $(\theta_f + 1)m = t_f m$. Similarly, one can construct the matrix $\tilde{\Pi}$, which is defined by discarding the corresponding columns from the matrix $[C^f \dots C^f D^f]$. Due to the genericity of matrix A , the selection process can be done such that the matrix $\tilde{\Pi}$ includes all the columns of the matrix C^f (and their linear combinations) and all columns of matrix D^f . Since $\tilde{\Gamma}$ is full-row rank, the rank of D_{τ_f} only depends on the rank of $\tilde{\Pi}$. Moreover, one can easily observe that the rank of $\tilde{\Pi}$ is equal to the rank of the matrix $\bar{\Pi} := [C^f D^f] \in \mathbb{R}^{p_f \times [n+m]}$. Under the assumption $n \geq (t_f - 1)m$, both the integers p_f and $n + m$ are greater than or equal to the number of columns of D_{τ_f} , namely $t_f m$, hence D_{τ_f} is full-column rank. ■

Remark 3: In the above proposition, for the particular choice $\tau_f = 1$, we provided sufficient conditions for D_{τ_f} to have full-column rank. This does not hold for all values of τ_f . The latter is due to fact that depending on the choice of τ , the highest time index of input sequences appearing in the blocked input vector becomes strictly larger than that of output sequences appearing in the blocked output vector.

Even though the conclusion of Proposition 2 is restricted to a particular choice of τ_f , we now provide the following lemma which holds for all $S_{\tau_f}(\zeta)$.

Lemma 3: Consider the system Σ_{τ_f} with a generic choice of the matrices $\{A, B, C^f, D^f\}$. Assume that $n \geq (t_f - 1)m$, $p_f > t_f m$; then the system matrix $S_{\tau_f}(\zeta)$ has full-column normal rank for any value of $\tau_f \in \{1, \dots, t_f\}$.

Proof: It was shown in Proposition 2 that for $\tau_f = 1$ the feedthrough matrix has full-column rank. Then with the help of Lemma 2.4 in [15], one can conclude the associated system matrix has full-column normal rank. Moreover, note that all $S_{\tau_f}(\zeta)$, for all possible values of τ_f , have the same normal rank (see Proposition A.1 in [13]). Hence, under the conditions established in the statement of the lemma, $S_{\tau_f}(\zeta)$ has full-column normal rank. ■

From Lemma 3 and Theorem 2, we provide the following proposition which has a trivial proof.

Proposition 3: Consider the system Σ_{τ_f} with a generic choice of the matrices $\{A, B, C^f, D^f\}$. Assume that $n \geq (t_f - 1)m$, $p_f > t_f m$; then the system matrix $S_{\tau_f}(\zeta)$ has full-column rank for all ζ .

We now can draw a conclusion about zero-freeness of the system Σ_{τ} .

Theorem 3: Consider the system Σ_{τ} defined by the sextuple $\{A, B, C^f, D^f, C^s, D^s\}$ with generic values for the entries of the defining matrices. Assume that $p_f > t_f m$ and $n \geq (t_f - 1)m$; then the system matrix $M_{\tau}(Z)$ has full-column rank for all $Z \in \mathbb{C} - \{0, \infty\}$, and accordingly the system Σ_{τ} has no finite nonzero zeros.

Proof: We first recall from [16] that the system Σ_{τ_f} has full-column normal rank if and only if the associated system matrix of $\tilde{\Sigma}_{\tau_f}^f$ has full-column normal rank. Then, with the help of Lemma 2 and Proposition 3, one can easily conclude that the system matrix of $\tilde{\Sigma}_{\tau_f}^f$ has full-column rank for all finite nonzero values. Hence, it is immediate that $M_{\tau}(Z)$ has full-column rank for all finite nonzero values as well. ■

Remark 4: In our earlier work [15], where $t_f = 1$, there was no parallel of the restriction on the state dimension. If we set $t_f = 1$ in Theorem 3, the condition simply becomes one of saying the state vector has dimension at least 0, which corresponds to giving no conditions on the state dimension.

As a parallel of the above theorem we can evidently also state the following theorem.

Theorem 4: Consider the system Σ_{τ} defined by the sextuple $\{A, B, C^f, D^f, C^s, D^s\}$ with generic values for the entries of the defining matrices. Assume that $p_s > t_s m$ and $n \geq (t_s - 1)m$; then the system matrix $M_{\tau}(Z)$ has full-column rank for all $Z \in \mathbb{C} - \{0, \infty\}$, and accordingly the system Σ_{τ} has no finite nonzero zeros.

V. CONCLUSIONS

Linear multirate systems with coprime measured output rates have been studied in this paper. For their blocked linear time-invariant version, we studied the generic observability and controllability using a two-step blocking approach. Moreover, under the assumption that the parameter matrices are chosen generically, we explored finite nonzero zeros.

In particular we focused on tall blocked systems. It was demonstrated that there exist three possible regions combinations of fast and slow output dimensions and sampling rates which lead to a tall blocked system. Here, we only studied two regions and showed conditions under which the blocked system generically has no finite nonzero zero. It is worthwhile mentioning that, as shown in [13], zeros at infinity and zeros at the origin require special treatment. Consequently, they are not treated here and we hope to consider them in future. Furthermore, as a part of our future work, we will study the generic zero-freeness of blocked systems lying in the third region.

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