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Threshold Phenomenon For Average Consensus

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Abstract—**In this paper, our main concern is to study the influence of the number of edges on the convergence rate and the total communication cost in distributed average consensus problems. We begin with the case of regular networks, i.e. networks for which the associated graph (vertices corresponding to sensors and edges being defined by communicating sensor pairs) has the same vertex degree for all vertices. In regular graphs, the number of edges is effectively determined by the common vertex degree. Therefore the problem is converted to one of analyzing the influence of the common vertex degree on the convergence rate and total communication cost. To evaluate the convergence rate we use the ratio of two eigenvalues of the Laplacian matrix of the graph, for which we obtain lower and upper bounds in terms of the common vertex degree. Using the bounds, we can illustrate the intuitively reasonable property that the convergence rate will increase with increase in the common vertex degree. However the increment in the convergence rate drops dramatically as the common vertex degree becomes progressively large. At the same time the total communication cost of the consensus process will become large. Based on these two observations we define a type of 'Magic Number' to help analyze the value or otherwise of adding more links to the network. The Monte Carlo simulation results are consistent with the theoretical analysis and demonstrate the existence of the magic number. Further we present the simulation results on irregular graphs in which the degree distribution is subject to a Poisson distribution. From the simulation results we observe that the magic number also exists in the irregular graphs. In general the magic number for irregular graphs is larger than the one for regular graphs.**

I. Introduction

The consensus problem has been extensively investigated in networked multi-agent systems as a distributed way to control coordinated motion [1], [2], [3]. Recently, consensus studies were extended to the sensor networks area. For example, in the sensor fusion problem, some distributed consensus algorithms have been proposed to achieve globally optimal decisions, detection or estimation [4], [5], [6].

In almost all consensus applications, the convergence rate has been considered as an important performance issue. The convergence rate is mainly influenced by the given consensus protocols and the topology of the network. Recently some mechanisms have been proposed to enhance the convergence rate achieving a good performance without changing the topology of the networks [7], [8]. In our work we do not focus on this type of promotion.

Another type of study has focussed on the influence of the network topology on the convergence rate which is also our concern in this paper. In [9], Jin et al. investigate the convergence behavior of discrete-time consensus protocols over large-scale sensor networks with uniformly random deployment. A random consensus protocol is proposed to locate certain chosen nodes and add shortcuts among them to decrease the number of iterations to achieve average consensus. In [10], Kar et al. investigate the influence of the network topology on the convergence rate of the average consensus problem. They show that in distributed inference and average consensus, the topology given by the nonbipartite Ramanujan graphs can optimize the convergence. However, from the practical application point of view, a sensor network may not be deployed as a Ramanujan graph and it is not clear to what extent the Ramanujan graph result is even an approximate pointer for an arbitrary sensor network. In [11], S. Boyd et al demonstrate how to exploit symmetries of a graph to efficiently compute the fastest mixing Markov chain on the graph. They describe two general approaches for symmetry exploitation to speed up the convergence rate, based on orbit theory and blockdiagonalization, respectively.

In our research work, we concentrate on the influence of the number of edges on the consensus convergence speed. Intuitively, for a connected sensor network, when the number of sensors is fixed but the edge count is adjustable, more edges will achieve a faster convergence rate. However when more edges are added to a network, the total communication consumption (TCC) may also increase[1], and in practical terms, will limit the benefit of edge addition. In studying this phenomenon and the associated trade-off, we will assume that: 1. The average consensus algorithm is used in a sensor network 2. The weight for each link/edge is equal 3. The communication cost incurred per iteration for each link is equal.

While undoubtedly, more general models of communication cost can be considered, the results we obtain under these special assumptions appear to be indicative of those obtained in more general situations. These assumptions mean that the TCC of the sensor network is proportionally related to the number of edges and the number of iterations required to secure practical consensus. The number of iterations and the convergence rate are inversely related. Therefore the TCC is proportional to the number of edges and inversely related to the convergence rate. However, the convergence rate itself depends on the number of edges, which means that ultimately the TCC

[1]Broadcast communication model is normally used in wireless sensor networks. However, due to some security issues, peer to peer model may also be applied. In peer to peer model, only if there is a link between two sensors they will communicate with each other. In this paper, we focus on peer to peer communication model. Therefore the total communication cost is related to the number of edges in the networks.

depends just on the number of edges. If the number of edges and the convergence rate were linearly related, by increasing the number of edges one could improve the convergence rate while maintaining the TCC. Otherwise the TCC will change by increasing the number of links. In many sensor networks, because of the limited energy of each sensor even a small increment of the communication cost may be burdensome. Therefore it is critical to know how the TCC behaves as the edge count changes and for this we need to investigate the relationship between the number of edges and the convergence rate.

In this paper, we begin with the case of *regular* graphs with average consensus algorithms. Therefore the problem is converted to one of analyzing the influence of the common vertex degree on the convergence rate. To evaluate the convergence rate we use the ratio of the second smallest eigenvalue and the largest eigenvalue of the Laplacian matrix of the networks. Lower and upper bounds for the convergence rate for regular graphs are given in the paper in terms of simply the common vertex. From the two bounds we can conclude that the relationship between the degrees and the convergence rate is logarithmic. We also define a type of *Magic Number* to analyze the convergence rate and the TCC of the network. In Monte Carlo simulations, we generate the regular graphs randomly and the simulation results are consistent with the theoretical analysis and demonstrate that the magic number exists in the regular graphs. Further we present simulations on irregular graphs in which the degree distribution is subject to a Poisson distribution. From the simulation results, we observe that the magic number also exists in irregular graphs. Further the magic number for irregular graphs is, in general, larger than the one for regular graphs.

The rest of the paper is organized as follows. In Section II background and the problem statement are summarized. The lower and upper bounds for the convergence rate and the definition of the magic number are presented in Section III. The results of Monte Carlo simulations are provided in Section IV. Section V summarizes the ideas of the paper and comments on future work.

II. BACKGROUND AND PROBLEM STATEMENT

A. Graph Theory Notations

In this paper, the topology of sensor networks is modeled by undirected graphs. Let $G = (V, E)$ be an undirected graph with the set of nodes $V = \{v_1, v_2, ..., v_N\}$ (N denoting the number of nodes), and edges the unordered pairs $e = (v_i, v_j) \in E$ ($E \subseteq V \times V$) (simply $e = (i, j)$). Here we also assume the graphs we consider in this paper are *connected* and *simple* (no self-loops or multiple edges). For every graph, there exists an $N \times N$ adjacency matrix A defined as: $a_{i,j} = 1$ if there is an edge between node i and node j. Otherwise $a_{i,j}$ is equal to 0. Let d_i denote the degree of node i, and define the degree matrix D to be the $N \times N$ diagonal matrix with i-th diagonal entry d_i.

The Laplacian matrix L of the graph is an $N \times N$ matrix defined as: $L = D - A$. It is well known that L is a symmetric matrix with zero row sums and thus a zero eigenvalue. For a connected graph, all other eigenvalues of L are positive. We suppose that the eigenvalues of the Laplacian matrix L are arranged in an increasing order: $0 = \lambda_1(L) < \lambda_2(L) \leq ... \leq \lambda_N(L)$

B. Consensus Algorithm

In this paper, we concentrate on the average consensus algorithm in a distributed manner to determine the average of the N initial values in all the nodes. Assume each node i holds an initial scalar value $x_i(0) \in R$ and $\mathbf{x}(0) = [x_1(0), x_2(0), ..., x_N(0)]^T$ denotes the vector of initial values. The average $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$ is computed via a distributed average consensus algorithm according to the following linear iterations [12], which have the form,

$$\mathbf{x}(t+1) = W\mathbf{x}(t) \tag{1}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), ..., x_N(t)]^T$ denotes the vector of values at the iteration numbered t with $\mathbf{x}(0) = [x_1(0), x_2(0), ..., x_N(0)]^T$. The matrix W is the weight matrix, with W_{ij} denoting the weight associated with link (i, j), and $x_i(t+1)$ is a linear combination of those $x_j(t)$ for which $j = i$ or node j is a neighbor of node i. The convergence of the average consensus algorithm occurs if $\lim_{t\to\infty} W^t = \frac{1}{N}\mathbf{1}\mathbf{1}^T$.

In this paper we postulate an equal weight α for each link. This means, according to [12] that

$$W_{ij} = \begin{cases} \alpha & \text{if} \quad e(i,j) \in E \\ 1 - \alpha d_i & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and the optimum value of α, i.e. the value for which convergence is fastest is $\alpha = \frac{2}{\lambda_2(L)+\lambda_N(L)}$. Observe that the weight matrix has the form,

$$W = I - \alpha L \tag{3}$$

where I is an $N \times N$ identity matrix and L is the Laplacian matrix of the graph.

From equation (3) we can express the eigenvalues of W in terms of those of the Laplacian matrix L as follows,

$$\lambda_i(W) = 1 - \alpha\lambda_i(L), \quad i = 1, ..., N \tag{4}$$

Now for any connected graph, we can derive the convergence rate of the above consensus algorithm from equation (1) and (4) and the formula for the optimum weight.

$$\|\mathbf{x}(T) - \bar{\mathbf{x}}\| \leq \|\mathbf{x}(0) - \bar{\mathbf{x}}\| \, \lambda_2^T(W) \tag{5}$$

where T denotes the largest number of iterations for which $\lambda_2(W)^T \geq 0.5$ or 0.1 or 0.01, etc. according to the practical requirements.

C. Problem Restatement

In this paper, we focus on applying average consensus algorithms to regular graphs. For a d-regular graph (i.e. all vertices have the same degree d) with N nodes, we have the $|E| = \frac{dN}{2}$. Notice for later reference that for an irregular graph if we know the average degree $\bar{d}$ of the graph, the equation is still valid after replacing d with $\bar{d}$.

Now consider the inter-sensor communication cost. For a network with N sensors, we can define a symmetric cost matrix C, where entry c_{ij} is the communication cost incurred per iteration when sensors i and j communicate with each other. When there is no edge between sensors i and j, $c_{ij} = 0$. Otherwise c_{ij} has a positive value. In this paper, in line with assuming the average consensus algorithm to have equal weight for each link, we assume the communication cost in each iteration for each link is equal [13]. In other words, the nonzero entries c_{ij} of the cost matrix C are all equal to a constant c. The TCC for the networks in the entire consensus process is therefore,

$$TCC_{G(V,E)} = c|E|T \tag{6}$$

where T has the same definition as above.

Since T is inversely related to the convergence rate (denoted by $\mathrm{CR_d}$), in a regular graph with fixed N, we have

$$TCC_{G(V,E,d)} \sim \frac{d}{\mathrm{CR_d}} \tag{7}$$

For sensor networks, because of the energy limitation even a small increase in the communication cost may be a significant negative. It is therefore important to understand the influence of d on both the convergence rate and thus the TCC.

Given a threshold ρ ($0 < \rho < 1$) which depends on the application requirement, from equation (5) we can obtain the following relationship:

$$T \sim \frac{|\ln \rho|}{-\ln \lambda_2(W)} \tag{8}$$

From the above equation, we can consider $-\ln \lambda_2(W)$ as the effective convergence rate in average consensus problems. Further from equation (4), we can obtain:

$$\text{Convergence Rate} = -\ln \lambda_2(W) = \ln \left[\frac{1 + \frac{\lambda_2(L)}{\lambda_N(L)}}{1 - \frac{\lambda_2(L)}{\lambda_N(L)}} \right] \tag{9}$$

From equation (9), we know that in order to obtain the optimal convergence rate over all possible networks within a certain class, we should maximize the ratio $\frac{\lambda_2(L)}{\lambda_N(L)}$: the larger the ratio is, the faster the convergence rate will be. Therefore the relationship between the common vertex degree d and $\frac{\lambda_2(L)}{\lambda_N(L)}$ is the main issue we want to investigate now.

III. MAGIC NUMBER IN CONSENSUS PROBLEM

From Section II, we know that a key goal is to find the relationship between d and $\frac{\lambda_2(L)}{\lambda_N(L)}$. However to analytically relate the ratio $\frac{\lambda_2(L)}{\lambda_N(L)}$ to the common vertex degree d is almost impossible. Instead, here we investigate an upper bound and a lower bound for the $\frac{\lambda_2(L)}{\lambda_N(L)}$ related to d. By using the lower and upper bounds for $\frac{\lambda_2(L)}{\lambda_N(L)}$ and equation (9), we can investigate the relationship between the common vertex degree d and the convergence rate.

A. An Upper Bound for $\frac{\lambda_2(L)}{\lambda_N(L)}$

In order to obtain an upper bound for the ratio $\frac{\lambda_2(L)}{\lambda_N(L)}$ in d-regular graphs, we will proceed by finding an upper bound for $\lambda_2(L)$ and a lower bound for $\lambda_N(L)$.

First from [14], we have the following theorem:

Theorem 1 Let $G = G_{N,d}$ be a d-regular graph on N vertices. Denote by $\lambda_A(G)$, the absolute value of the largest eigenvalue (in absolute value) of the adjacency matrix A, which is distinct from $\pm d$. Then

$$\liminf_{N\to\infty} \lambda_A(G) \geq 2\sqrt{d-1} \tag{10}$$

For non-bipartite connected regular graphs on which we concentrate in this paper, we can arrange the eigenvalues of the adjacency matrix A as $d = \lambda_1(A) > \lambda_2(A) \geq ... \geq \lambda_N(A) > -d$. Further for d-regular graphs, the eigenvalues of the adjacency matrix A and Laplacian matrix L have the following relationship[14],

$$\forall n \in [1, 2, 3, ..., N]: \quad \lambda_n(L) = d - \lambda_n(A) \tag{11}$$

Therefore from equation (10) and (11) we can obtain an upper bound on the second smallest eigenvalue of L in d-regular graphs as follows,

$$\liminf_{N\to\infty} \lambda_2(L) \leq d - 2\sqrt{d-1} \tag{12}$$

Next from [15] we can obtain a well-known lower bound for λ_N in d-regular graphs.

$$\lambda_N(L) \geq d \tag{13}$$

Therefore we can finally obtain an upper bound for convergence rate in d-regular graphs via combining equations (12) and (13).

$$\liminf_{N\to\infty} \frac{\lambda_2(L)}{\lambda_N(L)} \leq \frac{d - 2\sqrt{d-1}}{d} \tag{14}$$

B. A Lower Bound for $\frac{\lambda_2(L)}{\lambda_N(L)}$

In order to obtain a lower bound for convergence rate in regular graphs, we proceed by finding an upper bound for $\lambda_N(L)$ and a lower bound for $\lambda_2(L)$.

From [16] we obtain the following theorem:

Theorem 2 Consider the d-regular graphs where d is even. For N and d large, with high probability[2] the second smallest and the largest eigenvalues of the Laplacian matrix satisfy

$$\begin{array}{rcl} \lambda_2(L) & > & d - 2\sqrt{d-1} - \log d \\ \lambda_N(L) & < & d + 2\sqrt{d-1} + \log d \end{array} \tag{15}$$

Therefore we can obtain a lower bound for the ratio $\frac{\lambda_2(L)}{\lambda_N(L)}$ in d-regular graphs for large N and d (d is even):

$$\frac{\lambda_2(L)}{\lambda_N(L)} > \frac{d - 2\sqrt{d-1} - \log d}{d + 2\sqrt{d-1} + \log d} \tag{16}$$

[2]The precise probability is given in [17]. In this paper we assume these two bounds are always valid without further comment. Simulations evidence is consistent with the reasonableness of this assumption.

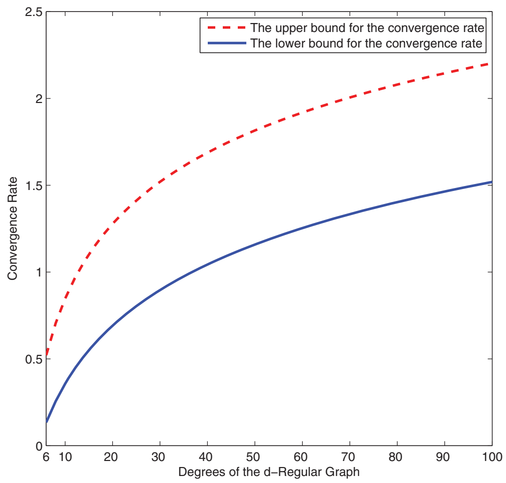

Fig. 1. The upper and lower bounds for the convergence rate

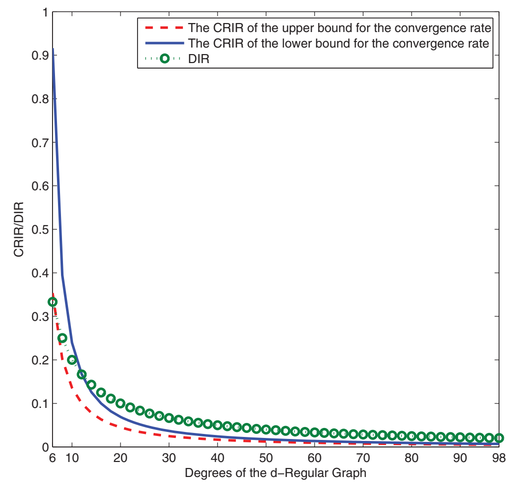

Fig. 2. The CRIR of the upper and lower bounds of the convergence rate and the DIR

C. Magic Number for Convergence Rate

From subsection A and B, we obtain the upper and lower bounds for the ratio $\frac{\lambda_2(L)}{\lambda_N(L)}$; these two bounds depend only on the common vertex degree d. Now by using the equation (9) and inequalities (14) and (16), we can obtain the following inequality,

$$\ln\left[\frac{d}{2\sqrt{d-1}+\log d}\right] < \text{Convergence Rate} \leq \ln\left[\frac{d-\sqrt{d-1}}{\sqrt{d-1}}\right] \tag{17}$$

where the common vertex degree d is even only.

Figure 1 illustrates the upper and lower bounds for the convergence rate related to the common vertex degree d increasing from 6 to 100 in steps of 2[3]. From the figure we can see that both the upper and lower bounds increase very rapidly with the common vertex degree increasing at first. After a certain period, the increment of the two bounds will become much smaller. This phenomenon demonstrates that the convergence rate does not linearly increase with the degree.

According to the equation (7) we obtain the following fact: each time the common vertex degree d increases by 2, only if the convergence rate increases by at least $\frac{2}{d} \times \mathrm{CR_d}$ will the TCC be reduced.

Therefore it is important to investigate the increment of the convergence rate while the common vertex degree d increases in the steps of 2. To this end we define the convergence rate increment ratio (CRIR) and the common vertex degree increment ratio (DIR) as follows:

$$\mathrm{CRIR} = \frac{\mathrm{CR}_{d+\delta d} - \mathrm{CR}_d}{\mathrm{CR}_d} \tag{18}$$

$$\mathrm{DIR} = \frac{(d+\delta d) - d}{d} = \frac{2}{d} \tag{19}$$

When the common vertex degree d increases in steps of 2 ($\delta d = 2$), only if DIR$\leq$CRIR the TCC will not be enlarged.

[3]Because the lower bound for the convergence rate is only for even degrees, the step is set as 2.

Figure 2 depicts the increment ratio of the upper and lower bounds for the convergence rate and the common vertex degree increment ratio with the common vertex degree d increasing in steps of 2. From figure 2 we obtain that the CRIR of the two bounds decreases very rapidly at first. Further we can observe both the lower and upper bound CRIR curves intersect with the DIR curve. Assume the degrees corresponding to the intersections are $d_{\text{intersection upper bound}}$ and $d_{\text{intersection lower bound}}$ for the lower and upper bounds. When the common vertex degree d is smaller than $d_{\text{intersection upper bound}}$ (or $d_{\text{intersection lower bound}}$), the DIR is always smaller than CRIR of the upper bound (or the lower bound) for the convergence rate. Thus by increasing the common vertex degree d, the convergence rate will increase without bringing extra communication cost. However when the common vertex degree d exceeds $d_{\text{intersection upper bound}}$ (or $d_{\text{intersection lower bound}}$), though the convergence rate can still be improved via increasing the common vertex degree d the TCC will also increase. Therefore at that time, we need to consider whether it is still desirable to increase the common vertex degree d.

Based on the above analysis, we suggest it makes sense to define a kind of magic number k for the common vertex degree d in regular graphs which are independent of the number of sensors in networks:

- The convergence rate can be improved without increasing the TCC when the common vertex degree d is smaller than a number k which is independent of the number of nodes. When the common vertex degree d exceeds the number k, though the convergence rate will continue increasing with further increase in d the TCC will also be enlarged. This number k is the magic number.

In the next section, we will use Monte Carlo simulation results to demonstrate the validity of the two bounds for the convergence rate and the existence the magic number.

IV. SIMULATION RESULTS

In this section, we will use Monte Carlo simulations to demonstrate that the phenomena described in the previous section exist in regular graphs and irregular graphs in which the degree distribution is subject to a Poisson distribution. First in subsection A some assumptions are presented. In subsection B we present the Monte Carlo simulation results both on regular graphs and irregular graphs.

A. Assumptions and Algorithmic Frameworks

- All graphs are connected (both regular and irregular graphs)
- The sizes of the graphs are set as 200, 500 and 1000 nodes.
- The common vertex degree d of each graph is adjusted from 6 to 100 in steps of 2.
- For the same vertex count and common vertex degree, we generate 1000 different regular/irregular graphs and calculate the CRIR. Finally, we use the average of 1000 simulation results to plot the figure.
- The CRIR and DIR are defined in the equation (18) and (19)

B. Numerical Results for Regular Graphs

Figure 3 depicts the simulation results for the CRIR in the networks with different number of nodes. We can see that the two bounds for the convergence rate are valid everywhere since the simulation curves always lie between them. Further the general shapes of the simulation curves for different networks are all similar in that the CRIR drops dramatically while the common vertex degree d increases.

Next we further analyze the simulation results in order to investigate whether the magic number defined in III.C exist or not. From the simulation results, we can obtain a table as follows.

Different Networks	N=200	N=500	N=1000
Interaction	8	8	8

TABLE I
THE MAGIC NUMBER FOR NETWORKS OF DIFFERENT NUMBER OF NODES

Further from Figure 3 we can observe that the CRIR curves always intersect with the DIR curve. Table I shows for different networks the common vertex degree corresponding to the intersection points which is the second type of magic number k. From the table we see that the magic number k is always equal to 8 which means it is independent of the number of nodes. *When the common vertex degree is smaller than 8, by increasing the common vertex degree a faster convergence rate can be achieved without incurring extra TCC. When the common vertex degree exceeds the magic number 8, the convergence rate can still be improved by increasing the common vertex degree d. However the TCC will also be enlarged.*

C. Numerical Results for Irregular Graphs

In this section, we present simulations indicating the extent to which the previous results carry over to irregular graphs. The irregular graphs are generated randomly in which the degree distribution is subject to a Poisson distribution of intensity λ. This means that λ is the average degree of the irregular graphs. Here we set the size of the connected irregular graphs as 200, 500 and 1000. The average degree (λ in Poisson distribution) of each graph is adjusted from 6 to 100 in steps of 2. Again, for the same vertex count and average vertex degree, we generate 1000 different irregular graphs and calculate the CRIR. Finally, we use the average of 1000 simulation results to plot the figure.

Different Networks	N=200	N=500	N=1000
Magic Number II	20	20	20

TABLE II
THE MAGIC NUMBER FOR NETWORKS OF DIFFERENT NUMBER OF NODES IN IRREGULAR GRAPHS

From Figure 4 we observe that the phenomena in regular graphs also occur in irregular graphs, in that the shapes of the simulation curves for different networks show a dramatic drop when the average degree λ increases. Moreover from the simulation results we obtain that the upper bound for regular graphs still can be considered as the upper bound for irregular graphs. The only difference is that the common vertex degree in the original upper bound now is replaced by the average degree (λ). The simulation results verify the validity of the upper bound. Now, similar to the situation in regular graphs, we can obtain the Table II for the magic number. From the table, we can obtain that the magic number also exists in the irregular graphs.

From the above analysis, we can obtain that the magic number not only exists in regular graphs but also occurs in irregular graphs with the degrees subject to a Poisson distribution. Further we can observe that the magic number for irregular graphs is larger than the one for regular graphs.

V. CONCLUSIONS AND FUTURE WORKS

In this paper, we analyze the influence of the number of edges on the convergence rate and total communication cost in average consensus problems. In the theoretical analysis, we restrict the underlying graphs of the networks for regular graphs in order to simplify the problem for investigating the relationship between the common vertex degree and the convergence rate. Here we use the ratio of the second smallest and the largest eigenvalues of the Lapalacian matrix of the graph to evaluate the convergence rate. We find a lower bound and an upper bound for the convergence rate in regular graphs related only to the common vertex degree d. From the analysis of two bounds we observe that by increasing the degrees the convergence rate will increase. However the convergence rate increment ratio will decrease significantly while the common vertex degree becomes progressively large. Meanwhile the total communication cost will be enlarged when the common vertex degree increases. Due to the energy limitation in sensor networks, any increase of energy cost should be taken in to consideration. Therefore we define a kind of magic number

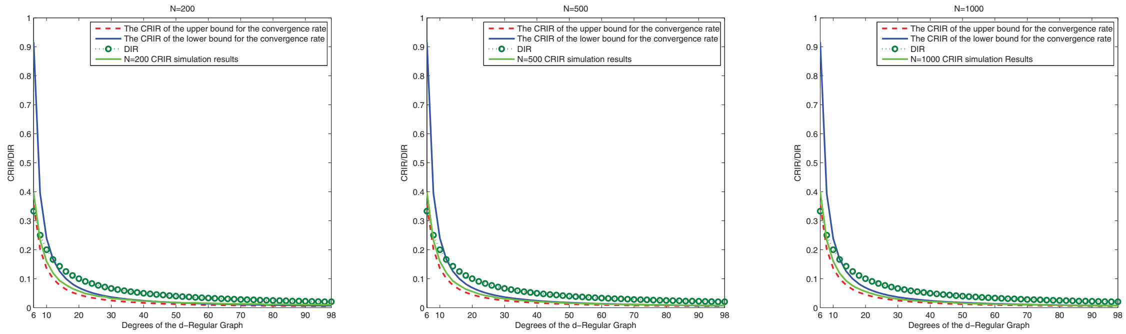

Fig. 3. Regular Networks with different number of nodes (a) N = 200 (b) N = 500 (c) N = 1000

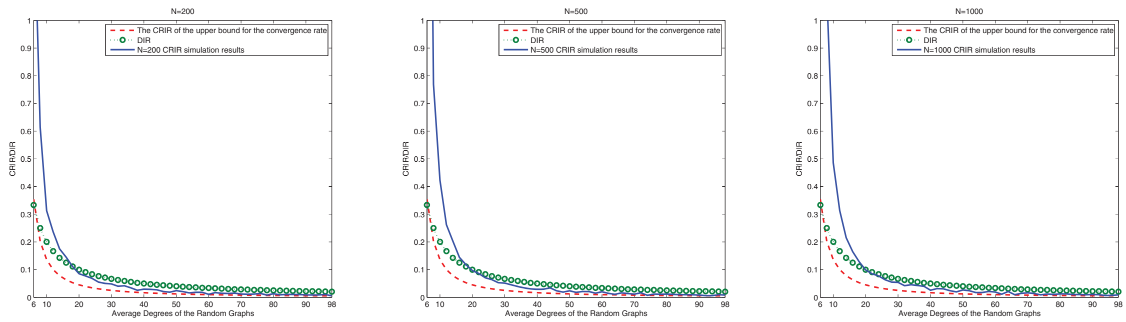

Fig. 4. Irregular Networks (degrees are subject to an Poisson distribution) with different number of nodes (a) N = 200 (b) N = 500 (c) N = 1000

to quantitatively analyze the influence of degrees increasing in regular graphs in order to help one decide how many links are desirable for a given sensor. The Monte Carlo simulation results demonstrate the valid of two analytical bounds and the existence of the magic number. Though all the theoretical analysis is mainly restricted to regular graphs in this paper, simulation results we presented show that the same phenomena also exist in irregular graphs (the degree distribution is subject to a Poisson distribution). Therefore our future work is to theoretical analyze the magic number in irregular graphs and some other types of networks.

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REFERENCES

[1] R. Olfati-Saber, J.A. Fax and R.M. Murray, "Consensus and Cooperation in Networked Multi-Agent Systems", *Proceedings of the IEEE*, vol. 95(1), Jan., 2007, pp 215-233

[2] R. Wei, R.W. Beard and E.M. Atkins, "A Survey of Consensus Problems in Multi-Agent Coordination", *Proceedings fo the 2005 American Control Conference*, vol. 3, Jun., 2005, pp 1859-1864

[3] R. Wei, R.W. Beard and E.M. Atkins, "Information Consensus in Multivehicle Cooperative Control", *IEEE Control Systems Magazine*, vol. 27(2), Apr., 2007, pp 71-82

[4] R. Olfati-Saber and J.S. Shamma, "Consensus Filters for Sensor Networks and Distributed Sensor Fusion", *44th IEEE Conference on Decision and Control, and the European Control Conference*, Seille, Spain, Dec., 2005, pp 6698-6703

[5] L. Xiao, S. Boyd and S. Lall, "A Scheme for Robust Distributed Sensor Fusion Based on Average Consensus" *Fourth International Symposiun on Information Processing in Sensor Networks (IPSN 2005)*, Apr., 2005

[6] I. D. Schizas, A. Ribeiro, and G. B. Giannakis, "Consensus in Ad Hoc WSNs with Noisy LinksłPart I: Distributed Estimation of Deterministic Signals", *IEEE Transaction on Signal Processing*, vol. 56, Jan., 2008, pp. 350-364

[7] H. Zhang, M. Z.Q. Chen, T. Zhou and G. Stan, "Ultrafast Consensus via Predictive Mechanisms", *Europhysics Letters*, vol. 83(4), 2008, pp. 40003

[8] S. Sardellitti, M. Giona and S. Barbarossa, "Fast Distributed Average Consensus Algorithms Based on Advection-Diffusion Processes", *IEEE Transaction on Signal Processing*, vol. 58(2), Feb., 2010, pp. 826-842

[9] Z. Jin and R. M. Murray, "Random Consensus Protocol in Large-Scale Networks", *46th IEEE Conference on Decision and Control, and the European Control Conference*, New Orleans, LA, USA, Dec., 2007, pp 4227-4232

[10] S. Kar, S. Aldosari and J. M.F. Moura, "Topology for Distributed Inference on Graphs", *IEEE Transaction on Signal Processing*, vol. 56(6), Jun., 2008, pp. 2609-2613

[11] S. Boyd, P. Diaconis, P. A. Parrilo, L. Xiao, "Fastest Mixing Markov Chain on Graphs with Symmetries", *SIAM Journal on Optimization*, vol. 20(2), 2009, pp. 792-819

[12] L. Xiao and S. Boyd, "Fast Linear Iteration for Distributed Averaging", *42nd IEEE Conference on Decision and Control*, vol. 53, 2003, pp. 4497-5002

[13] S. Kar, J. M.F. Moura, "Sensor Networks with Random Links: Topology Design for Distributed Consensus", *IEEE Transaction on Signal Processing*, vol. 56(7), Jul., 2008, pp. 3315-3326

[14] N. Alon, "Eigenvalues and Expanders", *Combinatorica*, vol. 6, 1986, pp. 83-96

[15] R. Merris, "Laplacian Matrices of a Graph: a Survey", *Linear Algebra and its Applications*, vol. 197, 1994, pp. 143-176

[16] C.W. Wu, "Synchronization in Complex Networks of Nonlinear Dynamical Systems", 2007 by *World Scientific Publishing Co. Pte. Ltd*

[17] J. Friedman, J. Kahn and E. Szemerédi, "On the Second Eigenvalue in Random Regular Graphs", *in Proceedings of the 21st Annual ACM Symposium on Theory of Computing (Seattle, 1989)*,Association for Computing Machinery, New York, 1989, pp. 587-598.