2012 IEEE International Symposium on Intelligent Control (ISIC)
Part of 2012 IEEE Multi-Conference on Systems and Control
October 3-5, 2012. Dubrovnik, Croatia

Dynamical Behavior, Cost and Magic Number in Average Consensus

Yiming Ji Changbin Yu Brian D.O. Anderson

Abstract— **In this paper, we mainly study the influence of the number of links on the convergence rate and the total number of message exchanges, a surrogate for cost, in average consensus problems. For a wireless sensor network with fixed number of nodes, the number of links is effectively determined by the average vertex degree. Therefore the problem is converted to one of analyzing the influence of the average vertex degree on the convergence rate and the total number of message exchanges. To evaluate the convergence rate we use the ratio of two eigenvalues of the Laplacian matrix of the graph, for which we find lower and upper bounds in terms of the average vertex degree in regular networks, i.e. networks for which the associated graph has the same vertex degree for all vertices. Through theoretical analysis, we first observe that, unsurprisingly, the convergence rate will increase with increase in the number of average vertex degree. However the increment in the convergence rate drops dramatically as the number of average vertex degree becomes progressively larger. At the same time the total number of message exchanges in the consensus process will become large. For such a phenomenon we define a kind of *Magic Number* to help analyze the value or otherwise of adding more links to the network. The Monte Carlo simulation results are consistent with the theoretical analysis and demonstrate the magic number we defined exists not only in regular networks but also in different kinds of networks such as Erdös-Renýi networks and scale-free networks. Further we observe the magic number can be considered as a guide to minimize the total number of message exchanges while achieving a satisfactory convergence rate.**

Key words: Consensus, Convergence rate, Magic number, Wireless sensor networks

I. Introduction

In networked multi-agent systems the consensus problem has been extensively investigated as a distributed way to control coordinated motion [1]. Recently, consensus studies were extended to the sensor networks area [2].

In consensus problems, the convergence rate is an important issue which can be considered to evaluate the performance of the consensus. Many mechanisms have been proposed to enhance the convergence rate see e.g. [3], [4], [5]. Though the convergence rate is a very relevant issue in the consensus problem, it is not the only important factor one should consider. Indeed, in the process of speeding up the convergence rate, some other significant issues may be affected such as cost of message exchanges (one exchange corresponds to two nodes exchanging their values). Intuitively, for a connected sensor network, when the number of sensors is fixed but the edge count is adjustable, the simplest and most direct method to achieve a faster convergence rate is to add more edges into the network[1]. The number of iterations for practical convergence and the convergence rate are inversely related. Therefore the total number of iterations required for practical convergence will decrease with the increase in the number of edges. However each link in a sensor network implies there is a message exchange in each iteration for a consensus problem. Therefore more links will lead to more message exchanges in each iteration. Further, in a given time interval the *total number of message exchanges (TME)* is equal to the product of the total number of iterations and the number of message exchanges in each single iteration. If the convergence rate is not strictly linearly related to the number of edges, the TME may change with the number of edges increasing. Therefore two questions naturally arise:

- **In the process of increasing the convergence rate (by increasing the number of links) will the TME be enlarged, stay the same, or decrease?**
- **Is there any possible method by which we can guarantee minimal TME while maintaining the convergence rate at a certain high level?**

In practical applications, these two questions are also important. In wireless sensor networks, the sensor energy is normally limited, and a small increment in energy cost may cause a sensor power supply to be overloaded and might even cause collapse of the whole network. Yet adding more edges will cost more energy simply to set up the links. Further if the TME is enlarged by using more links, the energy cost will further increase which may overload the burden of a wireless sensor network.

Therefore there is a practical motivation to seek the answers for the above two questions. In this paper, in order to answer these two questions, we investigate the influence of the number of edges on the convergence rate and the TME. For a wireless sensor network with a fixed number of sensors (assuming the weight for each link is equal), the number of edges is strictly linearly proportional to the average vertex degree. Therefore the problem is converted to one of analyzing the influence of the average vertex degree on the convergence rate and TME. To evaluate the convergence rate it is typical [4] to use the ratio of the second smallest eigenvalue and the largest eigenvalue of the

* Y. Ji, C. Yu and B. D.O. Anderson are with Research School of Engineering, The Australian National University, Canberra, ACT, Australia {yiming.ji, brad.yu, brian.anderson}@anu.edu.au

[1]Two types of communication models are presented in current literature [4], [5] to deal the message exchange between nodes . One is a broadcast model, which means all the sensors in the communication range of an emitter can receive its broadcasted messages. The other one envisages existence of an undirected radio link between certain node pairs through which they can exchange messages. Here in this paper, without further notice, we concentrate on the second type. Therefore the TME is proportional to the number of links.

Laplacian matrix of the networks. Lower and upper bounds for the convergence rate are given in the paper in terms of simply the average vertex degree for regular networks. From the theoretical analysis, we know that the convergence rate is not linearly proportional to the number of average vertex degree, and it turns out that the TME may increase after average vertex degree increases to a certain value which we regard as a *Magic Number* (the term has previously been used in studying random communication networks [6]). Monte Carlo simulation results demonstrate the existence of such a magic number not only in regular networks but also in different types of networks which have very general and practical topologies: 1. Random Erdös-Renýi networks [7] 2. Scale-Free networks[8]. Further from the numerical results we obtain the magic number can be taken as a guide for us to minimize the TME while achieving a certain high level of convergence rate.

The rest of the paper is organized as follows. In Section II backgrounds and the problem statement are summarized. The lower and upper bounds for the convergence rate and the definition of the magic number are presented in Section III. Section IV provides the Monte Carlo simulation results. V summarizes the ideas of the paper and comments on future work.

II. Background and the Problem Statement

A. Graph Theory Notations

In this paper, the topology of wireless sensor networks is modeled by undirected graphs. Let $G = (V, E)$ be an undirected graph with the set of nodes $V = \{v_1, v_2, ..., v_N\}$ (N denotes the total number of nodes), and edges the unordered pairs $e = (v_i, v_j) \in E$ ($E \subseteq V \times V$) (simply $e = (i, j)$). Here we also assume the graphs we consider in this paper are *connected* and *simple* (no self-loops or multiple edges). For every graph, there exists an $N \times N$ adjacency matrix A defined as: $a_{i,j} = 1$ if there is an edge between node i and node j. Otherwise $a_{i,j}$ is equal to 0. Let d_i denotes the degree of node i, and defines the degree matrix D to be the $N \times N$ diagonal matrix with i-th diagonal entry d_i.

The Laplacian matrix L of the graph is an $N \times N$ matrix defined as: $L = D - A$

It is well known that L is a symmetric matrix with zero row sums and thus a zero eigenvalue. For a connected graph, all other eigenvalues of L are positive. We suppose that the eigenvalues of the Laplacian matrix L are arranged in an increasing order:

$$0 = \lambda_1(L) < \lambda_2(L) \leq ... \leq \lambda_N(L) \tag{1}$$

B. Consensus Algorithm

In this paper, we concentrate on the average consensus algorithm operating in a distributed manner to determine the average of the N initial values in all the nodes. Assume each node i holds an initial scalar value $x_i(0) \in R$ and $\mathbf{x}(0) = [x_1(0), x_2(0), ..., x_N(0)]^T$ denotes the vector of initial values. The average $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$ is computed via a distributed average consensus algorithm according to the following linear iterations [9]:

$$\mathbf{x}(t+1) = W\mathbf{x}(t) \tag{2}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), ..., x_N(t)]^T$ denotes the vector of values at the iteration numbered t. The matrix W is a weight matrix, with W_{ij} denoting the weight associated with link (i, j), and $x_i(t+1)$ is a linear combination of those $x_j(t)$ for which $j = i$ or node j is a neighbor of node i.

In this paper we postulate an equal weight α for each link. This means according to [9] that

$$W_{ij} = \begin{cases} \alpha & \text{if} \quad e(i,j) \in E \\ 1 - \alpha d_i & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and the optimum value of α, i.e. the value for which convergence is fastest is $\alpha = \frac{2}{\lambda_2(L)+\lambda_N(L)}$. From equation (3) we can express the eigenvalues of W in terms of those of the Laplacian matrix L as follows,

$$\lambda_i(W) = 1 - \alpha\lambda_i(L), \quad i = 1, ..., N \tag{4}$$

where the eigenvalues of W satisfy,

$$1 = \lambda_1(W) > \lambda_2(W) \geq ... \geq \lambda_N(W) \tag{5}$$

Now for any connected graph, we can derive the convergence rate of the above consensus algorithm from equation (2), (3) and (4) and the formula for the optimum weight.

$$\|\mathbf{x}(T) - \bar{\mathbf{x}}\| \leq \|\mathbf{x}(0) - \bar{\mathbf{x}}\| \, \lambda_2^T(W) \tag{6}$$

where

$$\lambda_2(W) = 1 - \frac{2\lambda_2(L)}{\lambda_2(L) + \lambda_N(L)} \tag{7}$$

and T denotes the number of iterations for which $\lambda_2(W)^T \geq$ 0.5 or 0.1 or 0.01, etc. according to the practical requirements associated with defining when convergence has occurred.

C. Problem Statement

For a connected graph with N nodes and an average vertex degree $\bar{d}$ we can express the number of edges in terms of N and $\bar{d}$ (assuming the weight for each edge is equal): $|E| = \frac{\bar{d}N}{2}$. Therefore investigating the influence of the number of edges now is equivalent to investigating the influence of the average vertex degree $\bar{d}$.

Now consider the TME. For a network , one link denotes a message exchange process in each single iteration. Therefore the TME can be formulated as follows,

$$\text{TME}_{G(V,E)} = |E|T \tag{8}$$

Since T is inversely related to the convergence rate, in a network with fixed N, we have

$$\text{TME}_{G(V,E,\bar{d})} \sim \frac{\bar{d}}{\text{CR}_{\bar{d}}} \tag{9}$$

where $\text{CR}_{\bar{d}}$ denotes the convergence rate when the average degree is equal to $\bar{d}$.

Next given a required convergence threshold ρ ($0 < \rho < 1$) (a number for which $\|\mathbf{x}(T) - \bar{\mathbf{x}}\| \leq \rho \|\mathbf{x}(0) - \bar{\mathbf{x}}\|$) according to practical applications, from equation (6) we can define the convergence time T as follows:

$$T = \frac{|\ln \rho|}{-\ln \lambda_2(W)} \tag{10}$$

From the above equation, we can consider $-\ln \lambda_2(W)$ as the effective convergence rate in average consensus problems. Further from equation (7), we can obtain:

$$\begin{aligned} \text{CR}_{\bar{d}} &= -\ln \lambda_2(W) \\ &= \ln \left[\frac{1+\frac{\lambda_2(L)}{\lambda_N(L)}}{1-\frac{\lambda_2(L)}{\lambda_N(L)}} \right] \end{aligned} \tag{11}$$

From equation (11), we conclude that in order to obtain the optimal convergence rate over all possible networks within a certain class, we can use the ratio $\frac{\lambda_2(L)}{\lambda_N(L)}$ to evaluate the convergence rate: the larger the ratio is, the faster the convergence rate will be. The aim of our research work is to investigate the relationship between the number of links and the convergence rate. Therefore now our main issue is converted to investigate the relationship between the average vertex degree $\bar{d}$ and $\frac{\lambda_2(L)}{\lambda_N(L)}$.

III. MAGIC NUMBER IN CONSENSUS PROBLEM

From Section II, we know that a key goal is to find the relationship between $\bar{d}$ and $\frac{\lambda_2(L)}{\lambda_N(L)}$. However to analytically relate the ratio $\frac{\lambda_2(L)}{\lambda_N(L)}$ to the average vertex degree $\bar{d}$ is almost impossible. Instead, here we investigate an upper bound and a lower bound for the $\frac{\lambda_2(L)}{\lambda_N(L)}$ related to $\bar{d}$. To begin the work, here we focus on the simplified case of regular networks in which each vertex has the same degree $\bar{d}$.

A. An Upper Bound for $\frac{\lambda_2(L)}{\lambda_N(L)}$

In order to obtain an upper bound for the ratio $\frac{\lambda_2(L)}{\lambda_N(L)}$ in $\bar{d}$-regular networks, we will proceed by finding an upper bound for $\lambda_2(L)$ and a lower bound for $\lambda_N(L)$.

First from [10], we have the following theorem:

Theorem 1 Let $G = G_{N,d}$ be a $\bar{d}$-regular graph on N vertices. Denote by $\lambda_A(G)$, the absolute value of the largest eigenvalue (in absolute value) of the adjacency matrix A, which is distinct from $\pm\bar{d}$. Then

$$\liminf_{N\to\infty} \lambda_A(G) \geq 2\sqrt{\bar{d}-1} \tag{12}$$

For non-bipartite connected regular graphs on which we concentrate in this paper, we can arrange the eigenvalues of the adjacency matrix A as $\bar{d} = \lambda_1(A) > \lambda_2(A) \geq ... \geq \lambda_N(A) > -\bar{d}$. Further for $\bar{d}$-regular graphs, the eigenvalues of the adjacency matrix A and Laplacian matrix L have the following relationship,

$$\forall n \in [1,2,3,...,N]: \quad \lambda_n(L) = \bar{d} - \lambda_n(A) \tag{13}$$

Therefore from equation (12) and (13) we can obtain an upper bound on the second smallest eigenvalue of L in $\bar{d}$-regular graphs as follows,

$$\liminf_{N\to\infty} \lambda_2(L) \leq \bar{d} - 2\sqrt{\bar{d}-1} \tag{14}$$

Next from [11] we can obtain a well-known lower bound for λ_N in $\bar{d}$-regular graphs.

$$\lambda_N(L) \geq \bar{d} \tag{15}$$

Therefore we can finally obtain an upper bound for convergence rate in $\bar{d}$-regular graphs via combining equations (14) and (15).[2]

$$\liminf_{N\to\infty} \frac{\lambda_2(L)}{\lambda_N(L)} \leq \frac{\bar{d}-2\sqrt{\bar{d}-1}}{\bar{d}} \tag{16}$$

B. A Lower Bound for $\frac{\lambda_2(L)}{\lambda_N(L)}$

In order to obtain a lower bound for the convergence rate in regular networks, we proceed by finding a lower bound for $\lambda_2(L)$ and an upper bound for $\lambda_N(L)$.

From [12] we obtain the following theorem:

Theorem 2 Consider the $\bar{d}$-regular graphs where $\bar{d}$ is even. For N and $\bar{d}$ large, with high probability[3] the second smallest and the largest eigenvalues of the Laplacian matrix satisfy

$$\lambda_2(L) > \bar{d} - 2\sqrt{\bar{d}-1} - \log \bar{d} \tag{17}$$

$$\lambda_N(L) < \bar{d} + 2\sqrt{\bar{d}-1} + \log \bar{d} \tag{18}$$

Therefore we can obtain a lower bound for the ratio $\frac{\lambda_2(L)}{\lambda_N}(L)$ in $\bar{d}$-regular graphs for large N and $\bar{d}$ ($\bar{d}$ is even):

$$\frac{\lambda_2(L)}{\lambda_N(L)} > \frac{\bar{d}-2\sqrt{\bar{d}-1}-\log\bar{d}}{\bar{d}+2\sqrt{\bar{d}-1}+\log\bar{d}} \tag{19}$$

C. Magic Number for Convergence Rate

From subsection A and B, we obtain the upper and lower bounds for the ratio $\frac{\lambda_2(L)}{\lambda_N(L)}$ in $\bar{d}$-regular networks; these two bounds depend only on the vertex degree $\bar{d}$. Now by using the equation (11) and inequalities (16) and (19), we can obtain the following inequality,

$$\ln \left[\frac{\bar{d}}{2\sqrt{\bar{d}-1}+\log\bar{d}} \right] < \text{CR}_{\bar{d}} \leq \ln \left[\frac{\bar{d}-\sqrt{\bar{d}-1}}{\sqrt{\bar{d}-1}} \right] \tag{20}$$

where the vertex degree $\bar{d}$ is even only.

Figure 1 illustrates the upper and lower bounds for the convergence rate related to the vertex degree $\bar{d}$ increasing from 6 to 100 in steps of 2[4]. From the figure we can see that both the upper and lower bounds increase very rapidly with the vertex degree increasing at first. After a certain period, the increment of the two bounds will become much smaller. This phenomenon demonstrates that the convergence rate does not linearly increase with the vertex degree.

According to the equation (9) we can conclude the following fact: each time the average vertex degree $\bar{d}$ increases by

[2] Though the bound is asymptotic in N, it proves informative also for finite N.

[3] The precise probability is given in [13]. In this paper we assume these two bounds are always valid without further comment. Simulations evidence is consistent with the reasonableness of this assumption.

[4] Because the lower bound for the convergence rate is only for even degrees, the step is set as 2.

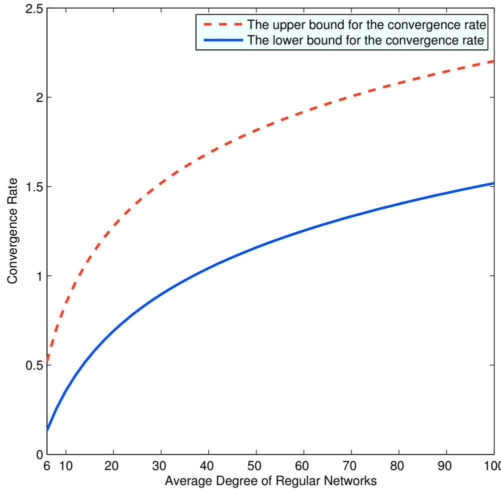

Fig. 1. The upper and lower bounds for the convergence rate in regular networks

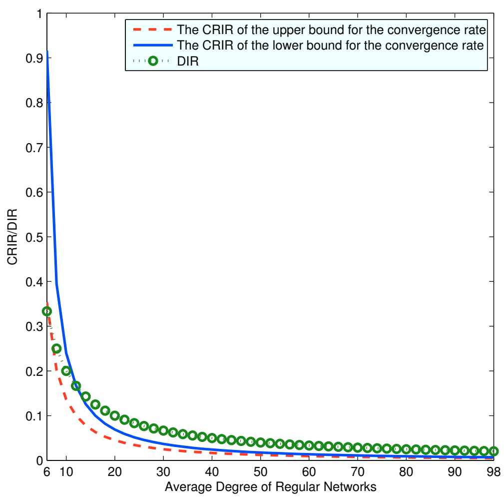

Fig. 2. The $\text{CRIR}_{\bar{d}}$ of the upper and lower bounds of the convergence rate and the $\text{DIR}_{\bar{d}}$ in regular networks

$\delta\bar{d}$, only if the convergence rate increases by at least $\frac{\delta\bar{d}}{\bar{d}} \times \text{CR}_{\bar{d}}$ will the TME not be enlarged.

Therefore it is important to investigate the increment of the convergence rate while the degree $\bar{d}$ increases in the steps of $\delta\bar{d}$. To this end we define the convergence rate increment ratio ($\text{CRIR}_{\bar{d}}$) and the average vertex degree increment ratio ($\text{DIR}_{\bar{d}}$) as follows:

$$\text{CRIR}_{\bar{d}} = \frac{\text{CR}_{\bar{d}+2} - \text{CR}_{\bar{d}}}{\text{CR}_{\bar{d}}} \tag{21}$$

$$\text{DIR}_{\bar{d}} = \frac{(\bar{d} + \delta\bar{d}) - \bar{d}}{\bar{d}} = \frac{\delta\bar{d}}{\bar{d}} \tag{22}$$

When the average vertex degree $\bar{d}$ increases in steps of $\delta\bar{d}$, only if $\text{DIR}_{\bar{d}} \leq \text{CRIR}_{\bar{d}}$ the TME will not be enlarged.

Figure 2 depicts the increment ratio of the upper and lower bounds for the convergence rate and the average vertex degree increment ratio with the average vertex degree $\bar{d}$ increasing in steps of 2. From figure 2 we obtain that the $\text{CRIR}_{\bar{d}}$ of the two bounds decreases very rapidly at first. Further we can observe both the lower and upper bound $\text{CRIR}_{\bar{d}}$ curves intersect with the $\text{DIR}_{\bar{d}}$ curve. Assume the degrees corresponding to the intersections are d_u and d_l for the lower and upper bounds. When the vertex degree $\bar{d}$ is smaller than d_u (or d_l), the $\text{DIR}_{\bar{d}}$ is always smaller than $\text{CRIR}_{\bar{d}}$ of the upper bound (or the lower bound) for the convergence rate. Thus by increasing the vertex degree $\bar{d}$, the convergence rate will increase without bringing extra message exchanges. However when the vertex degree $\bar{d}$ exceeds d_u (or d_l), though the convergence rate can still be improved via increasing the vertex degree $\bar{d}$ the TME will also increase. Therefore at that time, we need to consider whether it is still desirable to increase the vertex degree $\bar{d}$.

Based on the above analysis, we suggest it may make sense to define a magic number k for the average degree $\bar{d}$ in a connected network which is *independent of the number of sensors in the network*:

- The convergence rate can be improved without increasing the TME when the average vertex degree $\bar{d}$ is smaller than a number k which is independent of the number of nodes. When the average vertex degree $\bar{d}$ exceeds the number k, though the convergence rate will continue increasing with further increase in $\bar{d}$ the TME will also be enlarged. This number k is the magic number.

In other words, when the average degree is equal to the magic number k, the TME achieves the minimal value.

In the next section, we will use numerical results to demonstrate the validity of the existence of the magic number not only in regular networks but also in other types of networks. Moreover if we can demonstrate that when $\bar{d} = k$, by further increasing the degree the convergence rate will not be improved much (which means the $\text{CRIR}_{\bar{d}}$ becomes small, such as 0.2), we can consider the convergence rate has been improved to a certain high level. At that time the two questions we proposed at first are answered.

IV. Simulation Results

In this subsection, we will use the Monte Carlo simulation results to demonstrate that the magic number exists in different types of networks.

A. Assumptions

- Three types of networks are considered here: 1. Regular networks 2. Erdös-Renýi networks [7] 3. Scale-Free networks [8].
- All networks are *connected* and *simple* (no self-loops or multiple edges)
- The sizes of the graphs are set as 200, 500 and 1000 nodes[5].
- The vertex degree $\bar{d}$ of regular network is adjusted from 6 to 100 in steps of 2 ($\delta\bar{d} = 2$). For Erdös-Renýi networks and Scale-Free networks, $\bar{d}$ now denotes the average degree and is changed from 8 to 90 in steps of 2.

[5]Due to the space limitation, we do not show the simulation results for other sizes of the networks. However all the results remain essentially the same which means the magic number is independent of the number of nodes.

- In the simulations, we fix the number of nodes N, and for each average degree (hence corresponding to a fixed number of edges $N * \bar{d}$), we generate 1000 topologies for each of the three types of networks and calculate their $\mathrm{CRIR}_{\bar{d}}$ respectively
- The $\mathrm{CRIR}_{\bar{d}}$ and $\mathrm{DIR}_{\bar{d}}$ are defined in the equation (21) and (22)

B. Numerical Results

Figure 3 to Figure 5 depict the simulation results for the $\mathrm{CRIR}_{\bar{d}}$ in the different networks with different number of nodes. First we can see that the two bounds for the convergence rate in regular networks are valid everywhere since the simulation curves always lie between them (shown in Figure 3). Further we can observe the general shapes of the simulation curves for different networks are all similar, in that the $\mathrm{CRIR}_{\bar{d}}$ drops dramatically while the average vertex degree $\bar{d}$ increases. Further we can observe that the $\mathrm{CRIR}_{\bar{d}}$ curves always intersect the $\mathrm{DIR}_{\bar{d}}$ curve. Table I shows for different networks the average vertex degree corresponding to the intersection points which is the magic number k. *From the table we see that the magic number k is always equal to the same value for a given type of network and is independent of the number of nodes. Hence the table can be used to formulate a constraint on a network of prescribed type so that it has minimal TME. Further we can observe that $CRIR_k$ is always around 0.1 when the average vertex degree achieves the magic number in scale-free networks (0.18 for Erdös-Renýi networks and 0.25 for regular networks). This means when the TME becomes minimal the $CRIR_k$ is also small. In other words, at that time, by further increasing the average degree the convergence can not be improved significantly. Therefore we can consider the convergence rate has already achieved a satisfactory level. This observation further demonstrates the significance of the magic number.*

In some practical applications, knowing the magic number can be very useful. For example, there are wireless sensor networks to which the energy is of high concern. Node in such networks typically relays on dedicated temporary links on on-and-off basis with selected neighbors, rather than keeping awake and communicating on broadcast basis with all available neighbors within the communication range. In such cases, it is crucial then to know how many links are really necessary to execute an average consensus algorithm with optimal total energy consumption (in part measured by the TME) while maintaining a satisfactory convergence rate. The magic number (and the associated figures for different networks) exactly serve this purpose. When the wireless sensor network implements an average consensus algorithm, the magic number provides a guidance that each node only needs approximately k neighbors or equivalently links (assuming the network is connected). At that time, it may be that less links are required than could be established given the communication range, and minimal TME can be achieved resulting in less energy cost while a satisfactory convergence rate can be achieved.

V. Conclusion

In this paper, we analyze the influence of the number of edges on the convergence rate and total number of message exchanges in average consensus problems. In the theoretical analysis, we restrict the underlying graphs of the networks to regular graphs in order to simplify the problem for investigating the relationship between the average vertex degree and the convergence rate. Here we use the ratio of the second smallest and the largest eigenvalues of the Lapalacian matrix of the graph to evaluate the convergence rate. We find a lower bound and an upper bound for the convergence rate in regular graphs related only to the vertex degree $\bar{d}$. From the analysis of the two bounds we observe that by increasing the degree the convergence rate will increase. However the convergence rate increment ratio will decrease significantly while the vertex degree becomes progressively larger. Meanwhile the total number of message exchanges will be enlarged when the vertex degree increases. Due to the energy limitation in wireless sensor networks, any increase of energy cost should be taken in to consideration. Therefore we define an index to quantitatively analyze the influence of average vertex degrees increasing in networks in order to help one decide how many links are desirable for a given sensor. This leads us to conclude that there is a *Magic Number*, i.e. optimum node degree, independent of network size. Further, Monte Carlo simulation results demonstrate the magic number we defined exists not only in regular networks but also in other types of networks such as Erdös-Renýi networks and scale-free networks. We verify too from the Monte Carlo simulation results that when the magic number is achieved, the convergence rate also attains a high level which from another point of view demonstrates the significance of the magic number.

One goal of our future work is to theoretically analyze the existence of the magic number in different types of networks than regular graphs. Further from the analysis on the magic number presented in the paper , we can conclude that the problem we discussed is an optimization problem minimizing the total message exchanges of the whole network. In the future work, we aim to cast the problem as one of finding the minimal total message exchanges or other communication cost as a semidefinite program (SDP). Therefore by solving the SDP, we can obtain the accurate magic number for different networks, something which is very useful in a network design.

Acknowledgment

Y. Ji, C. Yu and B.D.O. Anderson are supported by the Australian Research Council under DP-110100538. C. Yu is an ARC Queen Elizabeth II Fellow. B.D.O. Anderson and Y. Ji are also supported by National ICT Australia-NICTA.

References

[1] R. Wei, R.W. Beard and E.M. Atkins, "A Survey of Consensus Problems in Multi-Agent Coordination" *2005 American Control Conference (ACC 2005)*, vol. 3, 2005, pp. 1859-1864

[2] L. Xiao, S. Boyd and S. Lall, "A Scheme for Robust Distributed Sensor Fusion Based on Average Consensus" *Fourth International Symposiun on Information Processing in Sensor Networks (IPSN 2005)*, Apr., 2005

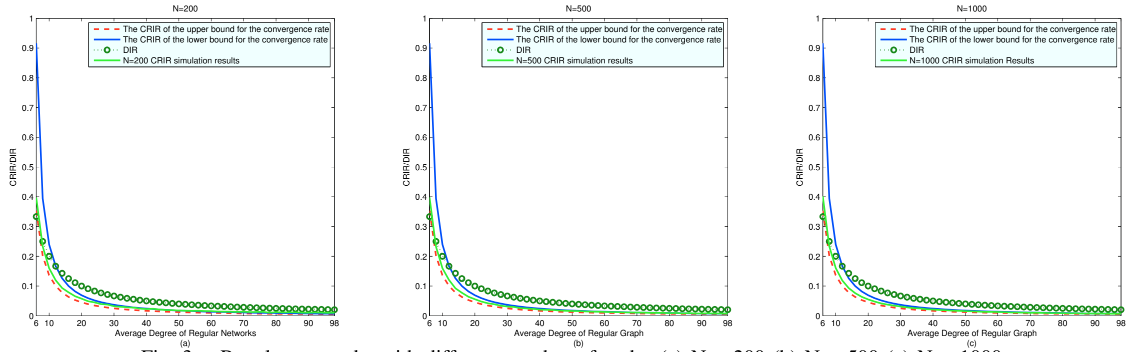

Fig. 3. Regular networks with different number of nodes (a) N = 200 (b) N = 500 (c) N = 1000

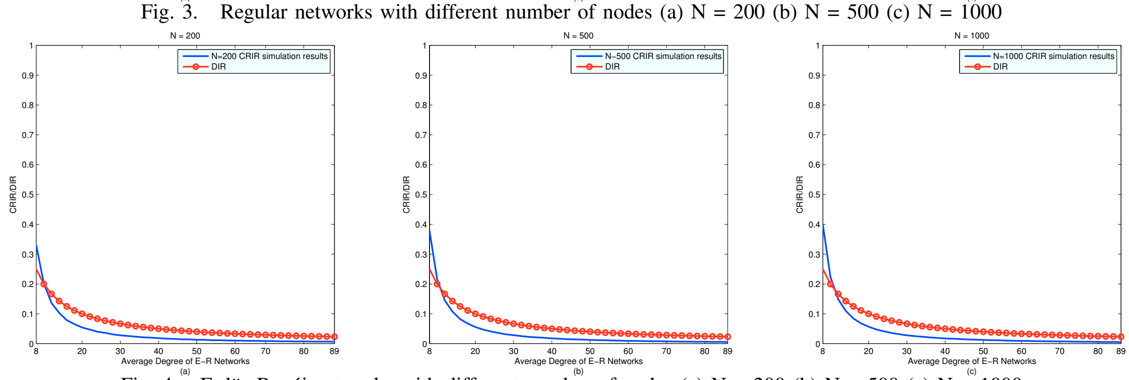

Fig. 4. Erdös-Renýi networks with different number of nodes (a) N = 200 (b) N = 500 (c) N = 1000

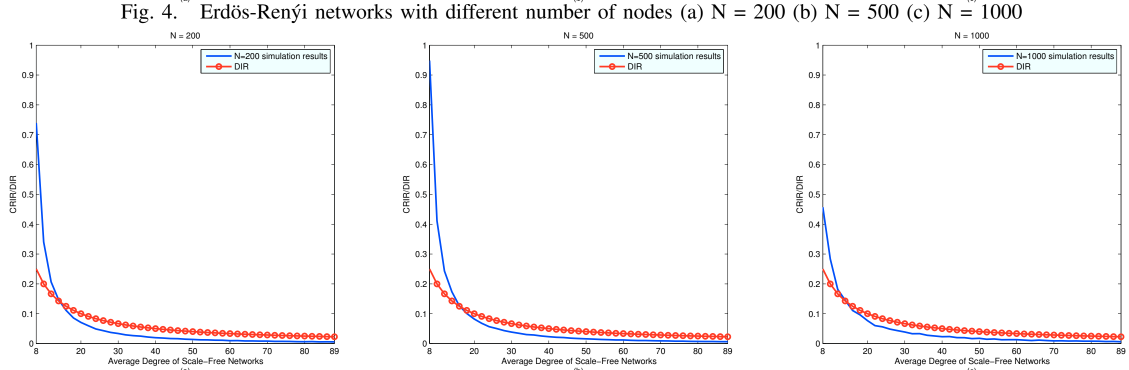

Fig. 5. Scale-free networks with different number of nodes (a) N = 200 (b) N = 500 (c) N = 1000

Magic Number	Regular Networks	E-R Networks	Scale-Free Networks
N=200	8	12	16
N=500	8	12	16
N=1000	8	12	16

TABLE I

THE MAGIC NUMBER FOR NETWORKS OF DIFFERENT NUMBER OF NODES

[3] S. Sardellitti, M. Giona and S. Barbarossa, ”Fast Distributed Average Consensus Algorithms Based on Advection-Diffusion Processes”, *IEEE Transaction on Signal Processing*, vol. 58(2), Feb., 2010, pp. 826-842

[4] S. Kar, S. Aldosari and J. M.F. Moura, ”Topology for Distributed Inference on Graphs”, *IEEE Transaction on Signal Processing*, vol. 56(6), Jun., 2008, pp. 2609-2613

[5] C. Asensio-Marco and B. Beferull-Lozano, ”Accelerating Consensus Gossip Algorithms: Sparsifying Networks Can be Good for You.”, ÎEEE International Conference on Communication (ICC), pp. 1-5, 2010

[6] T. C. Hou and V. O. K. Li, ”Transmission Range Control in Multihop Packet Radio Networks”, ÎEEE Transaction on Communication, vol COM-34, Jan., 1986, pp. 38-44

[7] B. Bollobás, ”Random Graphs(2nd edition)”, *Cambridge University Press*, 2001

[8] A. Clauset, C. R. Shalizi and M. E. J. Newman, ”Power-Law Distributions in Empirical Data”, *SIAM*, Review 51:661-703, 2009

[9] L. Xiao and S. Boyd, ”Fast Linear Iteration for Distributed Averaging”, *Systems & Control Letter*, vol. 53, pp. 65-78, 2004

[10] N. Alon, ”Eigenvalues and Expanders”, *Combinatorica*, vol. 6, 1986, pp. 83-96

[11] R. Merris, ”Laplacian Matrices of a Graph: a Survey”, *Linear Algebra and its Applications*, vol. 197, 1994, pp. 143-176

[12] C.W. Wu, ”Synchronization in Complex Networks of Nonlinear Dynamical Systems”, 2007 by *World Scientific Publishing Co. Pte. Ltd*

[13] J. Friedman, J. Kahn and E. Szemerédi, ”On the Second Eigenvalue in Random Regular Graphs”, *Proceeding of the 21st Annual ACM Symposium on Theory of Computing*, 1989, pp. 587-598