ON THE STABILIZABILITY OF MULTIVARIABLE SYSTEMS BY MINIMUM ORDER COMPENSATION

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Abstract

In this paper, we derive the necessary condition, $mp \geq n$, for stabilizability by constant gain feedback of the generic degree n, $p \times m$ system. This follows from another of our main results, which asserts that generic stabilizability is equivalent to generic solvability of a deadbeat control problem, provided $mp \leq n$. Taken together, these conclusions enable us to make some sharp statements concerning minimum order stabilization. The techniques are primarily drawn from decision algebra and classical algebraic geometry and have additional consequences for problems of stabilizability and pole-assignability. Among these are the decidability (by a Sturm test) of the equivalence of generic pole-assignability and generic stabilizability, the semi-algebraic nature of the minimum order, q, of a stabilizing compensator, and the nonexistence of formulae involving rational operations and extraction of square roots for pole-assigning gains when they exist, answering in the negative a question raised by Anderson, Bose, and Jury.

1. Introduction

One of the major open problems in algebraic system theory is to compute, for a given $p \times m$ system $G(s)$ of McMillan degree n, *the minimum order q of a compensator* $K(s)$ which (internally) stabilizes $G(s)$. Aside from its classical importance and appeal, it is fairly widely appreciated that a clean solution of this problem could potentially be applied to model reference adaptive control, particularly in the development of new "parameter adjustment" equations. In this paper, we give a survey of our results [7] yielding necessary conditions for the generic stabilizability by constant gain feedback of systems with m, n, and p fixed. Some new results are also given, including the use of these conditions to determine minimal orders of compensators for certain low-dimensional cases, and a new proof — based on the classical algebraic geometry of projective curves — of the high gain arguments we used in [7]. Since our proofs will also rely on the Tarski-Seidenberg Theorem, we begin with a discussion of the qualitative behavior of q in this context.

Fix m, n, and p and denote by U_q the set of systems for which q is the minimum order of a stabilizing compensator. Then, one solution of this problem would entail the explicit description of U_q which, by analogy with the Routh-Hurwitz criterion, one might expect to be given by inequalities in the "parameters" of the system. Since this latter concept is potentially a thorny one, we parameterize a system in terms of any one of its minimal realizations (F,G,H). That is, for m, n, and p fixed as usual, set

$$\tilde{\sum}^{n}_{m,p} = \left\{(F,G,H) \; : \; (F,G,H) \text{ is minimal}\right\}$$

Thus, $\tilde{\sum}^{n}_{m,p}$ is an open, dense subset of $\mathbb{R}^N$, $N = n^2 + n(m+p)$, which is the disjoint union of the subsets U_q, which one wishes to describe.

Using the quantifier *elimination theory of* Tarski-Seidenberg [17], Anderson, et al [1],[2] proved that membership in U_0 could be decided by a finite sequence of Sturm tests in analogy with the Routh-Hurwitz conditions. This result is equivalent to a qualitative geometric statement about the subset $U_0 \subset \tilde{\sum}^{n}_{m,p}$, following from the reinterpretation of Tarski's Theorem in logic as an assertion in real algebraic geometry which we now briefly describe. A semialgebraic subset of $\mathbb{R}^N$ is a subset defined by one or more polynomial equations (e.g., $f(x) = 0$), inequations ($f(x) \neq 0$), or inequalities ($f(x) > 0$), possibly taken in conjunction or disjunction. Thus, $\tilde{\sum}^{n}_{m,p}$ is an open semialgebraic subset of $\mathbb{R}^N$. The quantifier elimination theorem asserts that the image of a semialgebraic set $X \subset \mathbb{R}^N$ under a projection $p : \mathbb{R}^N \to \mathbb{R}^M$ is semialgebraic. Tarski's Theorem also implies that the closure, boundary, and interior of semialgebraic sets are semialgebraic. With this in mind, one can give a qualitative description of the function $q : \tilde{\sum}^{n}_{m,p} \to \mathbb{N}\cup\{0\}$ and its level sets U_q. According to Anderson et al. [1], [2], U_0 is semialgebraic. A modification of their argument, due to Ghosh [9], aserts that $U_0 \cup \ldots \cup U_j$ is semialgebraic for all j. From these results, one observes

Proposition 1.1 The function q is an upper semicontinuous semialgebraic function. That is, q at most rises under limits and the sets U_q are semialgebraic.

$\tilde{\sum}^{n}_{m,p}$ is therefore the increasing union of the open sets $V_q = U_0 \cup \ldots \cup U_q$, and it is also of interest to compute the minimal value of q such that V_q is dense in $\tilde{\sum}^{n}_{m,p}$; i.e., the minimal q for which the generic $p \times m$ system of order n can be stabilized by a

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compensator of order $\leq q$. Thus, modulo an arbitrarily small perturbation, any system is stabilizable by a compensator of order $\leq q$.

Our first result (Theorem 2.2) asserts that $mp \geq n$ is necessary for generic stabilizability by a 0th order compensator. Presumably, similar conditions hold for $q \geq 0$, and we would like to show how such conditions can be used to determine minimal orders for stabilizing the generic system. First, combining this with results [3] on pole-assignability, one easily deduces

Proposition 1.2 If $\min(m,p) = 1$ and $n \leq 2 \max(m,p)$, then $q = 1$ is the minimum order for stabilization of the generic system.

Note that $mp \geq n$ is also a necessary condition for pole-assignability by constant gain output feedback, raising the question as to whether — for fixed m, n, and p — generic stabilizability and generic pole-assignability are equivalent. This has been shown for $\min(m,p) = 1$ and q arbitrary in the thesis [9] and, as we write, we know of no counter-example. Moreover, Theorem 2.2 is itself derived from an equivalence (Theorem 2.1) for constant gain output feedback between generic stabilizability and generic solvability of the deadbeat control problem. From this equivalence, we are able to give an independent proof (§ 3) of results (n=4) due to Willems-Hesselink [18] and Molander (unpublished), illustrating again this equivalence:

Proposition 1.3 If $m=p=2$, and $n=4$ or 5, $q=1$ is the minimum order required for stabilization of the generic system and, a fortiori, for generic pole assignability.

We remark that for fixed m, n, and p the question of the equivalence of these two properties can be answered in the context of decision algebra (see [7]).

Our proofs also use quite heavily basic properties of the set of "nondegenerate systems," introduced in [5], [6]. In section 4, we conclude by giving an explicit construction of a nondegenerate system when $mp = n$, from which the desired properties hold for the generic system.

2. A Necessary Condition for Generic Stabilizability

In this section, we shall sketch, modulo a technicality discussed in section 4, proofs of two of the main results of [7]:

Theorem 2.1 If $mp \leq n$, the following statements are equivalent:

(i) m, n, and p are such that the generic (F,G,H) is stabilizable;

(ii) m, n, and p are such that for (F,G,H) the output feedback "deadbeat control" problem is solvable.

Theorem 2.2 $mp \geq n$ is necessary for generic stabilizability by constant gain output feedback.

Proof of Theorem 2.1: We begin by noting that (as expected) for m, n, and p fixed, generic stabilizability in discrete and in continuous time are equivalent properties; slightly more generally,

Lemma 2.3 [7] The following statements are equivalent:

(i) m, n, and p are such that for all (F,G,H) — except those contained in a proper algebraic set — there exists a stabilizing gain K (with respect to the left-half plane);

(ii) m, n, and p are such that for all (F,G,H) — except perhaps those in a proper algebraic set — for all real ρ and all $\varepsilon > 0$, there exists a gain K such that the eigenvalues of $F + GKH$ are contained in an ε-disc centered about ρ.

Our strategy is now to consider the behavior of a family of gains K_ε as in Lemma 2.3 (ii) in the "high gain limit" as $\varepsilon \to 0$. Explicitly, for $\sigma = (F,G,H)$ consider the function

$$\chi_\sigma : \mathbb{R}^{mp} \to \mathbb{R}^n$$

defined via $\chi_\sigma(K) = (p_1,\ldots,p_n)$ where

$$s^n + p_1 s^{n-1} + \ldots + p_n = \det(sI - F - GKH).$$

In section 4, we give a new proof of the fact ([6]) that provided $mp \leq n$ there exists an open dense set $W \subset \mathbb{R}^N$, $N = n^2 + n(m+p)$, of triples σ — the set of nondegenerate systems — such that image (χ_σ) is euclidean closed for $\sigma \in W$.

Lemma 2.4 If $mp \leq n$, the following statements are equivalent:

(i) m, n, and p are such that the generic (F,G,H) is stabilizable;

(ii) m, n, and p are such that for all real ρ and the generic (F,G,H), there exists a gain K such that the closed loop characteristic polynomial is $(s-\rho)^n$.

Proof: If (i) holds, for each r there exists an open dense subset $U_r \subset W$ such that, for $(F,G,H) \in U_r$, $(p_1,\ldots,p_n) \in$ image (χ_σ) where the roots of $s^n + p_1 s^{n-1} + \ldots + p_n$ lie in a $1/r$-disc centered about ρ. By the Baire Category Theorem, $U = \bigcap_{r=1}^{\infty} U_r$ is a dense subset of W and therefore of $\mathbb{R}^N$. Moreover, for $\sigma \in U$ there exists K such that $\chi_\sigma(K) = (s-\rho)^n$ since image χ_σ is closed. Now consider the real algebraic set

$$V^\rho = \{(F,G,H,K) : \det(sI - F - GKH) = (s-\rho)^n\}$$

and the projection $p_1(F,G,H,K) = (F,G,H)$. By the Tarski-Seidenberg Theorem, $p_1(V^\rho)$ is semialgebraic. Since $U \subset p_1(V^\rho)$ is dense, $p_1(V^\rho)$ may be defined by inequalities

$$f_1(F,G,H) > 0, \ \ldots, \ f_k(F,G,H) > 0$$

from which it follows that $p_1(V^\rho)$ is open and dense. Since $(F,G,H) \in p_1(V^\rho)$ if, and only if, there exists a K such that $\det(sI - F - GKH) = (s-\rho)^n$, the lemma is proved.

Q.E.D.

This proves Theorem 2.1, except for the sharper assertion that the open dense set in (ii) may be taken to be W. See the discussion in section 4.

We shall now prove Theorem 2.2. Note that it suffices to consider the case $mp \leq n$. Consider the algebraic set of $n \times n$ nilpotent matrices

$$\mathcal{N} = \{N : N^k = 0 \text{ for some } k\}$$

and the polynomial mapping

$$\Phi : \mathcal{N} \times \mathbb{R}^{nm} \times \mathbb{R}^{np} \times \mathbb{R}^{mp} \to \mathbb{R}^{n^2} \times \mathbb{R}^{nm} \times \mathbb{R}^{np}$$

defined via

$$\Phi(N,G,H,K) = (N - GKH, G, H).$$

Taking $\rho = 0$ in our preceding lemmata, we note that, for $mp \leq n$, to say that the generic system is stabilizable is to say that the image of Φ contains an open, dense set and therefore $\dim N + mp \geq n^2$. However, $\dim N = n^2 - n$ (see [13], also [7]) and therefore $mp \geq n$, proving Theorem 2.2. Q.E.D.

3. Example ($m = p = 2$, $n = 4$).

As in the Introduction, we consider the semialgebraic sets

$$U_i = \{(F,G,H) \quad \text{minimum order of a stabilizing compensator is } i\}$$

where we have fixed the parameters $m = p = 2$, and $n = 4$. If

$$X = \tilde{\Sigma}^4_{2,2} - (U_1 \cup U_0)$$

then we claim

Theorem 3.1 $q = 1$ is the minimum order required to stabilize the generic 2×2 system of degree 4. Explicitly, X is a closed semialgebraic set of dimension at most 30, and thus $U_1 \cup U_0$ is open and dense. Moreover, $\operatorname{int}(U_1) \neq \emptyset$.

Remark: For $n=5$, Proposition 1.3 follows from pole assignability results [3] and Theorem 1.2. Note that $\tilde{\Sigma}^4_{2,2}$ decomposes as

Fig. 3.1: $\tilde{\Sigma}^4_{2,2}$ decomposed into the level sets of q.

Proof. First note that the open semialgebraic set $U_1 \cup U_0$ is dense, by the Brasch-Pearson Theorem [3]. Indeed, $X \subset Y$ where Y is the algebraic set of triples with controllability and observability indices in the set $\{(4,0),(3,1)\}$ of partitions of 4. That Y has codimension at least 2 follows from the dimension formulae in [4]. That U_1 is non-empty follows from an unpublished modification by Molander of the techniques in [18]. We sketch our proof given in [7], viz., we *claim nonsolvability of the corresponding deadbeat control problem*

Lemma 3.2. If $m = p = 2$, $n = 4$, then the set

$$V = \{(F,G,H) : \exists K \text{ such that } \det(sI - F - GKH) = s^n\}$$

has a non-empty open complement in $\tilde{\Sigma}^4_{2,2}$.

By Theorem 2.1 and Lemma 3.2, Molander's result will follow. Again, we recall some basic facts (cf. §4) about the set W of nondegenerate systems:

(i) if $mp \leq n$, W is open and dense in $\tilde{\Sigma}^n_{m,p}$; and

(ii) for any monic polynomial $p(s)$ of degree n, the set

$$V_p = \{(F,G,H) : \exists K \text{ such that } \det(sI - F - GKH) = p(s)\}$$

is closed in W.

Thus, it suffices to find one nondegenerate system for which the deadbeat control problem is not solvable. According to [5], one has a frequency domain criterion for nondegeneracy: if $T(s) = H(sI - F)^{-1}G$, denote by $t_i(s)$ the i^{th}-column of

$$\mathcal{T}(s) = \begin{bmatrix} T(s) \\ I \end{bmatrix}$$

thus, $t_i(s)$ is a vector in $\mathbb{C}^{p+m}$. For any linearly independent set $\Phi = \{\phi_1,\ldots,\phi_p\}$ of linear functionals, form the determinant

$$\Phi(s) = \det[\phi_i(t_j(s))].$$

Lemma 3.3 ([5]) $T(s)$ is nondegenerate if, and only if, $\Phi(s) \neq 0$ for all Φ.

As an example (see [7])

$$T(s) = \begin{bmatrix} s^3-1 & -s \\ s & s^3 \end{bmatrix} / s^4 + s - 1$$

is nondegenerate. Moreover, a straightforward calculation [7] shows that there exist no K placing the poles of this $T(s)$ at 0. Q.E.D.

4. Nondegenerate Systems

Suppose $mp \leq n$. In the previous sections, we have made use of the following properties of nondegenerate systems:

(ND1) The subset W of $\tilde{\Sigma}^n_{m,p}$ of nondegenerate systems is is open and dense;

(ND2) If $\sigma \in W$, image (X_σ) is euclidean closed;

(ND3) For $p(s) = s^n + p_1 s^{n-1} + \ldots + p_n$, the subset

$$V_p = \{(F,G,H) : \exists K \text{ such that } \det(sI - F - GKH) = p(s)\}.$$

is a euclidean closed semialgebraic subset of W.

We remark that every scalar input-scalar output system is nondegenerate and, of course, (ND2) and (ND3) are well-known properties in the classical root-locus theory. For $\max(m,p) > 1$, there exist, however, minimal triples for which (ND2) and (ND3) fail to hold. For (ND3), this is quite analogous to the fact that $\{x \in \mathbb{R} : \exists y \text{ such that } xy = 1\}$ is not closed in $\mathbb{R}$, while its intersection with $W = \mathbb{R} - \{0\}$ is closed in W. In particular, nondegeneracy is a multivariable concept enabling one to generalize several nice technical facts concerning root-loci. In order to verify these properties, we proceed formally, at first assuming only the fact that the set D of degenerate systems over $\mathbb{C}$ is a proper algebraic set.

Thus, assume that D is defined as the *locus of* functions $f_i(F,G,H)$, where each f_i is a rational function on $\mathbb{C}^N$ having no poles on $\tilde{\Sigma}^n_{m,p}(\mathbb{C})$. We claim that then the set of real nondegenerate systems is open and dense in $\tilde{\Sigma}^n_{m,p}(\mathbb{R})$. For if $D \cap \tilde{\Sigma}^n_{m,p}(\mathbb{R})$ contained an open set U, then each f_j would vanish on an open subset $U \subset \mathbb{R}^N \subset \mathbb{C}^N$. A power series argument

[14] and which, of course, has application to real pole-assignment questions. Our techniques suggested that use of (5.1) together with Galois theory as in [11] would prove

Conjecture 5.1 For mp = n, there is no universal formula for pose-assigning gains involving only rational operations and extractions of roots, except in the cases

(1) min(m,p) = 1, when there exist linear formulae
(2) m = p = 2, when there exists a quadratic formula.

That is, we conjecture the Galois group is insolvable, except in the aforementioned cases. This is in marked contrast to the situation for state feedback, where rational formulae exist. Also the conjecture would, if true, answer in the negative the corresponding question -- raised in [1] -- for output feedback. In fact, the case m = p = 2 already contains enough information to settle this special case. More generally, in [7] it was proved that only in case (1) would there exist rational formulas and only in case (2) would there exist formulae involving only rational operations and square roots. More recently, the conjecture was affirmed in the case m = 2, p = 3 (see [ ]) where the Galois group was shown to be the full symmetric group S_5.

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(i.e., Osgood's Lemma) then implies that $f_j(F,G,H) \equiv 0$ on $\mathbb{C}^N$, contradicting the fact that D is a proper subset of $\mathbb{C}^N$. We now give a new, constructive proof of the result ([5],[6]).

Theorem 4.1 (i) D is an algebraic subset of $\tilde{\Sigma}^n_{m,p}(\mathbb{C})$;
(ii) There exists a minimum triple $(F,G,H) \notin D$;
(iii) The space of real nondegenerate systems is open and dense in $\tilde{\Sigma}^n_{m,p}(\mathbb{R})$

Proof: Assuming (i), assertion (ii) implies D is a proper algebraic subset of $\tilde{\Sigma}^n_{m,p}(\mathbb{C})$. By the remarks above, (i) and (ii) imply (iii) which is, of course, (ND1). For the proof, we use the Hermann-Martin representation [ ] of the transfer function T(s) as a holomorphic curve

$$T : \mathbb{CP}^1 \to \text{Grass}(m,m+p)$$

of degree n. By definition, T is nondegenerate just in case $G(\mathbb{CP}^1)$ lies in some Schubert hypersurface $\sigma(V)$, where V is a p-plane in $\mathbb{C}^{m+p}$.

In case $m = p = 1$, $\text{Grass}(m,m+p) \simeq \mathbb{CP}^1 \simeq S^2$ and the Hermann-Martin curve is the transfer function

$$T : S^2 \to S^2$$

while $\sigma(V)$ is simply a point. Thus, to say T is degenerate is to say $T(s) \equiv 0$.

Proof(i): Consider the algebraic subset

$$X = \{((F,G,H),V) : T(s) \subset \sigma(V)\}$$

of $\tilde{\Sigma}^n_{m,p} \times \text{Grass}(p,m+p)$. If p_1 is projection on the first factor, then by the Fundamental Theorem of Elimination Theory [15] $D = p_1(X)$ is an algebraic subset of $\tilde{\Sigma}^n_{m,p}$. This proves (i).

Lemma 4.2 If $m = 1$, there exists a minimal $(F,G,H) \notin D$.

Proof: Consider the twisted curve of degree p

$$G : [s,t] \to [t^p, st^{p-1}, \ldots, s^p] \in \mathbb{CP}^p$$

G has (McMillan) degree p and lies in no hyperplane; i.e.

$$a_1 t^p + \ldots + a_{p+1} s^p \equiv 0 \quad \Rightarrow a_i = 0. \quad \text{Q.E.D.}$$

Lemma 4.3 If $mp = n$, there exists a minimal $(F,G,H) \notin D$.

Proof: (J. Harris) Denote the derivative of G(s) by $G^1(s)$. Thus, for the twisted curve of degree r

$$G^1(s) : \mathbb{CP}^1 \to \text{Grass}(2,r+1)$$

which is easily seen to have degree 2r. For $k < r$, we denote by $G^k(s)$ the "k^{th} associated curve" of G(s) (see [ ]); i.e., for fixed s, $G^k(s)$ is the osculating k-plane to G(s) in $\mathbb{CP}^r$. Taking $r = m+p-1$ and $k=m$, one has

$$G^m : \mathbb{CP}^1 \to \text{Grass}(m,m+p).$$

If G is the twisted curve of degree m+p, then its m^{th} associated curve has degree mp (see [10]).

Suppose that $G^m(s)$ is degenerate. If V is a degenerate p-plane for $G^m(s)$, we consider the linear projection $\Phi : \mathbb{CP}^{m+p-1} \to \mathbb{CP}(B) \to \mathbb{CP}^{p-1}$ and the curve $\Phi(G(s))$ in $\mathbb{CP}^{p-1}$. Since $\Phi(G(s))$ is nondegenerate,

$$\Phi(G(s)) \wedge \Phi(G^1(s)) \wedge \ldots \wedge \Phi(G^{p-1}(s)) \not\equiv 0 \qquad (4.1)$$

However, the vanishing of (4.1) is precisely the condition for degeneracy derived in [5], contrary to hypothesis. Q.E.D.

Turning to (ND2) and (ND3), think of a gain K, qua graph(K) as a p-plane in $\mathbb{R}^p \oplus \mathbb{R}^m$; i.e., as a point in the compact manifold Grass(p,m+p). According to [6, Remarks p. 103] for $\sigma = (F,G,H) \in W$, X_σ extends to a continuous map

$$X_\sigma : \text{Grass}(p,m+p) \to \mathbb{R}\,\mathbb{P}^n$$

where $p = (p_1,\ldots,p_n) = X_\sigma(K)$ is regarded as a point $[p_1,\ldots,p_n,1]$ in real projective space $\mathbb{RP}^n$. Moreover, X_σ satisfies

$$X_\sigma(V) = [p_1,\ldots,p_n,1]$$

if, and only if, V = graph(K) for some K. Since Grass(p,m+p) is compact, and X_σ continuous, image $(X_\sigma) \cap \mathbb{R}^n$ is euclidean closed, proving (ND2). We now prove (ND3).

Proposition 4.3 For nondegenerate (F,G,H), (ND3) holds.

Proof: Consider the algebraic function

$$X : W_{\mathbb{R}} \times \text{Grass}(p,m+p) \to \mathbb{RP}^n$$

defined via $X(F,G,H,V) = X_{(F,G,H)}(V)$. For any $p \in \mathbb{RP}^n$, $Z = X^{-1}(p)$ is a closed, algebraic subset of $W_{\mathbb{R}} \times \text{Grass}(p,m+p)$. Applying the projection p_1, $p_1(Z)$ is semialgebraic by the Tarski-Seidenberg Theorem. Moreover, $p_1(Z)$ is closed in W, since Grass(p,m+p) is compact. But $p_1(Z_p) = V_p$. Q.E.D.

5. Concluding Remarks

In this paper, necessary conditions, in terms of m, n, and p, were derived for stabilizability of the generic system by constant gain output feedback. These conditions also yielded characterizations of the minimum order of stabilizing compensators in low dimensions. The techniques of proof, however, appear to be of independent interest. On the one hand, one of the ingredients -- the Tarski-Seidenberg Theorem -- also leads to assertions concerning the determination, by Sturm tests, of the minimum order of a stabilizing compensator and of the conjecture that generic stabilizability and generic pole assignability might be equivalent properties of m, n, and p. Our second ingredient, an application of classical algebraic geometry, had already been used in one form in [5] to study the problem of pole assignment by output feedback. Specifically, if $mp = n$, one knows [5] that for nondegenerate systems and any monic p(s) there are

$$d_{m,p} = \frac{(1 \ldots (p-1)! \ldots (mp)!}{m! \ldots (m+p-1)!} \qquad (5.1)$$

complex gains which place p(s), which agree with the calculations $d_{2,2} = 2$ and $d_{2,3} = 5$ made in [18] and