2012 12th International Conference on Control, Automation, Robotics & Vision
Guangzhou, China, 5-7th December 2012 (ICARCV 2012)

On the Performance Limit of Single-Hop TOA Localization

Baoqi Huang
College of Computer Science
Inner Mongolia University
Hohhot, 010021, China
Email: cshbq@imu.edu.cn

Tao Li
the Key Laboratory of Systems and Control
Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100190, China
Email: litao@amss.ac.cn

Brian D.O. Anderson
and Changbin Yu
the Australian National University
and NICTA Ltd.
Canberra, A.C.T., Australia

***Abstract*—In this paper, we analyze the performance limit of sensor localization from a novel perspective. We consider distance-based single-hop sensor localization with noisy distance measurements by time of arrival (TOA). Differently from the existing studies, the anchors are assumed to be randomly deployed, with the result that the trace of the associated Cramér-Rao Lower Bound (CRLB) matrix becomes a random variable. We adopt this random variable as a scalar metric for the performance limit and then focus on its statistical attributes. By the Central Limit Theorems for U-statistics, we show that as the number of anchors goes to infinity, this scalar metric converges to a random variable which is an affine transformation of a chi-square random variable of degree 2. In addition, we provide the quantitative relationship among the mean, the standard deviation, the number of anchors, parameters of communication channels and the distribution of the anchors. Extensive simulations are carried out to confirm the theoretical results. On the one hand, our study reveals some fundamental features of sensor localization; on the other hand, the conclusions we draw can in turn guide us in the design of wireless sensor networks.**

I. Introduction

Wireless sensor networks have a wide range of applications nowadays, including military operations, medical treatments, environmental sensing, water quality monitoring and many others [1]. Location information plays a vital role in those applications, for it is useful to report the geographic origin of events, to assist in target tracking, to achieve geographic aware routing, to manage sensor networks, to evaluate their coverage, and so on. A sensor network generally consists of two types of nodes: anchors and sensors. Anchor locations are known a priori through GPS or manual configurations, while sensor locations are not known and need to be determined through the procedures of sensor localization. Up to now, considerable efforts have been invested into the development of localization algorithms, see e.g. [2]–[5]. Take trilateration, the most basic localization technique, for example: in a two-dimensional plane, the location of a sensor is estimated from the known locations of at least three non-collinear anchors and the measured distance, derived e.g. from received signal strength (RSS) or time of arrival (TOA), to each anchor; this is also termed single-hop distance-based sensor localization.

Single-hop sensor localization can be found in many practical localization scenarios, such as source localization and target tracking. Effectively, another example arises in simultaneous localization and mapping (SLAM) [6], where a mobile robot equipped with a GPS receiver moves in a two-dimensional environment, measures relative location information to various objects, and then determines the locations of these objects; herein, the positions where the robot makes measurements can be abstracted as *anchors*, such that the localization procedure is single-hop. Additionally, in [7], a mobile anchor(s) is used to assist in sensor localization by providing relative location measurements to sensors at multiple positions, which is evidently single-hop. Therefore, it is meaningful to study single-hop sensor localization.

Apart from localization algorithms, the performance limit of sensor localization, namely the lowest achievable error bound for location estimates, also attracts much attention. On the one hand, it provides a measure of theoretically optimal performance no matter what sensor localization algorithm is applied; on the other hand, it reflects fundamentals of sensor localization. Since the Cramér-Rao lower bound (CRLB) establishes a lower limit (or bound) on the variance for any unbiased estimator, it has been widely used in the performance (limit) analysis of sensor localization, see e.g. [8], [9]. For single-hop sensor localization in a two-dimensional plane, the CRLB is a 2×2 matrix and turns out to be dependent on multiple factors, including measuring techniques, noise statistics of measurements and sensor-anchor geometries (i.e. relative node locations, or their coordinates). Since the trace of the CRLB matrix is the minimum *mean square estimation error (MSE)*, it is often used as a *scalar metric* for the performance limit [8], [9]. Provided that the measuring technique and the noise statistics of measurements are both known, the scalar metric can be regarded as a function of the sensor-anchor geometry. An obvious problem arising is to minimize the scalar metric, equivalent to identifying optimal sensor-anchor geometries for sensor localization, and it has been widely studied [10]–[14]. The metric can also give valuable qualitative information. It can be very large, implying that a localization problem is badly conditioned (e.g. the anchors are nearly collinear with the sensor) and localization algorithms almost fail. Evidently, we should avoid the situations where the scalar metric takes large values. In short, the CRLB provides much useful information regarding sensor localization.

978-1-4673-1872-3/12/$31.00 ©2012 IEEE

The conventional CRLB studies assume a deterministic sensor-anchor geometry. But, prior to deploying a sensor and several anchors in a domain, the sensor-anchor geometry is usually unknown; hence, it is difficult to evaluate the localization performance. Yet, a probability measure for the sensor-anchor geometry might be available. Indeed, it is natural to model the senor-anchor geometry by assuming a random and uniform distribution for the anchors' positions, and consequently, the scalar metric itself becomes random and offers a broad, statistical view on the localization performance, in contrast to one deterministic quantity for a given sensor-anchor geometry. For instance, if the scalar metric hardly takes large values, there is less reason to worry about the sensor-anchor geometry; otherwise, one must impose proper control on it. In addition, the mean of the scalar metric further establishes a lower limit on the performance of single-hop sensor localization given a fixed number of anchors with undetermined locations; in the situations where sensor-anchor geometries are unknown, e.g. prior to system deployment, this performance limit is certainly useful. Furthermore, the novel statistical sensor-anchor geometry modeling method not only provides insights into single-hop sensor localization and in turn guides us in the design and deployment of wireless sensor networks, but also as a prototype paves the way for dealing with more complicated scenarios of sensor localization. For instance, in a mobile environment, as may arise with ad-hoc networks, SLAM, mobile anchors assisting in sensor localization and so on, it is trivial to concentrate on localization performance in one particular time instant, whereas it is evidently more attractive to grasp the knowledge about the average localization performance over a period of time and/or in a wide region. Hopefully, these challenges can be addressed by the statistically modeling method. In summary, statistical sensor-anchor geometry modeling is a powerful method for investigating the performance limit of sensor localization. Hence, we are motivated to study the scalar metric based on the statistical sensor-anchor geometry modeling method.

In this paper, we take into account single-hop TOA-based sensor localization, and show that the scalar metric is essentially a function of U-statistics [15]. Based on the theory of U-statistics, we make the following contributions: (i) it is proved that as the number of the anchors increases, the scalar metric converges to a random variable which is an affine transformation of a *chi-square* random variable of degree 2 in the TOA case; (ii) the *asymptotic* formulas for the mean and standard deviation of the scalar metric are derived; (iii) last but not the least, these formulas are analyzed to demonstrate some properties of sensor localization and the conclusions are verified by extensive simulations. [The more extensive treatment of the ideas in this paper has been submitted to a journal, but a decision has not yet been delivered on the status of the journal submission.]

The remainder of this paper is organized as follows. The next section introduces the problem formulation. Section III presents the main results about statistical attributes of performance limits. Finally, we conclude this paper and shed light on future work in Section V.

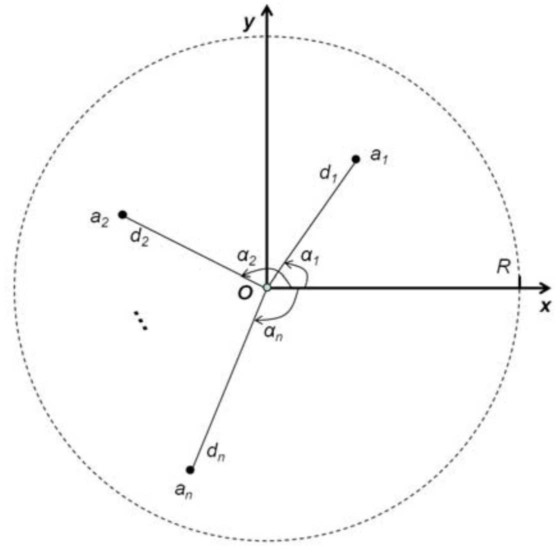


Fig. 1. Localizing a sensor using n anchors.

II. Problem Formulation

In this section, we formulate the scalar metric of the performance of single-hop sensor localization using TOA measurements and define a random sensor-anchor geometry model. Throughout this paper, we shall use the following mathematical notations: $Tr(\cdot)$ denotes the trace of a square matrix; $E(\cdot)$ denotes the expected value of a random variable; $Var(\cdot)$ denotes the variance; $Std(\cdot)$ denotes the standard deviation.

A. One-hop Sensor Localization Using TOA Measurements

In a two-dimensional plane, consider a single sensor (or source, target) located at some unknown position and n distance (or angle) measurements made to this sensor at n known locations, as illustrated in Figure 1. Here, the n known locations are abstracted as anchors and are labeled $1, \cdots, n$ with the i-th anchor's location denoted by $\mathbf{s}_i = [x_i, y_i]^T$. The true distance between the sensor and the i-th anchor is denoted by $d_i = \|\mathbf{s}_i\|$. The true angle subtended at the sensor by the i-th anchor and the positive x-axis is denoted α_i.

For a specific localization problem, the precise locations of the n anchors, i.e. $[x_i, y_i]^T$, are given in advance; pair-wise distance measurements $\{\hat{d}_i, i = 1, \cdots, n\}$ between the sensor and the anchors are made and obey certain error models. Then, the aim of single-hop sensor localization is finding an estimate of the true sensor position using the observable set of distance measurements $\{\hat{d}_i, i = 1, \cdots, n\}$. In this paper, we consider the performance limit of sensor localization over a family of random anchor locations other than a specific localization problem with given anchor locations.

TOA refers to the travel time of a radio signal through a medium from a transmitter to a remote receiver. By the relation between signal propagation speed in this medium, the time is a measure for the distance between the transmitter and the receiver. Without loss of generality, we let the sensor be a transmitter and the n anchors be receivers. Denote by $\{T_i, i = 1, \cdots, n\}$ the measured TOA between the sensor and the n anchors. As in both theoretical studies [16], [17] and experimental studies [8] on TOA-based sensor localization under line-of-sight conditions, we assume

Assumption 1: $\{T_i, i = 1, \cdots, n\}$ are statistically independent and Gaussian with means $\{\frac{d_i}{c}, i = 1, \cdots, n\}$ (c is the speed of propagation) and same variance σ_T^2.

Given the TOA measurements, one can obtain associated distance measurements to estimate the sensor location. Regarding the location estimation problem, the Fisher information matrix (FIM) F_{TOA} can be formulated as follows:

$$F_{TOA} = \frac{1}{\sigma_T^2 c^2} \begin{pmatrix} \sum_{i=1}^n \cos^2 \alpha_i & \sum_{i=1}^n \cos \alpha_i \sin \alpha_i \\ \sum_{i=1}^n \cos \alpha_i \sin \alpha_i & \sum_{i=1}^n \sin^2 \alpha_i \end{pmatrix}. \tag{1}$$

Let $(\hat{x}, \hat{y})$ be the unbiased sensor position estimate and C_{TOA} be the CRLB on the covariance of $(\hat{x}, \hat{y})$. If F_{TOA} is non-singular, C_{TOA} equals to F_{TOA}^{-1} and satisfies

$$E_\upsilon(\hat{x}^2 + \hat{y}^2) \geq Tr(C_{TOA}), \tag{2}$$

where $\upsilon = \{T_i, i = 1, \cdots, n\}$. That is, $Tr(C_{TOA})$ is the *scalar metric* for the performance limit with the expression:

$$Tr(C_{TOA}) = \frac{\sigma_T^2 c^2 n}{\sum_{1 \leq i < j \leq n} \sin^2(\alpha_i - \alpha_j)}. \tag{3}$$

B. A Random Sensor-Anchor Geometry Model

To carry out this study, we define the random sensor-anchor geometry model by assuming

Assumption 2: The n anchors are randomly and uniformly deployed within a circle of radius R ($R > 0$) centered at the sensor.

In Assumption 2, R is the upper bound on practical distances used in the wireless communication system. By Assumption 2, each possible sensor-anchor geometry is equiprobable, in the sense that the sensor-anchor geometry follows a "uniform" distribution. As a result, $\omega = \{d_i, \alpha_i, i = 1, \cdots, n\}$ are mutually independent random variables and the scalar metric $Tr(C_{TOA})$ is random. In what follows, we shall study the distribution, mean and standard deviation of $Tr(C_{TOA})$ and analyze the properties of sensor localization.

C. U-statistics

U-statistics are very natural in statistical work, particularly in the context of independent and identically distributed (i.i.d.) random variables, or more generally for exchangeable sequences, such as in simple random sampling from a finite population. The origins of the U-statistics theory are traceable to the seminal paper [15], which proved the Central Limit Theorems for U-statistics. Following the publication of this seminal paper, the interest in this class of statistics steadily increased, crystallizing into a well-defined and vigorously developing line of research in probability theory. Its formal definition is presented as follows:

Definition 1: Let $\{X_i, i = 1, \cdots, n\}$ be i.i.d. p-dimensional random vectors. Let $h(x_1, \cdots, x_r)$ be a Borel function on $\mathbb{R}^{r \times p}$ for a given positive integer $r \leq n$ and be symmetric in its arguments. A U-statistic U_n is defined as

$$U_n = \frac{r!(n-r)!}{n!} \sum_{1 \leq i_1 < \cdots < i_r \leq n} h(X_{i_1}, \cdots, X_{i_r}) \tag{4}$$

and $h(x_1, \cdots, x_r)$ is called the kernel of U_n.

It is obvious that $Tr(C_{TOA})$ equals to the inverse of a U-statistic according to (3), which motivates us to study $Tr(C_{TOA})$ through an asymptotic analysis based on the theory of U-statistics.

III. Main Results

Due to the complexity of $Tr(C_{TOA})$, it is very difficult to give its accurate distribution directly. As such, we endeavor to present an asymptotic analysis at first. Due to the space limit, proofs are omitted.

According to (3), a key property of $Tr(C_{TOA})$ is that it is the inverse of the sum of random variables, which can be processed by using the following lemma.

Lemma 1: Given i.i.d. random variables $\{X_i, i = 1, \cdots, n\}$ with bounded values, define a sequence of random variables

$$S_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \sin^2(X_i - X_j). \tag{5}$$

Then, as $n \to \infty$,

$$\frac{1}{S_n} = \frac{1}{m_X} + M_n + R_n \tag{6}$$

where $m_X = E(\sin^2(X_1 - X_2))$,

$$M_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} g(X_i, X_j), \tag{7}$$

$$g(X_i, X_j) = \frac{1}{m_X} - \frac{\sin^2(X_i - X_j)}{m_X^2}, \tag{8}$$

and R_n is the remainder term. For any $\varepsilon > 0$, R_n satisfies

$$Pr\{|nR_n| \geq \varepsilon\} = O(n^{-1}), \tag{9}$$

$$Pr\{|n(\ln n)R_n| \geq \varepsilon\} = o(1), \tag{10}$$

In Lemma 1, by letting $X_i = \alpha_i$, we have $m_X = 0.5$. By using Lemma 1, our main result is further summarized as follows.

Theorem 1: Let σ_T and $Tr(C_{TOA})$ be the same variables as defined in the previous subsection and define a sequence of random variables

$$V_n = \left(\frac{n(n-1)}{2\sigma_T^2 c^2}\right) Tr(C_{TOA}) - 2n + 2. \tag{11}$$

Then, as $n \to \infty$, V_n converges in distribution to a chi-square random variable of degree 2.

Remark 1: According to Theorem 1, if n is sufficiently large, the pdf of $Tr(C_{TOA})$ can be approximated by

$$\frac{n(n-1)}{2\sigma_T^2 c^2} f_\chi\left(\frac{n(n-1)}{2\sigma_T^2 c^2} x - 2n + 2\right), \tag{12}$$

where $f_\chi(\cdot)$ is the pdf of the chi-square random variable of degree 2. Therefore, we can approximate the moments of $Tr(C_{TOA})$ using (12), namely,

$$E_\omega(Tr(C_{TOA})) \approx \frac{4\sigma_T^2 c^2}{n-1}, \quad (13)$$

$$Std_\omega(Tr(C_{TOA})) \approx \frac{4\sigma_T^2 c^2}{n(n-1)}. \quad (14)$$

IV. Simulations

In this section, we report on simulations to verify Theorem 1 and the accuracy of using (13) and (14). In the simulations, we let $\sigma_T c = 1$ m which was measured in a typical environment in [8].

In the first place, we plot the distribution functions and pdfs of $Tr(C_{TOA})$ from both simulations[1] (with the legend "Simulation") and the random variable defined by (12) (with the legend "Formula") for $n = 5, 10, 15, 20$ in Fig. 2. It can be seen that the discrepancies between the pairs of distribution functions (or pdfs) are quite small, and vanish with n increasing, which is certainly consistent with Theorem 1. Note that every curve of the pdf from the simulations contains a sharp curvature, but the corresponding curve from the given random variable is perfectly smooth; this does not invalidate Theorem 1, due to the fact that as n increases, the curve associated with $Tr(C_{TOA})$ tends to become an impulse such that the minimum of $Tr(C_{TOA})$ will overlap with the value of $Tr(C_{TOA})$ attaining the maximum of the pdf.

From Fig. 2(a), we can obtain a clear understanding about $Tr(C_{TOA})$. When $n = 5$, $Tr(C_{TOA})$ has a lower limit around 0.8 m^2 and can be far greater than $(\sigma_T c)^2 = 1$ m^2 ($\sigma_T c$ denotes the accuracy of distance measurements from TOA); when $n \geq 10$, the lower limit of $Tr(C_{TOA})$ reduces to be less than 0.4 m^2 and $Tr(C_{TOA})$ could hardly take values above 1 m^2.

In the second place, we plot $E_\omega(Tr(C_{TOA}))$ and $Std_\omega(Tr(C_{TOA}))$ from simulations[2] and formulas (13) and (14) in Fig. 3(a), as well as the associated relative errors[3] in Fig. 3(b). It is clear that the accuracies of both formulas improve with n increasing; in addition,, both formulas produce overestimates and the relative error of (13) is much smaller than that of (14). Assuming that a relative error below 10% is acceptable, (13) is applicable if $n \geq 6$, whereas (14) is not applicable even if $n = 20$.

In what follows, we present some useful remarks on the properties of sensor localization provided that (13) and (14) are applicable.

Remark 2: Equation (13) quantitatively characterizes the average performance limit over all possible sensor-anchor geometries and is indicative for evaluating the average localization performance over a period of time and/or in a wide region. In addition, because the mean is in inverse proportion to n, a *critical* value n^* differing from the parameters $\sigma_T c$ can be determined, such that having more anchors than n^* contributes little to the quality of sensor localization.

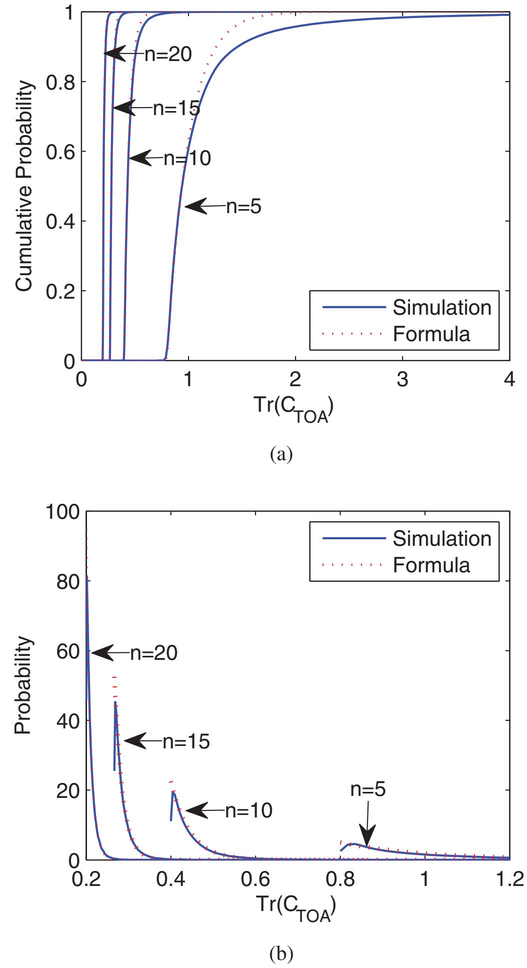


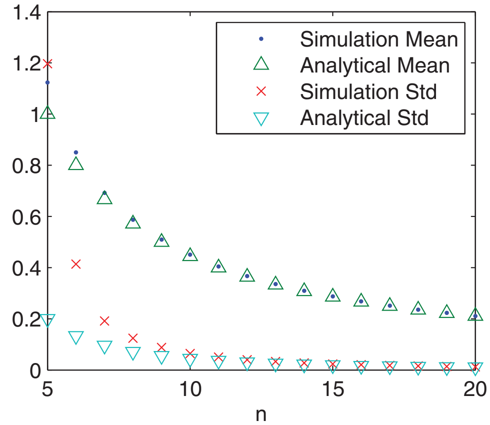
Fig. 2. The distribution functions and pdfs of $Tr(C_{TOA})$ with $R = 10$m and $\sigma_T c = 1$m.

Remark 3: The dispersion of $Tr(C_{TOA})$ reflects its sensitivity to sensor-anchor geometries. Specifically, with a large dispersion, the chance of having two different sensor-anchor geometries which leads to a big difference in the resulting values of $Tr(C_{TOA})$ is large, implying a large sensitivity, and we should be careful about sensor-anchor geometries; by contrast, with a small dispersion, the chance is certainly small, so is the sensitivity, and there is less reason to worry about sensor-anchor geometries even if the anchors are randomly deployed. Given a random variable, the coefficient of variation, defined to be the ratio of its standard deviation to its mean, is a normalized measure of dispersion of its distribution. Therefore, the coefficient associated with $Tr(C_{TOA})$ is $\frac{1}{n}$ and the less is the coefficient, the smaller is the sensitivity. In particular, if the coefficient equals its minimum, i.e. 0, all the sensor-anchor

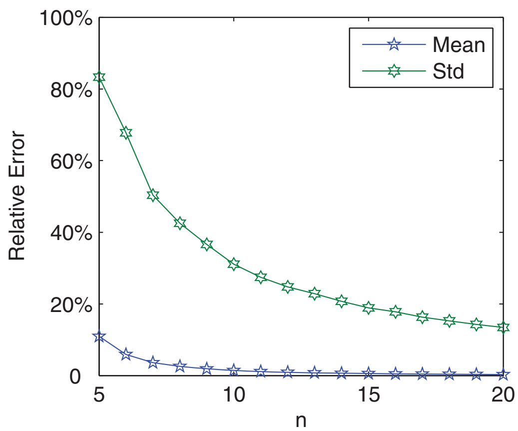
[1] 100000 single-hop sensor localization scenarios are generated, the CRLB in each scenario is then evaluated, and the distribution function of $Tr(C_{TOA})$ is approximately computed using the Matlab routine "ksdensity".

[2] $E_\omega(Tr(C_{TOA}))$ and $Std_\omega(Tr(C_{TOA}))$ are respectively approximated by the sample mean and sample standard deviation of $Tr(C_{TOA})$ which are evaluated based on the simulation results in the previous step.

[3] The *relative error* is defined to be the ratio of the difference between the quantity from the simulations and that from the corresponding formula to the former one.



(a)



(b)

Fig. 3. The means and the standard deviations of $Tr(C_{TOA})$ from the simulations and formulas, and the corresponding relative errors, with $R =$ 10m and $\sigma_T c = 1$m.

geometries will result in one unique value of $Tr(C_{TOA})$, so that the minimum sensitivity is attained. Alternatively, we can observe the sensitivity from Fig. 2(b): the range of $Tr(C_{TOA})$ with a non-trivial probability becomes narrower and narrower with n increasing, implying that the sensitivity is reducing.

V. Conclusion and Future Work

In this paper, we investigated the performance limit of single-hop sensor localization with the TOA measurements by statistically sensor-anchor geometry modeling. That is, the positions of anchors are assumed to be random and the statistical attributes of the trace of the CRLB matrix embodies essential features of sensor localization. With strict mathematical proofs, we showed that the trace of the CRLB matrix converges to a random variable which is an affine transformation of a chi-square random variable of degree 2. Based on this study, we analyzed the features of sensor localization and carried out extensive simulations.

In future work, we would like to take into account other distributions of anchor positions other than the uniform distributions, as well as considering other types of measuring techniques, including time difference of arrival (TDOA), etc. In addition, it is more attractive, but of course extremely difficult, to conduct similar studies for multi-hop sensor localization.

Acknowledgment

B. Huang is supported by the Program of Higher-level talents of Inner Mongolia University under 125124 and the Inner Mongolia Autonomous Region Science and Technology Innovation Guide Reward Fund Project. T. Li is supported by the National Natural Science Foundation of China under grant 61004029. C. Yu and B.D.O. Anderson are supported by the ARC (Australian Research Council) under DP-110100538 and NICTA. C. Yu is an ARC Queen Elizabeth II Fellow and is also supported by Overseas Expert Program of Shandong Province. This material is based on research sponsored by the Air Force Research Laboratory, under agreement number FA2386-10-1-4102.

References

[1] I. F. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci. Wireless sensor networks: a survey. *Computer Networks*, 38:393–422, 2002.

[2] D. Niculescu and B. Nath. Ad hoc positioning system (aps). In *Proc. IEEE Globecom*, volume 5, pages 2926–2931, Roppongi Hills, Tokyo, Japan, Nov. 2001.

[3] A. Savvides, H. Park, and M. Srivastava. The n-hop multilateration primitive for node localization problems. *ACM Mobile networks and applications*, 8(4):443–451, 2003.

[4] M. Cao, B. D. O. Anderson, and A. S. Morse. Sensor network localization with imprecise distances. *Systems & Control Letters*, 55(11):887–893, 2006.

[5] J. Fang, M. Cao, A. S. Morse, and B. D. O. Anderson. Sequential localization of sensor networks. *SIAM Journal on Control and Optimization*, 48(1):321–350, 2009.

[6] M.W.M.G. Dissanayake, P. Newman, S. Clark, H.F. Durrant-Whyte, and M. Csorba. A solution to the simultaneous localization and map building (slam) problem. *IEEE Transactions on Robotics and Automation*, 17(3):229 –241, June 2001.

[7] P.N. Pathirana, N. Bulusu, A.V. Savkin, and S. Jha. Node localization using mobile robots in delay-tolerant sensor networks. *IEEE Transactions on Mobile Computing*, 4(3):285–296, May-Jun. 2005.

[8] N. Patwari, A.O. III Hero, M. Perkins, N.S. Correal, and R.J. O'Dea. Relative location estimation in wireless sensor networks. *IEEE Transactions on Signal Processing*, 51(8):2137 – 2148, Aug. 2003.

[9] C. Chang and A. Sahai. Estimation bounds for localization. In *Proc. IEEE SECON*, pages 415–424, Santa Clara, CA, USA, Oct. 2004.

[10] S.C. Nardone, A.G. Lindgren, and K.F. Gong. Fundamental properties and performance of conventional bearings-only target motion analysis. *IEEE Transactions on Automatic Control*, 29(9):775 – 787, September 1984.

[11] S. Martinez and F. Bullo. Optimal sensor placement and motion coordination for target tracking. *Automatica*, 42(4):661 – 668, 2006.

[12] K. Doğançay and H. Hmam. Optimal angular sensor separation for aoa localization. *Signal Process.*, 88:1248–1260, May 2008.

[13] A. N. Bishop, B. Fidan, B. D. O. Anderson, K. Dogcancay, and P. N. Pathirana. Optimality analysis of sensor-target localization geometries. *Automatica*, 46:479C492, 2010.

[14] M. Cimino and P. R. Pagilla. Optimal location of mouse sensors on mobile robots for position sensing. *Automatica*, 47(10):2267–2272, 2011.

[15] W. Hoeffding. A class of statistics with asymptotically normal distribution. *The Annals of Mathematical Statistics*, 19(3):293–325, 1948.

[16] S. Zhu and Z. Ding. Joint synchronization and localization using toas: A linearization based wls solution. *IEEE Journal on Selected Areas in Communications*, 28(7):1017–1025, Sep. 2010.

[17] E.g Xu, Z. Ding, and S. Dasgupta. Source localization in wireless sensor networks from signal time-of-arrival measurements. *IEEE Transactions on Signal Processing*, 59(6):2887–2897, Jun. 2011.