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Robustness Issues with Undirected Formations

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***Abstract*— It is shown for any rigidity-based, undirected triangular formation of the type studied in [1], that if neighboring agents in the formation have slightly different understandings of what the desired distance between them is suppose to be, then almost for certain, the trajectory of the resulting distorted but rigid formation will converge exponentially fast to a closed orbit in $\mathbb{R}^2$ which is traversed periodically at a single sinusoidal frequency.**

I. INTRODUCTION

The problem of coordinating a large, network of mobile autonomous agents by means of distributed control has raised a number of issues concerned with the forming, maintenance and real-time modification of multi-agent networks of all types. One of the most natural and useful tasks along these lines is to organize a network agents into an application specific "formation" which might be used for such tasks as environmental monitoring, search, or simply moving the agents efficiently from one location to another. By an multi-agent formation is usually meant a collection of agents in real two or three dimensional space whose inter-agent distances are all essentially constant over time, at least under ideal conditions. One approach to maintaining such formations is based on the idea of "graph rigidity" [2]. Rigid formations can be "directed" [3], "undirected" [1], or some combination of the two. By an *undirected rigid formation* of mobile autonomous agents is meant a formation based on graph rigidity in which each pair of "neighboring" agents i and j is responsible for maintaining the prescribed target distance d_{ij} between them. Recent research [1], [4] has led to the development of an elegant potential function based theory of formation control which provides gradient laws for asymptotically stabilizing a large class of rigid, undirected formations in two-dimensional space assuming all agents are described by kinematic point models. This particular methodology is perhaps the most comprehensive currently in existence for maintaining undirected formations based on graph rigidity. The aim of this paper is to explain what happens to such formations if neighboring agents i and j have slightly different understandings of what the desired distance d_{ij} between them is suppose to be. The question is relevant because no two positioning controls can be expected to move agents to precisely specified positions because of inevitable imprecision in the physical comparators used to compute the positioning errors. The question is also relevant because it is mathematically equivalent to determining what happens if neighboring agents i and j have differing estimates of what the actual distance between them is. In either case, what one would hope for would be a gradual distortion of the formation from its target shape as discrepancies in desired or sensed distances increase. While this is observed for the gradient laws in question, something else quite unexpected happens at the same time. In particular it turns out for any rigidity-based, undirected formation of the type considered in [1] which is comprised of three or more agents, that if some neighboring agents have slightly different understandings of what the desired distances between them are suppose to be, then almost for certain, the trajectory of the resulting distorted but rigid formation will converge exponentially fast to a closed orbit in $\mathbb{R}^2$ which is then traversed periodically at a single sinusoidal frequency. The aim of this paper is to explain why this is so in the special case of a three agent triangular formation. In a full-length version of this paper it will be shown that this same phenomenon also occurs with any undirected rigid formation in the plane consisting of three or more agents. It will also be shown that in three dimensional space a similar phenomenon occurs where instead of convergence to closed orbits, the trajectory of each distorted but rigid formation converges exponentially fast to a helical orbit determined by a single sinusoidal signal and a constant drift.

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II. TRIANGULAR FORMATIONS

We consider a formation in the plane consisting of three mobile autonomous agents labeled $1,2,3$. We adopt the notation $[1] = 2$, $[2] = 3$ and $[3] = 1$. We assume that the desired *target distance* between agents i and $[i]$ is d_i where d_i is a positive number. We further assume that for $i \in \{1,2,3\}$, agent i is tasked with the job of maintaining the specified target distances to both of its *neighbors*, namely agents $[i]$ and $[[i]]$. However we do not assume that each agent i has exact knowledge of both of the target distances d_i and $d_{[[i]]}$ which it is to maintain. In particular we assume that agent i knows the values d_i exactly but instead of knowing $d_{[[i]]}$, agent i knows an approximately correct target distance $\widehat{d}_{[[i]]}$

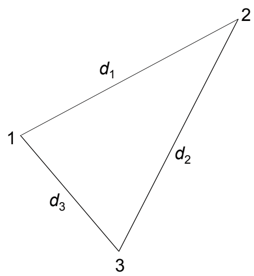

Fig. 1. Undirected Triangular Point Formation

where each $\widehat{d_i}$ is a real number satisfying $|d_i - \widehat{d_i}| \leq \varepsilon_i$ for some small positive number ε_i. Thus $d_{[[i]]} - \widehat{d}_{[[i]]}$ is the discrepancy between agent i's understanding of what the desired distance between itself and neighbor $[[i]]$ is suppose to be and neighbor $[[i]]$'s understanding of what the same desired distance is. We assume for all possible values of the $\widehat{d_i}$ satisfying $|d_i - \widehat{d_i}| \leq \varepsilon_i$, that the following triangle inequalities hold:

$$\widehat{d_i} < \widehat{d}_{[i]} + \widehat{d}_{[[i]]}, \qquad i \in \{1,2,3\} \tag{1}$$

In the sequel we write x_i for the Cartesian coordinate vector of agent i in some fixed global coordinate system in the plane; x then denotes the *multi-point* $x = [x_1' \quad x_2' \quad x_3']'$ in $\mathbb{R}^6$. Let $\mathbb{G}$ denote the simple graph with vertex set $\{1,2,3\}$ and edge set $\{(1,2),(2,3),(3,1)\}$. Note that if x is any multi-point for which $||x_i - x_{[[i]]}|| = \widehat{d_i}$, $i \in \{1,2,3\}$, then $\{x, \mathbb{G}\}$ must be an infinitesimally rigid framework [2]. This is a consequence of the (1).

We assume that agent i's motion is described in global coordinates by a simple kinematic point model of the form

$$\dot{x}_i = u_i, \quad i \in \{1,2,3\} \tag{2}$$

We further assume that for $i \in \{1,2,3\}$, agent i can measure the relative position of its two neighboring agents $[i]$ and $[[i]]$, namely $z_{[[i]]}$ and $-z_i$ where

$$z_i = x_i - x_{[i]}, \quad i \in \{1,2,3\} \tag{3}$$

As controls we consider the feedback laws

$$u_i = -z_i e_i + z_{[[i]]}\widehat{e}_{[[i]]},\ i \in \{1,2,3\}$$

where

$$e_i = ||z_i||^2 - d_i^2,\ i \in \{1,2,3\} \tag{4}$$

and

$$\widehat{e_i} = ||z_i||^2 - \widehat{d}_i^2,\ i \in \{1,2,3\}$$

Note that if the $\widehat{d_i}$ were exactly equal to the d_i, then the $\widehat{e_i}$ would be exactly equal to the corresponding e_i and the preceding controls would be precisely the gradient controls considered in [1]. Application of these perturbed controls to (2) then yields the equations

$$\begin{aligned} \dot{x}_1 &= -z_1 e_1 + z_3 e_3 + z_3 \mu_3 \\ \dot{x}_2 &= -z_2 e_2 + z_1 e_1 + z_1 \mu_1 \\ \dot{x}_3 &= -z_3 e_3 + z_2 e_2 + z_2 \mu_2 \end{aligned} \tag{5}$$

where for simplicity we've set $\mu_i = \widehat{d}_i^2 - d_i^2$ and replaced $\widehat{e_i}$ by $e_i + \mu_i$. In the light of (3) and (4) it is clear that the preceding constitutes a smooth, time-invariant dynamical system of the form

$$\dot{x} = f(x, \mu)$$

where μ is the perturbation vector $\mu = [\mu_1 \quad \mu_2 \quad \mu_3]'$. In the sequel we refer this system as the *overall system*.

A. Error System

Our aim is to study the behavior of the overall system. Towards this end let $z = [z_1' \quad z_2' \quad z_3']'$ and note that

$$z = Lx \tag{6}$$

where L is a 6×6 matrix determined by (3). Note also that

$$z_1 + z_2 + z_3 = 0 \tag{7}$$

and that

$$\dot{z}_1 = -2z_1 e_1 + z_2 e_2 + z_3 e_3 + z_3 \mu_3 - z_1 \mu_1 \tag{8}$$

$$\dot{z}_2 = -2z_2 e_2 + z_1 e_1 + z_3 e_3 + z_1 \mu_1 - z_2 \mu_2 \tag{9}$$

$$\dot{z}_3 = -2z_3 e_3 + z_1 e_1 + z_2 e_2 + z_2 \mu_2 - z_3 \mu_3 \tag{10}$$

These equations and (4) enable one to write

$$\begin{aligned} \dot{e}_1 &= -4||z_1||^2 e_1 + 2z_1' z_2 e_2 + 2z_1' z_3 e_3 + 2z_1' z_3 \mu_3 - 2||z_1||^2 \mu_1 \\ \dot{e}_2 &= -4||z_2||^2 e_2 + 2z_2' z_1 e_1 + 2z_2' z_3 e_3 + 2z_2' z_1 \mu_1 - 2||z_2||^2 \mu_2 \\ \dot{e}_3 &= -4||z_3||^2 e_3 + 2z_3' z_1 e_1 + 2z_3' z_2 e_2 + 2z_3' z_2 \mu_2 - 2||z_3||^2 \mu_3 \end{aligned} \tag{11}$$

A key step in the analysis of the overall system (5) is to show that the error vector $e = [e_1 \quad e_2 \quad e_3]'$ satisfies a differential equation of the form

$$\dot{e} = g(e, \mu) \tag{12}$$

where g is a smooth function of just e and μ and not z. While for more general formations such a differential equation can only be shown to be valid locally, for the simple triangular formation under consideration here one can derive a globally valid model using very elementary reasoning. Here's how. Note that for distinct $i,j,k \in \{1,2,3\}$, $||z_i + z_j||^2 = ||z_i||^2 + ||z_j||^2 + 2z_i' z_j$; from this and (7) it follows that for distinct $i,j,k \in \{1,2,3\}$, $z_i' z_j = \frac{1}{2}(||z_k||^2 - ||z_i||^2 - ||z_j||^2)$. Using these relations, and the fact that $||z_i||^2 = e_i + d_i^2, \quad i \in \{1,2,3\}$, it is clear that the differential equation for e in (11) can be re-written without z in the form shown in (12). We call (12) the *error system*.

The next step in the analysis is to show that for small $||\mu||$, this error system has an exponentially stable equilibrium close to the value $e = 0$. To do this it is clearly enough to prove that the *unperturbed error system*

$$\dot{e} = g(e, 0) \tag{13}$$

has an exponentially stable equilibrium at $e = 0$. To accomplish this, note from (11) that $\dot{e} = 0$ if e and μ are both zero. Thus $e = 0$ is in fact an equilibrium state of the unperturbed error system described by (13). In addition (5) shows that when $e = 0$, the unperturbed overall system $\dot{x} = f(x, 0)$ must be in some equilibrium state in the closed manifold $\{x : e = 0, z = Lx\}$.

To proceed, we need to make use of infinitesimal rigidity [2]. For this, let

$$S(z) = \begin{bmatrix} z_1' & -z_1' & 0 \\ 0 & z_2' & -z_2' \\ -z_3' & 0 & z_3' \end{bmatrix}$$

As is well known, the set of multi-points $x \in \mathbb{R}^6$ for which the framework $\{x, \mathbb{G}\}$ is infinitesimally rigid is the set of multi-points at which the 3×6 *rigidity matrix* $R(x) = S(z)|_{z=Lx}$ attains its maximal rank of 3 [2]. Thus the set of multi-points for which $\{x, \mathbb{G}\}$ is not infinitesimally rigid is precisely the closed manifold $\mathcal{N} = \{x : \text{rank}\, R(x) < 3\}$. In this special case it can easily be shown [3] that $\mathcal{N}$ admits the more explicit description

$$\mathcal{N} = \{x : \text{rank}\, [\, z_1 \quad z_2 \quad z_3 \,] < 2,\ z_1 + z_2 + z_3 = 0, z = Lx\} \tag{14}$$

As we've already noted in the last section, $\{x, \mathbb{G}\}$ is infinitesimally rigid whenever x is such that $e = 0$. Thus the matrix $S(z)$ must have rank 3 whenever z is such that $e = 0$. Thus for such values of z, the 3×3 symmetric matrix $S(z)S'(z)$ must be positive definite. Observe that all nonzero entries in $S(z)S'(z)$ are terms of the form $z_i' z_j$ for $i, j \in \{1, 2, 3\}$. Using the same ideas used to convert (11) into the self-contained error system (12), it is possible to write $S(z)S'(z)$ as a matrix Q depending smoothly on e and only on e. Note that $Q(e)$ is positive semi-definite for all e because $S(z)S'(z)$ is positive semi-definite for all z. In addition $Q(0)$ must be positive definite because $S(z)S'(z)$ is positive definite whenever z is such that $e = 0$. Since $Q(e)$ depends smoothly on e and $Q(0)$ is positive definite, there must be a positive number ρ sufficiently small so that $Q(e)$ is positive definite for all e in the closed bounded set $\mathcal{E} = \{e : ||e||^2 \leq \rho\}$

We are now ready to show that $e = 0$ is an exponentially stable equilibrium of the unperturbed error system (13). Towards this end, let $e(t)$ be the solution to the unperturbed error system starting at any given state $e(0) \in \mathcal{E}$. Let $[0, T)$ denote this solution's maximal interval of existence. It is easy to verify from the equations defining the system (11) with $\mu = 0$, that the rate of change of the function $V = \frac{1}{2}||e||^2$ along this solution must satisfy $\dot{V} = -2e'Q(e)e$. Since $Q(e)$ is at least positive semi-definite no matter what the value of e, V must be non-increasing. Thus $V(t) \leq V(0)$. In view of the definition of V and the assumption that $e(0) \in \mathcal{E}$ it must be true that $V(0) \leq \frac{1}{2}\rho$. This implies that $V(t) \leq \frac{1}{2}\rho$ and thus that $e(t) \in \mathcal{E}$. Thus the solution in question is bounded on $[0, T)$. From this it follows by a standard argument that $T = \infty$. Moreover since $e(t) \in \mathcal{E}$ for all $t \geq 0$, $Q(e)$ must be positive definite all along $e(t)$ for all $t \geq 0$. Thus if for $p \in \mathcal{E}$ we let $\lambda(Q(p))$ denote the smallest eigenvalue of $Q(p)$ and define

$$\bar{\lambda} = 4 \min_{p \in \mathcal{E}} \lambda(Q(p))$$

then $\bar{\lambda} > 0$ and at all points e along the trajectory in question, $-2e'Q(e)e \leq -\frac{\bar{\lambda}}{2}||e||^2$. This means that V must be a bona fide Lyapunov function for the unperturbed error system (13) and moreover $\dot{V} \leq -\bar{\lambda}V$. This clearly implies for the unperturbed error system (13) that any trajectory starting inside of the set $\mathcal{E}$ must approach the equilibrium $e = 0$ as fast as $e^{-\bar{\lambda}t}$ approaches 0. We summarize.

Proposition 1: The equilibrium $e = 0$ of the unperturbed error system (13) is exponentially stable.

We now turn our attention to the perturbed error system (12). We have two points to make. First, as is well known, exponential stability is a robust property with respect to parametric perturbations such as μ. In view of Proposition 1, this means that for any value of μ inside a sufficiently small open ball $\mathcal{B}$ about $\mu = 0$, the perturbed error system must have an exponentially stable equilibrium $\bar{e}(\mu)$ which is close to $e = 0$. Second, as is also well known, infinitesimal rigidity of a framework is a robust property with respect to small perturbations in the framework's edge lengths. Moreover, because of (1) $\{x, \mathbb{G}\}$ is infinitesimally rigid for any $x \in \{x : e = 0\}$ since the edge lengths in this case are d_1, d_2 and d_3. Therefore by picking a sufficiently small neighborhood $\mathcal{B}$ about $\mu = 0$, we can guarantee that for any $\mu \in \mathcal{B}$, $\bar{e}(\mu)$ is an exponentially stable equilibrium of the error system (12) and $\{x, \mathbb{G}\}$ is an infinitesimal rigid framework for every $x \in \{x : e = \bar{e}(\mu)\}$. We summarize:

Proposition 2: There exists an open ball $\mathcal{B}$ about $\mu = 0$ in $\mathbb{R}^3$ and a vector $\bar{e}(\mu) \in \mathbb{R}^3$ depending continuously on $\mu \in \mathcal{B}$ such that $\bar{e}(0) = 0$ and for each $\mu \in \mathcal{B}$, $\bar{e}(\mu)$ is an exponentially stable equilibrium of the perturbed error system (12); moreover $\{x, \mathbb{G}\}$ is an infinitesimally rigid framework for all $x \in \{x : e = \bar{e}(\mu)\}$.

B. The z System

We now turn our attention to the self-contained z system defined by (7) - (10), and $e_i = ||z_i||^2 - d_i^2,\ i \in \{1, 2, 3\}$. Our aim is to study its behavior assuming $\mu \in \mathcal{B}$ and the error system is in an equilibrium state $\bar{e} = \bar{e}(\mu)$. Towards this end, let $\bar{z} = [\bar{z}_1' \quad \bar{z}_2' \quad \bar{z}_3']'$ denote *any* solution to (7) - (10) along which $e = \bar{e}$ and $||\bar{z}_i||^2 = \bar{e}_i + d_i^2, \quad i \in \{1, 2, 3\}$. Our first objective is to show that the $\bar{z}_i$ will be nonconstant for almost every value of $\mu \in \mathcal{B}$. This is a direct consequence of the following result.

Lemma 1: Suppose the system defined by (7) - (10) is in equilibrium at $z = q$ when $e = \bar{e}$. Then either $q \in \{z : \text{rank}\, [\, z_1 \quad z_2 \quad z_3 \,] < 2,\ z_1 + z_2 + z_3\}$ or $\mu_1 + \mu_2 + \mu_3 = 0$.

Proof of Lemma 1: By hypothesis, q is an equilibrium state of the system (7) - (10) when $e = \bar{e}$ so

$$0 = -2q_1\bar{e}_1 + q_2\bar{e}_2 + q_3\bar{e}_3 + q_3\mu_3 - q_1\mu_1 \quad (15)$$

$$0 = -2q_2\bar{e}_2 + q_1\bar{e}_1 + q_3\bar{e}_3 + q_1\mu_1 - q_2\mu_2 \quad (16)$$

$$0 = -2q_3\bar{e}_3 + q_1\bar{e}_1 + q_2\bar{e}_2 + q_2\mu_2 - q_3\mu_3 \quad (17)$$

where $[q_1' \quad q_2' \quad q_3'] = q'$. Suppose $q \notin \{z : \text{rank } [z_1 \quad z_2 \quad z_3] < 2,\ z_1 + z_2 + z_3\}$. From (7) one has $q_3 = -q_1 - q_2$. Substitution into (15) and (16) thus yields

$$0 = -q_1(2\bar{e}_1 + \bar{e}_3 + \mu_1 + \mu_3) + q_2(\bar{e}_2 - \bar{e}_3 - \mu_3)$$

$$0 = -q_2(2\bar{e}_2 + \bar{e}_3 + \mu_2) + q_1(\bar{e}_1 - \bar{e}_3 + \mu_1)$$

In view of (14) and the hypothesis that $q \notin \{z : \text{rank } [z_1 \quad z_2 \quad z_3] < 2,\ z_1 + z_2 + z_3\}$, it must be true that q_1 and q_2 are linearly independent. Therefore

$$2\bar{e}_1 + \bar{e}_3 + \mu_1 + \mu_3 = 0 \quad (18)$$

$$\bar{e}_2 - \bar{e}_3 - \mu_3 = 0 \quad (19)$$

$$\bar{e}_1 - \bar{e}_3 + \mu_1 = 0 \quad (20)$$

$$2\bar{e}_2 + \bar{e}_3 + \mu_2 = 0 \quad (21)$$

Eliminating μ_1 and μ_3 from (18) - (20), one obtains $\bar{e}_1 + \bar{e}_2 + \bar{e}_3 = 0$. This and (21) then yield $\bar{e}_1 - \bar{e}_2 - \mu_2 = 0$. It follows from this, (19) and (20) that $\mu_1 + \mu_2 + \mu_3 = 0$ ■

Fix $\mu \in \mathcal{B}$ and let z denote any solution to (7) - (10). Note that (7) may be used to eliminate z_3 from (8) and (9) thereby obtaining self-contained differential equations for z_1 and z_2. From this it follows that the 2×2 matrix $Z = [z_1 \quad z_2]$ must satisfy a differential equation of the form

$$\dot{Z} = ZA(e, \mu) \quad (22)$$

where $A(e, \mu)$ is a 2×2 matrix depending only on e and μ. Since the $\bar{z}_i$ have been defined to satisfy (7) - (10) with $e = \bar{e}$,

$$\dot{\bar{Z}} = \bar{Z}A(\bar{e}, \mu) \quad (23)$$

where $\bar{Z} = [\bar{z}_1 \quad \bar{z}_2]$. It follows that the Gramian $\bar{Z}'\bar{Z}$ must satisfy

$$\dot{\overline{\bar{Z}'\bar{Z}}} = A'(\bar{e}, \mu)\bar{Z}'\bar{Z} + \bar{Z}'\bar{Z}A(\bar{e}, \mu) \quad (24)$$

In view of Proposition 2, for any x satisfying $||x_i - x_{[i]}||^2 = \bar{e}_i(\mu) + d_i^2,\ i \in \{1, 2, 3\}$, the framework $\{x, \mathbb{G}\}$ is infinitesimally rigid. Infinitesimal rigidity in turn implies that x_1, x_2, x_3 are not collinear points which is equivalent to saying that $x \notin \mathcal{N} = \{x : z = Lx,\ \text{rank } [z_1 \quad z_2 \quad z_3] < 2,\ z_1 + z_2 + z_3 = 0\}$. Thus for any $z_i \in \mathbb{R}^2$ satisfying $||z_i||^2 = \bar{e}_i(\mu) + d_i^2,\ i \in \{1, 2, 3\}$ and $z_1 + z_2 + z_3 = 0$, it must be true that rank $[z_1 \quad z_2 \quad z_3] = 2$. But $z_1 + z_2 + z_3 = 0$ and rank $[z_1 \quad z_2 \quad z_3] = 2$ imply rank $[z_1 \quad z_2] = 2$. In summary, if $||z_i||^2 = \bar{e}_i(\mu) + d_i^2,\ i \in \{1, 2, 3\}$ and $z_1 + z_2 + z_3 = 0$, then rank $[z_1 \quad z_2] = 2$. But by definition, the $\bar{z}_i$ satisfy these conditions so rank $[\bar{z}_1 \quad \bar{z}_2] = 2$. In other words, $\bar{Z}$ must be nonsingular.

Suppose that $\mu_1 + \mu_2 + \mu_3 \neq 0$. Then by Lemma 1, $\dot{\bar{Z}} \neq 0$ because $\bar{z} \notin \{z : \text{rank } [z_1 \quad z_2 \quad z_3] < 2,\ z_1 + z_2 + z_3\}$. Since we've assumed that $||\bar{z}_i||^2 = \bar{e}_i + d_i^2,\ i \in \{1, 2, 3\}$, each $||\bar{z}_i||^2$ must be constant. Moreover the cross product $\bar{z}_1'\bar{z}_2$ must also be constant because $\bar{z}_1'\bar{z}_2 = \frac{1}{2}(||\bar{z}_3||^2 - ||\bar{z}_1||^2 - ||\bar{z}_2||^2)$. Therefore the Gramian $\bar{Z}'\bar{Z}$ must be constant. Thus

$$A'(\bar{e}, \mu)\bar{Z}'\bar{Z} + \bar{Z}'\bar{Z}A(\bar{e}, \mu) = 0$$

because of (24). Clearly

$$(\bar{Z}A(\bar{e}, \mu)\bar{Z}^{-1})' + \bar{Z}A(\bar{e}, \mu)\bar{Z}^{-1} = 0$$

Evidently the 2×2 matrix $\bar{Z}A(\bar{e}, \mu)\bar{Z}^{-1}$ is skew symmetric as is the similar matrix $A(\bar{e}, \mu)$. Therefore the spectrum of $A(\bar{e}, \mu)$ must be $\{j\omega, -j\omega\}$ for some real number $\omega \geq 0$. But $\omega \neq 0$ because $\dot{\bar{Z}} = \bar{Z}A(\bar{e}, \mu)$ and $\dot{\bar{Z}} \neq 0$. Thus $\bar{z}_1$ and $\bar{z}_2$ must be constant norm, sinusoidal vectors varying at a single positive frequency ω. In addition, the same must also be true of $\bar{z}_3$ because $\bar{z}_3 = -\bar{z}_1 - \bar{z}_2$. We are led to the following observation.

Proposition 3: Let μ be a point in $\mathcal{B}$ at which $\mu_1 + \mu_2 + \mu_3 \neq 0$. Suppose the error system is in equilibrium at $e = \bar{e}(\mu)$. Suppose $\bar{z}(t)$ is a solution to (7) - (10) along which $e = \bar{e}(\mu)$ and $||\bar{z}_i||^2 = \bar{e}_i(\mu) + d_i^2,\ i \in \{1, 2, 3\}$. Then there exist phase angles $\phi_i,\ i \in \{1, 2, 3\}$, and a positive frequency ω such that for $i \in \{1, 2, 3\}$

$$\bar{z}_i(t) = (\bar{e}_i(\mu) + d_i^2)^{\frac{1}{2}} [\cos(\omega t + \phi_i) \quad \sin(\omega t + \phi_i)]'$$

Suppose that x is a solution the overall system along which $\bar{z} = Lx$. Then the differential equations (5) which the x_i solve have right hand sides which are bounded, sinusoidal functions of t with average values zero over a period. This means that x too must be bounded and sinusoidal at the same frequency ω. We summarize.

Corollary 1: Let $\bar{z}$ be as in Proposition 3. If x is a solution to the overall system equations (5) along which $e = \bar{e}(\mu)$ and $z = Lx$, then the coordinate vectors $\bar{x}_i,\ i \in \{1, 2, 3\}$ of the three agents in the formation all vary sinusoidally at the same frequency ω as does the rigid triangular formation with edge lengths $(\bar{e}_i(\mu) + d_i^2)^{\frac{1}{2}},\ i \in \{1, 2, 3\}$ which they define.

It can be shown that

$$\omega = \frac{(\sin\theta_1)(\sin\theta_2)(\sin\theta_3)}{2(1 + (\cos\theta_1)(\cos\theta_2)(\cos\theta_3))}(\mu_1 + \mu_2 + \mu_3)$$

where the θ_i are the interior angles of the equilibrium triangle with edge lengths $(\bar{e}_i(\mu) + d_i^2)^{\frac{1}{2}},\ i \in \{1, 2, 3\}$. For small $||\mu||$ this triangle will be approximately the same as the target triangle with edge lengths $d_i,\ i \in \{1, 2, 3\}$ so for small $||\mu||$, ω is approximately equal to a function of just the d_i times the sum of the μ_i. For an equilateral target triangle, $\omega \approx \frac{1}{2\sqrt{3}}(\mu_1 + \mu_2 + \mu_3)$.

III. Main Result

Fix $\mu \in \mathcal{B}$; in view of Proposition 2, the perturbed error system (12) has an exponentially stable equilibrium $\bar{e} = \bar{e}(\mu)$.

There may of course be many solutions $\bar{x}(t)$ to the overall system along which $e=\bar{e}(\mu)$. We refer to each of these as an *equilibrium solution*. Each equilibrium solution $\bar{x}(t)$ determines an *equilibrium trajectory* $\{\bar{x}(t): x\geq 0\}$ which, by Corollary 1, is a closed orbit in $\mathbb{R}^6$ if $\mu_1+\mu_2+\mu_3\neq 0$.

Note that because of the error system's exponential stability, there must be a positive number γ such that any trajectory of the error system starting in the open set $\mathcal{V}=\{e: ||e-\bar{e}||<\gamma\}$ converges to $\bar{e}$ exponentially fast. Let $\mathcal{W}=\{x: e\in\mathcal{V}\}$. Note that $\mathcal{W}$ is an open set containing the equilibrium trajectories of the overall system. Our aim is to show that any trajectory $\{x(t), t\geq 0\}$ of the overall system starting in $\mathcal{W}$ converges exponentially fast to an equilibrium trajectory of the overall system.

Towards this end, let $x(t)$ be any solution to the overall system starting with an initial state $x(0)\in\mathcal{V}$ and let z and e be as defined by (3) and (4) respectively. Then e tends to $\bar{e}(\mu)$ exponentially fast and z is bounded for all time. This implies that $Z=[z_1 \quad z_2]$ is bounded on $[0,\infty)$ and that $A(e,\mu)$ tends to $A(\bar{e},\mu)$ exponentially fast. Note in addition that $e^{A(\bar{e},\mu)t}$ is bounded on the whole real line $(-\infty,\infty)$ because of $A(\bar{e},\mu)$'s spectrum.

Let $\bar{Z}$ be that solution to $\dot{\bar{Z}}=\bar{Z}A(\bar{e},\mu)$ with initial state

$$\bar{Z}(0)=Z(0)+\int_0^{\infty}U(\tau)e^{-A(\bar{e},\mu)\tau}d\tau \qquad (25)$$

where $U=Z(A(e,\mu)-A(\bar{e},\mu))$. Note that U tends to zero exponentially fast because Z is bounded and because $(A(e,\mu)-A(\bar{e},\mu))$ tends to zero exponentially fast. Observe that $\bar{Z}(0)$ exists because $e^{-A(\bar{e},\mu)t}$ is bounded on $[0,\infty)$ and because U tends to zero exponentially fast. We claim that Z converges exponentially fast to $\bar{Z}$ as $t\to\infty$. To understand why this is so, consider the error $E=Z-\bar{Z}$ and note that

$$\dot{E}=EA(\bar{e},\mu)+U$$

By the variation of constants formula

$$E(t)=E(0)e^{A(\bar{e},\mu)t}+\int_0^t U(\tau)e^{A(\bar{e},\mu)(t-\tau)}d\tau$$

In view of (25)

$$E(t)=-\int_t^{\infty}U(\tau)e^{A(\bar{e},\mu)(t-\tau)}d\tau$$

Now since $e^{A(\bar{e},\mu)(t-\tau)}$ is bounded for all t and τ and $U(\tau)$ tends to zero exponentially fast, there must exist positive constants c and λ such that $||U(\tau)e^{A(\bar{e},\mu)(t-\tau)}||\leq ce^{-\lambda\tau}$. Clearly $||E(t)||\leq\int_t^{\infty}ce^{-\lambda\tau}d\tau=\frac{c}{\lambda}e^{-\lambda t}$ so $E(t)\to 0$ as $t\to\infty$ as fast as $e^{-\lambda t}$ does. It follows that Z converges exponentially fast to $\bar{Z}$ as claimed.

Let $\bar{z}_1$ and $\bar{z}_2$ denote the columns of $\bar{Z}$ and define $\bar{z}_3=-\bar{z}_1-\bar{z}_2$. Proposition 3 will apply to the $\bar{z}_i$ provided $||\bar{z}_i(t)||^2=\bar{e}_i+d_i^2,\ i\in\{1,2,3\}$. To show that this is indeed the case, fix $i\in\{1,2,3\}$ and note that $||||z_i||^2-||\bar{z}_i||^2||=||(z_i-\bar{z}_i)'(z_i+\bar{z}_i)||$. Thus by the Cauchy-Schwarz inequality $||||z_i||^2-||\bar{z}_i||^2||\leq||(z_i-\bar{z}_i)||||(z_i+\bar{z}_i)||$ But z_i and $\bar{z}_i$ are bounded signals and $z_i\to\bar{z}_i$ so clearly $||||z_i||^2-||\bar{z}_i||^2||\to 0$ which implies that $||z_i||\to||\bar{z}_i||$. But $||||\bar{z}_i||^2-\bar{e}_i-d_i^2||=||||\bar{z}_i||^2-||z_i||^2+e_i-\bar{e}_i||$ so $||||\bar{z}_i||^2-\bar{e}_i-d_i^2||\leq||||\bar{z}_i||^2-||z_i||^2||+||e_i-\bar{e}_i||$. Since $||||\bar{z}_i||^2-||z_i||^2||$ and $||e_i-\bar{e}_i||$ both converge to zero as $t\to\infty$, $||\bar{z}_i||^2$ must converge to $\bar{e}_i+d_i^2$. But $\bar{z}_i(t)$ has a constant norm so $||\bar{z}_i(t)||^2=\bar{e}_i+d_i^2$ for all $t\geq 0$. Hence Proposition 3 applies so the $\bar{z}_i$ are vector-valued sinusoidal signals with frequency ω.

To conclude we need to construct an equilibrium solution $y(t)$ to the overall system to which x converges. As a first step let us note that the differential equation describing the overall system (5) can be written as $\dot{x}=B(e,\mu)z$ where $B(e,\mu)$ is continuous in e and $z=Lx$. Define

$$y(t)=x(0)+\int_0^{\infty}w(\tau)d\tau+\int_0^t B(\bar{e},\mu)\bar{z}(\tau)d\tau$$

where $w(\tau)=B(e(\tau),\mu)z(\tau)-B(\bar{e},\mu)\bar{z}(\tau)$. The integral $\int_0^{\infty}w(\tau)d\tau$ is well defined and finite because $z-\bar{z}$ and $B(e,\mu)-B(\bar{e},\mu)$ tend to zero exponentially fast. Our goals are to show that $x-y$ converges to zero exponentially fast and also that $y(t)$ is an equilibrium solution to the overall system along which $e=\bar{e}$. To deal with the first issue observe that

$$\dot{y}=B(\bar{e},\mu)\bar{z} \qquad (26)$$

and thus that the error vector $q=x-y$ satisfies $\dot{q}=w(t)$. Thus $q(t)=x(0)-y(0)+\int_0^t w(\tau)d\tau$ so $q=-\int_t^{\infty}w(\tau)d\tau$. Recall that w converges to zero exponentially fast; therefore by the same reasoning which was used to show that $E(t)$ converges to zero exponentially fast, one concludes that q must converge to zero exponentially fast. Thus x converges to y exponentially fast.

In view of (26), to show that y is in fact an equilibrium solution, it is enough to show that $Ly=\bar{z}$. From (26) we know that $L\dot{y}=LB(\bar{e},\mu)\bar{z}$. But $\bar{z}$ is a solution to the z system with $e=\bar{e}$, so $\dot{\bar{z}}=LB(\bar{e},\mu)\bar{z}$. It follows that y and $\bar{z}$ differ by at most a constant vector p. But we've already shown that y converges to x. Since $z=Lx$, it must be that Ly converges to z. But we've also already shown that z converges to $\bar{z}$, so Ly must converge to $\bar{z}$. This means that $p=0$ and thus that $Ly=\bar{z}$. Therefore y is an equilibrium solution of the overall system along which $e=\bar{e}$. We are led to the following theorem which is the main result of this paper.

Theorem 1: Let μ be any perturbation of the overall system (5) with sufficiently small norm. Suppose that $\mu_1+\mu_2+\mu_3\neq 0$. Let $\{x(t): t\geq 0\}$ be any trajectory of the overall system starting in a state for which the corresponding error $e(0)$ is in the domain of attraction of an exponentially stable equilibrium state $\bar{e}(\mu)$ of the error system $\dot{e}=g(e,\mu)$. Then $x(t)$ converges exponentially fast to a closed orbit of the overall system along which $e=\bar{e}(\mu)$.

IV. Concluding Remarks

The robustness issues raised here have broader implications extending well beyond formation maintenance to the entire field of distributed control. In particular, this research illustrates that when assessing the efficacy of a particular distributed control, one must consider the consequences of distinct agents having slightly different understandings of what the values of shared data between them suppose to be. For without the protection of exponential stability/convergence, it is likely that such discrepancies will cause significant misbehavior to occur.

The findings of this paper raise a number of questions. Among these, the most obvious one is this. Is there a way to modify the Krick et al. control laws developed in [1] to eliminate the robustness issue which this paper has uncovered? Given the systematic and broadly applicable nature of the Krick et al. approach, there would seem to be plenty of motivation to seek results which affirmatively answer this question.

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