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On distributed cluster consensus for multiple double-integrator agents

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Abstract— **This paper investigates the consensus for multiple interacting clusters of double-integrator agents under two different frameworks, viz, the framework that all agents share the same position and velocity interaction topology and the framework that the position and velocity topologies are modeled by totally independent graphs. Different cluster consensus algorithms are designed and analyzed accordingly. A consistent structural result is shown for both frameworks that cluster consensus can be reached if the interaction topologies satisfy some connectivity assumptions and further, compared to the interactions among different clusters, the interactions within each cluster are sufficiently strong. Some lower bounds for such strengths are specified as well.**

I. INTRODUCTION

The (complete) consensus/synchronization problem, as one of the central and fundamental problems in multi-agent coordination, has attracted great attention in different disciplines of science and engineering in the last decade, see, e.g., [2], [7], [9], [10]. In the networks of agents, the aim of "consensus" is to reach an agreement regarding the state of all agents. However, a real-world complex network may be composed of multiple smaller subnetworks, e.g., communities of natural oscillators are usually composed of interacting sub-populations [12], and such a network in general exhibits more abundant scenarios than just consensus or synchronization. Also, in multi-agent systems, several different clusters of agents may merge into a bigger one due to certain cooperation task while the agents within the same clusters are still required to achieve the state agreement. Such a problem, termed group/cluster consensus/synchronization has received increasing over the past few years [13]–[16]. A basic question which is still not fully addressed is that under what conditions each cluster of agents can keep their consensus/synchronization behavior in the presence of interaction between different clusters.

Cluster synchronization with the presence of negative couplings among different clusters for coupled oscillators is considered in [13] via a pinning control techniques. The general idea used therein to realize the desired cluster pattern is that the positively weighted interactions/couplings among the nodes functioned as a synchronizing scheme while the interactions among nodes from different clusters which are negatively weighted serve as an inhibitory mechanism to desynchronize such nodes. This idea is later applied to the classical complete consensus model of single-integrator agent [4], [7] to consider the group consensus [1] of homogeneous multi-agent systems, see, e.g., [15], [16]. Some necessary and/or sufficient conditions in terms of a series of linear matrix inequalities (LMIs) for guaranteeing the group consensus are proposed in [15], [16]; note though that the feasibility of such LMIs may be difficult to check. Very recently, this scheme was revisited in [14] for single-integrator agents, where an algebraic condition of much simpler form is proposed to guarantee the cluster consensus. However, such an algebraic condition still cannot specify the conditions on the interaction topology and strengths of the interactions under which group consensus can be reached. Current works concerning group/cluster consensus problem is still far from complete.

As a separate issue, (complete) consensus of multiple agents with double-integrator dynamics has also received much attention recently, due to its ability to model a broader class of complicated dynamical agents in real applications. For example, unmanned aerial vehicles and underwater vehicles are adjusted for their desired motion by their accelerations instead of directly by their speeds. To date, a number of attempts have been made towards the design and analysis of consensus for double-integrator agents, see, e.g., [8], [10]. But there are very few results reported investigating the clustering scenarios for multiple agents with double-integrator dynamics.

This gap is essentially the motivation of this paper, which aims to consider the cluster consensus for a group of N double-integrator agents moving in $\mathbb{R}^n$ with each agent dynamics taking in the following form:

$$\dot{x}_i = v_i,\ \dot{v}_i = u_i,\ i \in \mathcal{V} = \{1, 2, \ldots, N\}, \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the position state, $v_i \in \mathbb{R}^n$ is the velocity state, and $u_i \in \mathbb{R}^n$ is the distributed control input (or called algorithm) of agent i, under two different frameworks, viz, the leaderless case that position and velocity interactions

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[1]To clarify the difference between group and cluster consensus, throughout the paper, by group consensus is meant that for any initial states of the agents, all the agents within the same cluster finally reach complete consensus, while there may or may not be consensus between different clusters, depending on the initial values of the agents [14], [16]. If for some given initial states, not only all the agents within the same cluster reach complete consensus, but also there is no consensus between any two different clusters, then cluster consensus is said to be achieved for such initial states. Obviously, cluster consensus implies group consensus if it can be achieved for any initial states.

among agents share the same topology and the leader-following case that position and velocity interaction topology are modeled by independent graphs. Two different cluster consensus algorithms are designed and analyzed accordingly. Similarly to [13], [16], we also employ the idea of negatively weighted interactions among different clusters to help design the cluster algorithms.

Differently from the convergence analysis in [15], [16], this paper presents not only the convergence analysis but also the design of appropriate weights for the interactions within each cluster to guarantee the cluster consensus and finally comes out with a structural conclusion that complete consensus for each cluster can be realized if the interaction topologies satisfy some connectivity assumptions while at the same time, compared to the interactions among different clusters, the interactions within the clusters are sufficiently strong. Some lower bounds of such strengths, in terms of the weights of the interactions, are also analyzed. Some simulations examples are also provided to illustrate the theoretical findings.

Notations: Let $\|x\|$ denote the Euclidean norm of a finite dimensional vector x. Denote by I_n the identity matrix and by $0_{n\times n}$ the zero matrix in $\in \mathbb{R}^{n\times n}$. Let $\mathbf{1}_m$ ($\mathbf{0}_m$) be the column vector with all entries equal to 1 (0). When the subscripts m and n are dropped, the dimensions of these vectors and matrices are assumed to be compatible with the context. Let $diag\{\Xi_1,\dots,\Xi_p\}$ denote the block diagonal matrix with the i-th main diagonal block being square matrix Ξ_i, $\lambda_{min}(M)$ and $\lambda_{max}(M)$ denote respectively the smallest and largest eigenvalues of a symmetric matrix M, and $\lambda_2(M)$ denotes the second smallest eigenvalue of M if M is a symmetric matrix. $\otimes$ denotes the *Kronecker product*.

II. Background and Preliminaries

For a group of N agents in $\mathbb{R}^n$, the interactions among agents can be modeled by a weighted directed graph (digraph) or undirected graph. Let $G=(\mathcal{V},\varepsilon,\mathcal{A})$ be a weighted graph consisting of a node set $\mathcal{V}=\{1,2,\dots,N\}$, a set of edges $\varepsilon\subset\mathcal{V}\times\mathcal{V}$, and a weighted *adjacency matrix* $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ which is defined as $a_{ij}\neq 0$ if $(j,i)\in\varepsilon$ and $a_{ij}=0$ otherwise. Moreover, we assume $a_{ii}=0$ for all $i\in\mathcal{V}$. The Laplacian matrix $L=[l_{ij}]$ of a weighted graph is defined by

$$l_{ij}=\begin{cases}\sum_{k=1,k\neq i}^{N} a_{ik} & j=i\\ -a_{ij} & j\neq i\end{cases}, .$$

Note that since there may exist negative element in $\mathcal{A}$, the property for the standard L associated with non-negatively weighted graph does not apply here. However, there still holds the fact that $L\mathbf{1}_N=0$.

A digraph *has a directed spanning tree* if there exists at least one node, called the root, having a directed path to all of the other nodes. An undirected graph is connected if there is an undirected path between every pair of distinct nodes.

Let $\mathcal{V}_1,\dots,\mathcal{V}_q$ denote the node sets of the q ($q>1$) interacting clusters of agents ($\cup_{\ell=1}^{q}\mathcal{V}_\ell=\mathcal{V}$). For $i\in\mathcal{V}$, let $\bar{i}$ denote the subscript of the subset to which the integer i belongs, i.e. $i\in\mathcal{V}_{\bar{i}}$. Let G_ℓ denote the underlying topology of cluster $\mathcal{V}_\ell$, $\ell=1,\dots,q$, i.e., $\mathcal{V}_\ell=\mathcal{V}(G_\ell)$. Further, without loss of generality, assume the number of agents in a cluster, say $\mathcal{V}(G_\ell)$, is n_ℓ, $1\le\ell\le q$, and the n_ℓ agents in $\mathcal{V}(G_\ell)$ are respectively indexed as $\sum_{j=0}^{\ell-1}n_j+1,\dots,\sum_{j=0}^{\ell}n_j$, where $n_0=0$ ($N=n_1+\cdots+n_q$).

When investigating the group consensus for single-integrator agents, the following algorithm [14], [16] is considered

$$u_i(t)=\sum_{j\in\mathcal{V}_{\bar{i}}}a_{ij}\left(x_j(t)-x_i(t)\right)+\sum_{j\in\mathcal{V}-\mathcal{V}_{\bar{i}}}a_{ij}x_j(t). \quad (2)$$

Apparently, to guarantee the group consensus, the following group consensus manifold

$$\mathcal{S}(n)=\left\{[x_1^{\mathrm{T}},\dots,x_N^{\mathrm{T}}]^{\mathrm{T}}:x_i=x_j,\forall\bar{i}=\bar{j}\right\}$$

should be invariant through Eq. (2), which leads to the following condition.

Assumption 1:

$$\sum_{j\in\mathcal{V}(G_\ell)}a_{ij}=0,\ \ \forall i=1,\dots,N,\ i\in\mathcal{V}\backslash\mathcal{V}(G_\ell),\ \ell=1,\dots,q.$$

Algorithm (2) will be extended in the paper to deal with the double-integrator case and the above condition will be imposed throughout the paper to help design and analyze the group/cluster consensus algorithm as well. Note that in terms of the Laplacian matrix, the in-degree balanced condition is equivalent to the condition that each row sum of L_{ij}, $i\neq j$, in matrix $L=\begin{bmatrix}L_{11}&\cdots&L_{1q}\\ \vdots&\ddots&\vdots\\ L_{q1}&\cdots&L_{qq}\end{bmatrix}$ is zero. From the definition of the Laplacian matrix, it follows that each L_{ii} is the Laplacian matrix of G_i, the underlying graph of cluster $\mathcal{V}_i$, and moreover, it is symmetric positive semi-definite if G_i is undirected.

For future reference, we record the following result.

Lemma 1: (cf. [9]) Assume L is a Laplacian matrix of a non-negatively weighted connected undirected graph of order N. Let $x=[x_1^{\mathrm{T}},x_2^{\mathrm{T}},\dots,x_N^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{nN}$, where $x_i\in\mathbb{R}^n$, $i=1,\dots,N$, be any column vector satisfying $\sum_{i=1}^{N}x_i=0_{n\times 1}$. Then for any symmetric positive-semidefinite matrix $B\in\mathbb{R}^{n\times n}$, one has

$$x^{\mathrm{T}}(L\otimes B)x\ge\lambda_2(L)x^{\mathrm{T}}(I_N\otimes B)x,$$

where $\lambda_2(L)=\min_{x\neq 0,\mathbf{1}^{\mathrm{T}}x=0}\frac{x^{\mathrm{T}}Lx}{x^{\mathrm{T}}x}$.

III. Same Position and Velocity Graph Topologies with No Leaders

In this section, by cluster consensus we mean that a group of agents will finally evolve into different clusters, in which the agents within the same cluster reach both the position and velocity consensus while there is no consensus of position trajectories among different clusters.

Definition 1: For any given initial states $x(0)=[x_1^{\mathrm{T}}(0),\dots,x_N^{\mathrm{T}}(0)]^{\mathrm{T}}$ and $v(0)=[v_1^{\mathrm{T}}(0),\dots,v_N^{\mathrm{T}}(0)]^{\mathrm{T}}$, *cluster consensus* for the interacting q clusters of agents is said to be achieved by employing algorithm u_i for each agent i if the states of the agents satisfy $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=$

0 and $\lim_{t\to\infty}\|v_i(t)-v_j(t)\|=0$ when $\bar{i}=\bar{j}$, while $\limsup_{t\to\infty}\|x_i(t)-x_j(t)\|>0$ when $\bar{i}\neq\bar{j}$. The velocities of agents from different clusters may or may not be consistent, depending on the initial velocities of the agents.

group consensus for the interacting q clusters of agents is said to be achieved by employing algorithm u_i for each agent i if, for any initial states of the agents, there hold $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0$ and $\lim_{t\to\infty}\|v_i(t)-v_j(t)\|=0$, $\forall \bar{i}=\bar{j}$.

For each agent i, the following cluster consensus algorithm will be considered:

$$\begin{aligned} u_i(t)=&\sum_{j\in\mathcal{V}_{\bar{i}}} a_{ij}\left[(x_j(t)-x_i(t))+\gamma(v_j(t)-v_i(t))\right]\\ &+\sum_{j\in\mathcal{V}-\mathcal{V}_{\bar{i}}} a_{ij}\left(x_j(t)+\gamma v_j(t)\right)\end{aligned}$$

where $\gamma>0$ is the velocity coupling gain; $a_{ij}\geq 0$ if $\bar{i}=\bar{j}$ and $a_{ij}\in\mathbb{R}$ otherwise. Under Assumption 1, the consensus algorithm can be rewritten as

$$u_i(t)=\sum_{j=1}^{N} a_{ij}\left[(x_j(t)-x_i(t))+\gamma(v_j(t)-v_i(t))\right]. \quad (3)$$

We are interesting in studying when for such a law group/cluster consensus is achieved.

Theorem 1: Assume the interaction topology G is undirected and G_i, the underlying topology of cluster $\mathcal{V}_i$, $i=1,\ldots,q$, is connected and the interactions between different clusters satisfy Assumption 1. If

$$\Phi=\begin{bmatrix}\lambda_2(L_{11})I_{n_1} & \cdots & L_{1q}\\ \vdots & \ddots & \vdots\\ L_{q1} & \cdots & \lambda_2(L_{qq})I_{n_q}\end{bmatrix}>0,$$

then group consensus for all the agents can be reached by employing algorithm (3) for any $\gamma>0$.

If $\Phi>0$ does not hold, scaling the weights of edges among the agents within the same cluster, say G_i, with a scalar parameter c_i, $i=1,\ldots,q$, such that

$$c_i>\frac{-\lambda_{min}\left(\begin{bmatrix}0 & L_{12} & \cdots & L_{1q}\\ L_{21} & 0 & \cdots & L_{2q}\\ \vdots & \vdots & \ddots & \vdots\\ L_{q1} & L_{q2} & \cdots & 0\end{bmatrix}\right)}{\lambda_2(L_{ii})},\quad i=1,\ldots,q, \quad (4)$$

then group consensus for all the agents can be reached by employing algorithm (3) for any $\gamma>0$.

Furthermore, for both cases, $v_i(t)\to\frac{1}{n_{\bar{i}}}\sum_{k\in\mathcal{V}(G_{\bar{i}})}v_k(0)$ and $x_i(t)\to\left(\frac{1}{n_{\bar{i}}}\sum_{k\in\mathcal{V}(G_{\bar{i}})}v_k(0)\right)t+\frac{1}{n_{\bar{i}}}\sum_{k\in\mathcal{V}(G_{\bar{i}})}x_k(0)$, $i=1,\ldots,N$, as $t\to\infty$. Finally, cluster consensus for all the agents can be reached if the initial states are chosen such that either $\frac{1}{n_\ell}\sum_{k\in\mathcal{V}(G_\ell)}x_k(0)$ with $\ell=1,\ldots,q$, are not equal to each other or $\frac{1}{n_\ell}\sum_{k\in\mathcal{V}(G_\ell)}v_k(0)$ with $\ell=1,\ldots,q$, are not equal to each other.

Proof: Let $\bar{x}_i(t)=x_i(t)-\frac{1}{n_{\bar{i}}}\sum_{k\in\mathcal{V}(G_{\bar{i}})}x_k(t)$, $\bar{v}_i(t)=v_i(t)-\frac{1}{n_{\bar{i}}}\sum_{k\in\mathcal{V}(G_{\bar{i}})}v_k(t)$, $i=1,\ldots,N$, and $\bar{\mathbf{x}}(t)=[\bar{x}_1^{\mathrm{T}}(t),\ldots,\bar{x}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\bar{\mathbf{v}}(t)=[\bar{v}_1^{\mathrm{T}}(t),\ldots,\bar{v}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$.

It is not difficult to observe from Assumption 1 together with the condition that G is undirected that both the row sums and column sums of each L_{ij}, $i,j=1,\ldots,q$, are zeros, which yields straightforwardly that

$$\begin{aligned}&\dot{\bar{v}}_i(t)\\ =&\dot{v}_i(t)-\frac{1}{n_{\bar{i}}}\sum_{k\in\mathcal{V}(G_{\bar{i}})}\left\{\sum_{j=1}^{N}a_{kj}\left[(x_j(t)-x_k(t))\right.\right.\\ &\left.\left.+\gamma(v_j(t)-v_k(t))\right]\right\}=\dot{v}_i(t).\end{aligned}$$

On the other hand, it follows from the definition of Laplacian matrix and Assumption 1 that

$$\begin{aligned}&\sum_{j=1}^{N}a_{ij}(x_j(t)-x_i(t))\\ =&-\sum_{\ell=1}^{q}\sum_{j\in\mathcal{V}(G_\ell)}l_{ij}x_j(t)\\ =&-\sum_{\ell=1}^{q}\sum_{j\in\mathcal{V}(G_\ell)}l_{ij}\left[\left(x_j(t)-\frac{1}{n_\ell}\sum_{k\in\mathcal{V}(G_\ell)}x_k(t)\right)\right.\\ &\left.+\frac{1}{n_\ell}\sum_{k\in\mathcal{V}(G_\ell)}x_k(t)\right]\\ =&-\sum_{j=1}^{N}l_{ij}\bar{x}_j(t)-\sum_{\ell=1}^{p}\left[\left(\sum_{j\in\mathcal{V}(G_k)}l_{ij}\right)\times\right.\\ &\left.\left(\frac{1}{n_\ell}\sum_{k\in\mathcal{V}(G_\ell)}x_k(t)\right)\right]\\ =&\sum_{j=1}^{N}a_{ij}(\bar{x}_j(t)-\bar{x}_i(t)).\end{aligned}$$

This in turn yields the following compact system dynamics

$$\begin{cases}\dot{\bar{\mathbf{x}}}(t)=\bar{\mathbf{v}}(t)\\ \dot{\bar{\mathbf{v}}}(t)=-\left(L\otimes I_n\right)\bar{\mathbf{x}}(t)-\gamma\left(L\otimes I_n\right)\bar{\mathbf{v}}(t)\end{cases}. \quad (5)$$

Notice that for $\bar{\mathbf{x}}(t)$, since $\sum_{i\in\mathcal{V}(G_{\bar{i}})}\bar{x}_i(t)\equiv 0$, it follows from Lemma 1 that

$$\begin{aligned}&\bar{\mathbf{x}}^{\mathrm{T}}(t)(L\otimes I_n)\bar{\mathbf{x}}(t)\\ =&\bar{\mathbf{x}}^{\mathrm{T}}(t)\left(diag\{L_{11},\ldots,L_{qq}\}\otimes I_n\right)\bar{\mathbf{x}}(t)\\ &+\bar{\mathbf{x}}^{\mathrm{T}}(t)\left[(L-diag\{L_{11},\ldots,L_{qq}\})\otimes I_n\right]\bar{\mathbf{x}}(t)\\ \geq&\bar{\mathbf{x}}^{\mathrm{T}}(t)\left(diag\{\lambda_2(L_{11})I_{n_1},\ldots,\lambda_2(L_{qq})I_{n_q}\}\otimes I_n\right)\bar{\mathbf{x}}(t)\\ &+\bar{\mathbf{x}}^{\mathrm{T}}(t)\left[(L-diag\{L_{11},\ldots,L_{qq}\})\otimes I_n\right]\bar{\mathbf{x}}(t)\\ =&\bar{\mathbf{x}}^{\mathrm{T}}(t)\left(\Phi\otimes I_n\right)\bar{\mathbf{x}}(t)>0,\ \forall\bar{\mathbf{x}}(t)\neq 0.\end{aligned}$$

Similarly, for $\bar{\mathbf{v}}(t)$ we have

$$\bar{\mathbf{v}}^{\mathrm{T}}(t)(L\otimes I_n)\bar{\mathbf{v}}(t)\geq\bar{\mathbf{v}}^{\mathrm{T}}(t)\left(\Phi\otimes I_n\right)\bar{\mathbf{v}}(t)$$

since $\sum_{i\in\mathcal{V}(G_{\bar{i}})}\bar{v}_i(t)\equiv 0$ as well.

Since $\Phi > 0$, consider the following Lyapunov function candidate for system (5):

$$V(t) = \frac{1}{2}\bar{\mathbf{x}}^{\mathrm{T}}(L \otimes I_n)\bar{\mathbf{x}}(t) + \frac{1}{2}\bar{\mathbf{v}}^{\mathrm{T}}(t)\bar{\mathbf{v}}(t)$$
$$\geq \frac{1}{2}\bar{\mathbf{x}}^{\mathrm{T}}(\Phi \otimes I_n)\bar{\mathbf{x}}(t) + \frac{1}{2}\bar{\mathbf{v}}^{\mathrm{T}}(t)\bar{\mathbf{v}}(t)$$

Differentiating $V(t)$ along (5) gives

$$\dot{V}(t) = -\gamma\bar{\mathbf{v}}^{\mathrm{T}}(t)\left(L \otimes I_n\right)\bar{\mathbf{v}}(t)$$
$$\leq -\gamma\bar{\mathbf{v}}^{\mathrm{T}}(t)\left(\Phi \otimes I_n\right)\bar{\mathbf{v}}(t) \leq 0. \qquad (6)$$

Let $S = \left\{(\bar{\mathbf{x}}(t), \bar{\mathbf{v}}(t)) : \dot{V}(t) = 0\right\}$. From inequality (6), we know that $\dot{V}(t) \equiv 0$ implies that $\bar{\mathbf{v}}(t) \equiv 0$, and thus $\dot{\bar{\mathbf{v}}}(t) \equiv 0$. It then follows (5) that $\left(L \otimes I_n\right)\bar{\mathbf{x}}(t) \equiv 0$, and thus $0 \leq \bar{\mathbf{x}}^{\mathrm{T}}(t)\left(\Phi \otimes I_n\right)\bar{\mathbf{x}}(t) \leq \bar{\mathbf{x}}^{\mathrm{T}}(t)\left(L \otimes I_n\right)\bar{\mathbf{x}}(t) \equiv 0$, which in turn implies that $\bar{\mathbf{x}}(t) \equiv 0$. As a result, invoking the Lasalle's Invariance Principle [6] yields that $\bar{x}_i(t) \to 0$ and $\bar{v}_i(t) \to 0$, $\forall i = 1, \ldots, N$ as $t \to \infty$. This completes the proof for the first statement.

As to the second statement, after scaling the weights of the edges within each cluster as that described in the first statement of the theorem, L and Φ corresponds respectively to the following forms

$$L^{'} = \begin{bmatrix} c_1 L_{11} & \cdots & L_{1q} \\ \vdots & \ddots & \vdots \\ L_{q1} & \cdots & c_q L_{qq} \end{bmatrix}$$

and

$$\Phi^{'} = \begin{bmatrix} c_1\lambda_2(L_{11})I_{n_1} & \cdots & L_{1q} \\ \vdots & \ddots & \vdots \\ L_{q1} & \cdots & c_q\lambda_2(L_{qq})I_{n_q} \end{bmatrix}.$$

To guarantee that $\Phi^{'} > 0$, it suffices to choose c_i such that

$$\lambda_{min}\left(diag\{c_1\lambda_2(L_{11})I_{n_1}, \ldots, c_q\lambda_2(L_{qq})I_{n_q}\}\right)$$
$$+\lambda_{min}\left(\begin{bmatrix} 0 & L_{12} & \cdots & L_{1q} \\ L_{21} & 0 & \cdots & L_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ L_{q1} & L_{q2} & \cdots & 0 \end{bmatrix}\right) > 0,$$

which can be guaranteed if c_i satisfies the condition in (4).

The consensus value can be computed by noting that $\frac{1}{n_{\bar{i}}}\sum_{k\in\mathcal{V}(G_{\bar{i}})}\dot{v}_k(t) = 0$ and further, $\frac{1}{n_{\bar{i}}}\sum_{k\in\mathcal{V}(G_{\bar{i}})}\dot{x}_k(t) = \frac{1}{n_{\bar{i}}}\sum_{k\in\mathcal{V}(G_{\bar{i}})}v_k(0)$. ∎

Remark 1: Note that in the above proof, the condition that $\sum_{i\in\mathcal{V}(G_{\bar{i}})}\bar{x}_i(t) \equiv 0$ is crucial in constructing the proposed Lyapunov function $V(t)$ since, in general, $V(t)$ is only positive semi-definite.

IV. Independent Position and Velocity Topologies with Multiple Leaders of Constant Velocity

In this section, we will consider the scenario that a group of N agents (termed follower agents in what follows) will finally evolve into different clusters, (see Definition 2 below for the details), where there is no consensus of position trajectories among different clusters, but all the agents asymptotically move with the same velocity due to the existence of leader agents with the same constant velocity v_r for every cluster. Moreover, considering in real applications, some interactions among agents may be used while the others may be neglected in order to optimize the coordination behavior or reduce the computational complexity, it is assumed that the graphs modeling respectively the position and velocity interactions among agents are totally independent (see, e.g., [11] for a flocking algorithm example where different position and velocity graph neighbors are required to achieve collision avoidance and cohesion as well as the velocity alignment).

Consider the q leader agents evolving according to the following dynamics

$$\dot{s}_\ell(t) = v_r,\ s_\ell(0) = s_d^\ell,\ \ell = 1, \ldots, q,$$

where s_d^ℓ is the initial position state of leader ℓ.

The following is the definition of cluster and group consensus for all the follower agents in the presence of leader agents.

Definition 2: For any given initial states $x(0) = [x_1^{\mathrm{T}}(0), \ldots, x_N^{\mathrm{T}}(0)]^{\mathrm{T}}$ and $v(0) = [v_1^{\mathrm{T}}(0), \ldots, v_N^{\mathrm{T}}(0)]^{\mathrm{T}}$, *cluster consensus* for the interacting q clusters of follower agents is said to be achieved by employing algorithm u_i for each agent i if the position states of the follower agents satisfy $\lim_{t\to\infty}\|x_i(t) - s_{\bar{i}}(t)\| = 0$ and $\limsup_{t\to\infty}\|x_i(t) - x_j(t)\| > 0$ if $\bar{i} \neq \bar{j}$; and the velocity states of the follower agents satisfy $\lim_{t\to\infty}\|v_i(t) - v_r(t)\| = 0$, $\forall i, j \in \mathcal{V}$.

group consensus for the interacting q clusters of follower agents is said to be achieved by employing algorithm u_i for each agent i if, for any initial states of the follower agents, the position states of the follower agents satisfy $\lim_{t\to\infty}\|x_i(t) - s_{\bar{i}}(t)\| = 0$ and the velocity states of the follower agents satisfy $\lim_{t\to\infty}\|v_i(t) - v_r(t)\| = 0$, $\forall i, j \in \mathcal{V}$.

To realize the desired cluster pattern, it is assumed that the leaders are moving from different initial positions which will ensure that $\limsup_{t\to\infty}\|s_i(t) - s_i(t)\| > 0$, $\forall i \neq j$.

In what follows, $G^p = (\mathcal{V}, \varepsilon^p, \mathcal{A}^p = [a_{ij}])$ and $G^v = (\mathcal{V}, \varepsilon^v, \mathcal{A}^v = [b_{ij}])$ are employed to represent respectively the position and velocity interaction topologies of all the N follower agents, where G^v is non-negatively weighted while G^p may have negatively weighted edges between agents from different clusters, i.e., $b_{ij} \geq 0$; $a_{ij} \geq 0$ if $\bar{i} = \bar{j}$ while $a_{ij} \in \mathbb{R}$ if $\bar{i} \neq \bar{j}$. The Laplacian matrices of G^p and G^v are denoted by L^p and L^v respectively. Further, $\mathcal{N}_i^p = \{j \in \mathcal{V} : a_{ij} \neq 0\}$ and $\mathcal{N}_i^v = \{j \in \mathcal{V} : b_{ij} > 0\}$ denote the position neighbor set and velocity neighbor set of agent i in G^p and G^v, respectively. G^p is undirected and G^v is directed in this section. Denote by G_i^p, $i = 1, \ldots, p$, the underlying position topology of cluster $\mathcal{V}_i$.

The nonnegative numbers d_i, $i = 1, \ldots, N$, are used to specify the accessibility of leader agents' velocity to the follower agents in $\mathcal{V}$, i.e., $d_i > 0$ if the follower agent i can receive the velocity state information of leader agent $\bar{i}$ and otherwise $d_i = 0$. Let $\bar{G}^v$ denote the digraph consisting of G^v, the q leader agents and the directed edges from these leader agents to the follower agents in $\mathcal{V}$ which have access to their velocity information v_r. Suppose that the initially given $\bar{G}^v$ *has a united directed spanning tree* (i.e., for each of the N followers, there exists at least one leader agent that has a directed path in $\bar{G}^v$) to the follower agent.

Differently from $\bar{G}^v$, let $\tilde{G}^v$ denote the digraph consisting of G^v, the sole velocity leader v_r and the directed edges from this leader to the follower agents in G^v which have access to its velocity information. Since all the leader agents are with the same velocity v_r, then saying that $\bar{G}^v$ has a united directed spanning tree is equivalent to saying that $\tilde{G}^v$ has a directed spanning tree. This can be observed by contracting all the leader agents in $\bar{G}^v$ into one node while keeping all the edges in $\bar{G}^v$ unchanged, which will come out with graph $\tilde{G}^v$. See, for example, Figure 4 in the simulation section.

If $\tilde{G}^v$ has a directed spanning tree and the velocity leader v_r is indexed as agent 0, it is not difficult to derive that the Laplacian matrix of $\tilde{G}^v$ is $\begin{bmatrix} 0 & 0 \\ -\mathbf{1}D^v & L^v + D^v \end{bmatrix}$, where $D^v = diag\{d_1, \ldots, d_N\}$. Then, from [9] we know there must exist a diagonal positive-definite matrix Ξ such that

$$E_v = \Xi(L^v + D^v) + (L^v + D^v)^{\mathrm{T}}\Xi > 0. \qquad (7)$$

Let $\Xi = diag\{\eta_1, \ldots, \eta_N\}$, where $\eta_i > 0$, $i = 1, \ldots, N$.

Remark 2: Details about how to find such a matrix Ξ can be found in the proof for Theorem 1 in [9] .

Now the distributed cluster consensus algorithm is designed as

$$\begin{aligned} u_i(t) =& \eta_i^{-1}\left[\sum_{j \in N_i^p} a_{ij}(x_j(t) - x_i(t)) + \epsilon_i(s_{\bar{i}}(t) - x_i(t))\right] \\ &+ \sum_{j \in N_i^v} b_{ij}(v_j(t) - v_i(t)) + d_i(v_r - v_i(t)), \qquad (8) \end{aligned}$$

where $\epsilon_i > 0$ if agent i can receive the position state information of leader $\bar{i}$ and otherwise $\epsilon_i = 0$; $\eta_i > 0$ is as that designed above.

Let $D^p = diag\{\epsilon_1, \ldots, \epsilon_N\}$. Further, denote by $\bar{G}^p_\ell$, $\ell = 1, \ldots, q$, the graph consisting of G^p_ℓ, the leader agent ℓ, and the edges from this leader agent to those follower agents in cluster $\mathcal{V}_\ell$ which have access to its position information $s_\ell(t)$ (see the graph topology in Figure 3 for example). Now we have

Theorem 2: Assume the the position interaction topology G^p is undirected and $\bar{G}^p_i$, $i = 1, \ldots, q$, is weakly connected and the position state interactions between different clusters satisfy Assumption 1; assume that $\bar{G}^v$ has a united directed spanning tree. Then there holds:

(a) If $L^p + D^p > 0$, cluster consensus can be reached for any initial states of the follower agents;

(b) If $L^p + D^p = \begin{bmatrix} \bar{L}_{11} & \cdots & L_{1q} \\ \vdots & \ddots & \vdots \\ L_{q1} & \cdots & \bar{L}_{qq} \end{bmatrix} > 0$ does not hold, scaling the weights of position state interactions among the follower agents within same cluster as well as their leader agent, say $\mathcal{V}\left(\bar{G}^p_\ell\right) = \mathcal{V}_\ell \cup \{\text{leader agent } \ell\}$ (i.e., scale the weights a_{ij} and c_i for any $i, j \in \mathcal{V}$ satisfying $\bar{i} = \bar{j} = \ell$), with a scalar parameter c_ℓ such that

$$c_\ell > \frac{-\lambda_{min}\left(\begin{bmatrix} 0 & L_{12} & \cdots & L_{1q} \\ L_{21} & 0 & \cdots & L_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ L_{q1} & L_{q2} & \cdots & 0 \end{bmatrix}\right)}{\lambda_{min}(\bar{L}_{ii})}, \quad \ell = 1, \ldots, q,$$

then cluster consensus can be reached for any initial states of the follower agents.

Remark 3: Obviously, Theorem 2 shows also that group consensus can be reached for all the N follower agents.

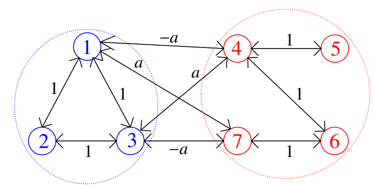

Fig. 1. Same position and velocity interaction topologies

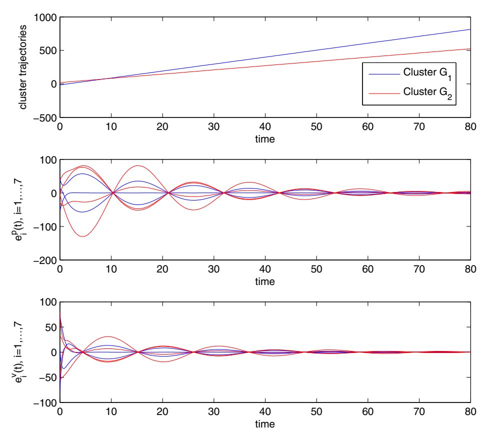

Fig. 2. Agents reach cluster consensus under algorithm (3) for $a = 0.8$: $G^p = G^v = G$

V. Simulation Example

Example 1: To illustrate the result in Theorem 1, consider a group of 7 agents with the underlying weighted interaction topology as shown in Figure 1, where a and $-a$ are the weights of the edges between agents from different clusters.

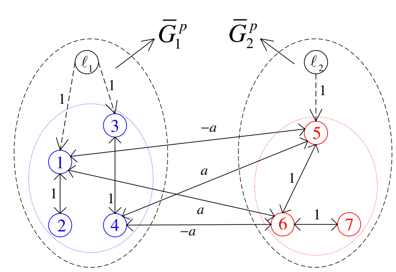

Fig. 3. Different position and velocity interactions: position interactions

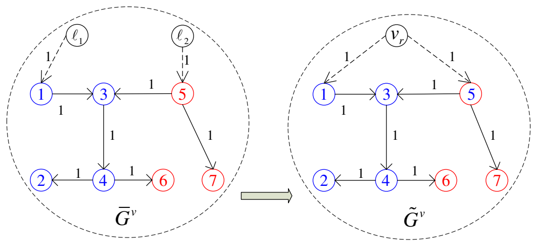

Fig. 4. Different position and velocity interactions: velocity interactions

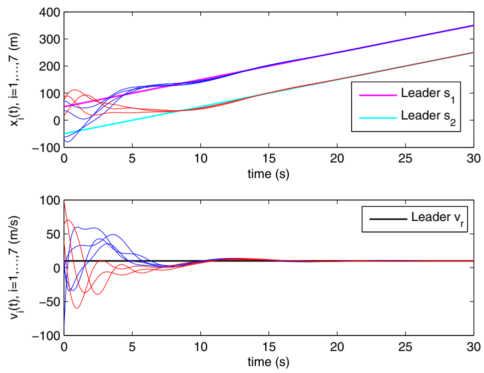

Fig. 5. Agents reach cluster consensus under algorithm (8) for $a = 0.4$: $G^p \neq G^v$

Let $e_i^v(t) = v_i(t) - \frac{1}{3}\sum_{k=1}^{3} v_k(0)$, when $i = 1, 2, 3$, and $e_i^v(t) = v_i(t) - \frac{1}{4}\sum_{k=1}^{4} v_k(0)$, when $i = 4, \ldots, 7$; $e_i^p(t) = x_i(t) - \left(\frac{1}{3}\sum_{k=1}^{3} v_k(0)\right)t - \frac{1}{3}\sum_{k=1}^{3} x_k(0)$, when $i = 1, 2, 3$, and $e_i^p(t) = x_i(t) - \left(\frac{1}{4}\sum_{k=4}^{7} v_k(0)\right)t - \frac{1}{4}\sum_{k=4}^{7} x_k(0)$, when $i = 4, \ldots, 7$.

Subplot 1 in Figure 2 shows the trajectories of $\left(\frac{1}{3}\sum_{k=1}^{3} v_k(0)\right)t - \frac{1}{3}\sum_{k=1}^{3} x_k(0)$ and $\left(\frac{1}{4}\sum_{k=4}^{7} v_k(0)\right)t - \frac{1}{4}\sum_{k=4}^{7} x_k(0)$ for clusters G_1 and G_2 respectively. Figure 2 shows that cluster consensus can be reached by using algorithm (3) if we choose $a = 0.8$; and the agents' position and velocity trajectories will be divergent if we choose $a = 1$.

Example 2: To illustrate the result in Theorem 2, consider also a group of 7 agents with underlying position and velocity interaction topologies as shown in Figure 3 and Figure 4 respectively. Figure 3 also shows how the agents have access to the two leaders, from which it is obvious that both $\bar{G}_1^p$ and $\bar{G}_2^p$ are weakly connected and $\bar{G}^v$ has a united directed spanning tree. For illustration, choose $v_r = 10$ (m/s) and the initial position values of leader 1 and leader are respectively $s_1(0) = 50$ (m) and $s_2(0) = -50$ (m). Initial values of all the 7 agents are randomly chosen from the interval $[-100, 100] \subset \mathbb{R}$.

From Figure 5, which plots the trajectories of the 7 agents under algorithm (8) with $a = 0.4$, we know that each cluster of agents follow their leader; while one can easily check that trajectories of all the agents are divergent for a larger a, e.g., $a = 0.6$.

VI. Conclusion

In this paper, we have designed and analyzed respectively two different cluster consensus algorithms under two different frameworks: in one, the underlying position and velocity interaction topology are the same and in the other, the position and velocity topologies are modeled by totally independent graphs. For both frameworks, through rigorous analysis we have obtained a consistent structural result that cluster consensus can be reached if the interaction topologies satisfy some connectivity assumptions and further, compared to the interactions among different clusters, the interactions within each cluster are strong enough.

For future work, one direction of interest would be to consider the case that all the leader agents are moving with time-varying velocities which may be different and still there is only a subset of the follower agents which have access to the leaders' velocity information.

References

[1] B. D. O. Anderson, Z. Lin, and M. Deghat, "Combining distance-based formation shape Control with formation translation," in : Developments in Control Theory Towards Glocal Control, Chapter 13, L. Qiu, J. Chen, T. Iwasaki, and H. Fujioka (Eds.), IET, 2012.

[2] M. Cao, A. S. Morse, and B. D. O. Anderson, "Reaching a consensus in a dynamically changing environment: Convergence rates, measurement delays, and asynchronous events," *SIAM J. Control Optim.*, vol. 47, no. 2, pp. 601–623, 2008.

[3] C. Godsil and G. Doyle, *Algebraic Graph Theory*. New York: Springer, 2001.

[4] A. Jadbabaie, J. Lin, and S. A. Morse, "Coordination of groups of mobile agents using nearest neighbor rules," *IEEE Trans. Automat. Contr.*, vol. 48, no. 6, pp. 988–1001, 2003.

[5] K. Kaneko, "Relevance of dynamic clustering to biological networks," *Phys. D*, vol. 75, nos. 1–3, pp. 55-73, 1994.

[6] H. K. Khalil. *Nonlinear Systems*. New Jersey: Second Edition, Prentice-Hall, 1996.

[7] R. Olfati-saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Automat. Contr.*, vol. 49, no. 9, pp. 1520–1533, 2004.

[8] J. Qin and H. Gao, "A sufficient condition for convergence of sampled-data consensus for double-integrator dynamics with nonuniform and time-varying communication delays," *IEEE Trans. Automat. Contr.*, vol. 57, no. 9, pp. 2417–2422, 2012.

[9] J. Qin, C. Yu, H. Gao, and X. Wang, "Leaderless consensus control of dynamical agents under directed interaction topology," in *Proc. IEEE Conf. Decision Control and European Control Conf.*, Orlando, FL, 2011, pp. 1455–1460.

[10] W. Ren, "On consensus algorithms for double-integrator dynamics," *IEEE Trans. Automat. Contr.*, vol. 53, no. 6, pp. 1503–1509, 2008.

[11] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in Fixed and Switching Networks," *IEEE Trans. Automat. Contr.*, vol. 52, no. 5, pp. 863–868, 2007.

[12] A. T. Winfree. *The Geometry of Biological Time*. New York: Springer-Verlag, 1980.

[13] W. Wu, W. Zhou, and T. Chen, "Cluster synchronization of linearly coupled complex networks under pinning control," *IEEE Trans. Circuits Syst. I*, vol. 56, no. 4, pp. 829-839, 2009.

[14] W. Xia and M. Cao, "Clustering in diffusively coupled networks," *Automatica*, vol. 47, no. 11, pp. 2395–2405, 2011.

[15] J. Yu and L. Wang, "Group consensus of multi-agent systems with directed information exchange," *International Journal of Systems Science*, vol. 43, no. 2, pp. 334–348, 2012.

[16] J. Yu and L. Wang, "Group consensus in multi-agent systems with swithcing topologies and communication delays," *Systems Control Lett.*, vol. 59, no. 6, pp. 340–348, 2011.