On Zeros of Tall, Fat and Square Blocked Time-Invariant Systems [*] [†] [‡]

Mohsen Zamani [§], ***Brian D. O. Anderson*** [¶], ***Uwe Helmke*** [‖], ***and Weitian Chen*** [**]

1 Introduction

The well-known technique of blocking or lifting has been used in systems and control [5], [2]. This method has mostly been exploited to transform linear discrete-time periodic systems to linear time-invariant systems so that the well-developed tools for linear time-invariant systems can be extended for design and analysis of linear discrete-time periodic systems [3], [1] and [2]. For example, the authors in [3] have extended the notions of poles and zeros of linear time-invariant systems to linear periodic systems. Some necessary and sufficient conditions for structural properties such as observability and reachability have been studied in [7]. The blocking technique has been applied to linear time-invariant systems as well, see e.g. [5], [10] and the references therein. For instance, in [10], linear time-invariant

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[§] M. Zamani is with Research School of Engineering, Australian National University, Canberra, ACT 0200, Australia. mohsen.zamani@anu.edu.au.

[¶] B. D. O. Anderson is with Research School of Engineering, Australian National University, Canberra, ACT 0200, Australia. B. D. O. Anderson is with National ICT Australia (NICTA). brian.anderson@anu.edu.au.

[‖] U. Helmke is with Department of Mathematics, University of Würzburg, 97074 Würzburg, Germany. helmke@mathematik.uni-wuerzburg.de.

[**] W. Chen is with Department of Electrical and Computer Engineering, Faculty of Engineering, University of Windsor, Windsor, ON N9B 3P4, Canada. cwt6122@uwindsor.ca.

systems have been blocked for the purpose of designing periodic controllers while the authors in [5] have performed the blocking technique on linear time-invariant systems for the purpose of dealing with multirate sampled-data systems.

In this paper, we examine the zero properties of the blocked systems resulting from blocking of linear time-invariant systems. This study is motivated from both application and theoretical perspectives. As mentioned above, the blocking of linear time-invariant systems is useful in the multirate sampled-data systems controller design as shown by [5] and [10]. Furthermore, from a theoretical perspective, the pole properties of the blocked systems are well understood [2] and [10], whereas less little known about the zero properties of the blocked systems. In our previous works [6] and [12], we have introduced some important results about the zero properties of blocked systems. For instance, in [6] matrix fraction descriptions (MFDs) have been used to establish a relation between the zero properties of blocked systems and the zero properties of their corresponding unblocked systems. Moreover, in [12], the time domain approach has been exploited to explore the zero properties of blocked systems. In both [6] and [12] only *tall* blocked systems i.e. blocked systems with more whose outputs than inputs, have been considered. Furthermore, in [6] and [12] only blocked systems for which their associated transfer functions have *full-column normal rank*, have been studied.

In this paper, we generalize the results of [6] and [12]. The zero properties of a general blocked system are studied. Here, *there exists no assumption such as tallness or fatness on the structure of blocked systems. Furthermore, we relax the assumption put by [6] and [12] on the normal rank of the transfer function associated with the blocked system.* In the current paper, the normal rank of the transfer function associated with the blocked system can either be equal to the minimum of number of its rows and columns or be less than that minimum value.

2 Blocked Systems and Unblocked Systems

2.1 State space approach

The linear time-invariant unblocked system under consideration is described as

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \end{aligned} \tag{1}$$

where $k \in \mathbb{Z}$, $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$ and $u_k \in \mathbb{R}^m$. Also, the transfer function associated with system (1) is defined as

$$W(z) = D + C(zI - A)^{-1}B, \tag{2}$$

where z is a forward shift operator i.e. $zu_k = u_{k+1}$ and $zy_k = y_{k+1}$, and also represents a complex number.

Now we define

$$\begin{aligned} U_k &= \begin{bmatrix} u_k^T & u_{k+1}^T & \dots & u_{k+N-1}^T \end{bmatrix}^T, \\ Y_k &= \begin{bmatrix} y_k^T & y_{k+1}^T & \dots & y_{k+N-1}^T \end{bmatrix}^T, \end{aligned} \tag{3}$$

where $k = 0, N, 2N, \ldots$.

Then the blocked system is given by [3]

$$\begin{aligned} x_{k+N} &= A_b x_k + B_b U_k, \\ Y_k &= C_b x_k + D_b U_k, \end{aligned} \tag{4}$$

where

$$\begin{aligned} A_b &= A^N, \quad B_b = \left[\begin{array}{cccc} A^{N-1}B & A^{N-2}B & \ldots & B \end{array} \right], \\ C_b &= \left[\begin{array}{cccc} C^T & A^T C^T & \ldots & A^{(N-1)^T} C^T \end{array} \right]^T, \\ D_b &= \left[\begin{array}{cccc} D & 0 & \ldots & 0 \\ CB & D & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \ldots & D \end{array} \right]. \end{aligned} \tag{5}$$

An operator Z is defined such that $Zx_k = x_{k+N}$, $ZU_k = U_{k+N}$, $ZY_k = Y_{k+N}$. The symbol Z is also used to denote a complex value. Then the transfer function of (4) is denoted by

$$V(Z) = D_b + C_b(ZI - A_b)^{-1} B_b. \tag{6}$$

Furthermore, it is worthwhile remarking that the unblocked system (1) is a minimal realization of $W(z)$ if and only if the blocked system (4) is a minimal realization of $V(Z)$ [6].

2.2 Transfer function description

In the previous subsection the state space representation for both unblocked and blocked systems was recalled [10], [4] and [6]. The aim of this subsection is to recall a relation between $V(Z)$ and $W(z)$. The well-known result of [2], [10] is summarized as the theorem below.

Theorem 1. *Consider the unblocked system (1) with transfer function $W(z)$ and the blocked system (4) with transfer function $V(Z)$. Then*

$$V(Z) = \left[\begin{array}{ccccc} V_1(Z) & Z^{-1}V_N(Z) & Z^{-1}V_{N-1}(Z) & \cdots & Z^{-1}V_2(Z) \\ V_2(Z) & V_1(Z) & Z^{-1}V_N(Z) & \ddots & Z^{-1}V_3(Z) \\ V_3(Z) & V_2(Z) & V_1(Z) & \ddots & Z^{-1}V_4(Z) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ V_N(Z) & V_{N-1}(Z) & V_{N-2}(Z) & \cdots & V_1(Z) \end{array} \right] \tag{7}$$

and

$$W(z) = V_1(z^N) + z^{-1}V_2(z^N) + \cdots + z^{-(N-1)}V_N(z^N). \tag{8}$$

where $V_1(Z) = D + C(ZI - A^N)^{-1}A^{N-1}B$ *and* $V_l(Z) = CA^{l-2}B + C(ZI - A^N)^{-1}A^{N+l-2}B, l = 2, \cdots, N.$

Another important result regarding the relation between $V(Z)$ and $W(z)$ is recorded in [4], [6]. Assume that the transfer function of the unblocked system (1) is denoted by a left coprime matrix fraction description (MFD) as

$$W(z) = P^{-1}(z)Q(z), \tag{9}$$

where

$$P(z) = P_\mu + P_{\mu-1}z + \cdots + P_0 z^\mu,\ Q(z) = Q_\mu + Q_{\mu-1}z + \cdots + Q_0 z^\mu. \tag{10}$$

where μ is defined so that P_0 and Q_0 are not both zero. By coprimeness, P_μ and Q_μ are not both zero. Then it can be easily shown that associated with the blocked system there exists a transfer function with a left matrix fraction description as below

$$Y_k = V(Z)U_k,\ V(Z) = \mathcal{A}^{-1}(Z)\mathcal{B}(Z), \tag{11}$$

where

$$\begin{aligned} \mathcal{A}(Z) &= \mathcal{A}_0 + \mathcal{A}_1 Z + \cdots + \mathcal{A}_\alpha Z^\alpha + \mathcal{A}_{\alpha+1}Z^{\alpha+1}, \\ \mathcal{B}(Z) &= \mathcal{B}_0 + \mathcal{B}_1 Z + \cdots + \mathcal{B}_\alpha Z^\alpha + \mathcal{B}_{\alpha+1}Z^{\alpha+1}, \end{aligned} \tag{12}$$

where α is the greatest integer less than μ/N and $\mathcal{A}_i$, $\mathcal{B}_i$ $i \in \{0, 1, \ldots, \alpha+1\}$ are constant coefficient matrices of size $N(p \times m)$ obtained by a certain procedure from the coefficient matrices P_i, Q_i, $i \in \{0, 1, \ldots, \mu\}$, respectively [6].

In the above, we related $V(Z)$ and $W(z)$. However, by using the above calculation relating the $\mathcal{B}_i$ to Q_l, we are able to relate $\mathcal{B}(Z)$ and $Q(z)$ as well.

Lemma 2. *For a nonzero complex number* Z_0*, let* $z_i, i = 1, 2, \cdots, N$ *be* N *distinct complex numbers such that* $z_i^N = Z_0, i = 1, 2, \cdots, N.$

$$\Upsilon = \begin{bmatrix} I_m & I_m & \cdots & I_m \\ z_1 I_m & z_2 I_m & \cdots & z_N I_m \\ \vdots & \ddots & \ddots & \vdots \\ z_1^{N-1} I_m & z_2^{N-1} I_m & \cdots & z_N^{N-1} I_m \end{bmatrix}, \Lambda = \begin{bmatrix} Q(z_1) & Q(z_2) & \cdots & Q(z_N) \\ z_1 Q(z_1) & z_2 Q(z_2) & \cdots & z_N Q(z_N) \\ \vdots & \ddots & \ddots & \vdots \\ z_1^{N-1} Q(z_1) & z_2^{N-1} Q(z_2) & \cdots & z_N^{N-1} Q(z_N) \end{bmatrix}. \tag{13}$$

Then

$$\mathcal{B}(Z_0)\Upsilon = \Lambda. \tag{14}$$

Proof. The proof is omitted due to page limitation. □

The results obtained in this section help us to analyze the zero properties of the blocked system (4) in the following section.

3 Zero Properties of Blocked Systems

In this section, the definitions for zeros of the systems (4) and (1) are first reviewed. Then, the zero properties of blocked systems are studied. Since the analysis of the zero properties for blocked systems is quite complicated, we consider three cases separately, that is, 1) finite nonzero system zeros; 2) system zeros at infinity; and 3) system zeros at zero.

3.1 Definition

In order to study the zero properties of the system (4), we need to provide a proper definition for zeros. Here, we first recall the following definition for zeros of the unblocked system (1) from [9] and [8] (page 178).

Definition 3. *The finite zeros of the transfer function* $W(z) = C(zI - A)^{-1}B + D$ *with minimal realization* $\{A, B, C, D\}$ *are defined to be the finite values of* z *for which the rank of the following system matrix falls below its normal rank*

$$M(z) = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}. \tag{15}$$

Further, $W(z)$ *is said to have an infinite zero when* $n + rank(D)$ *is less than the normal rank of* $M(z)$*, or equivalently the rank of* D *is less than the normal rank of* $W(z)$.

Similar to the above definition, we state the following definition for zeros of the blocked system (4).

Definition 4. *The finite zeros of the transfer function* $V(Z) = C_b(ZI - A_b)^{-1}B_b + D_b$ *with minimal realization* $\{A_b, B_b, C_b, D_b\}$ *are defined to be the finite values of* Z *for which the rank of the following system matrix falls below its normal rank*

$$M_b(Z) = \begin{bmatrix} ZI - A_b & -B_b \\ C_b & D_b \end{bmatrix}. \tag{16}$$

Further, $V(Z)$ *is said to have an infinite zero when* $n + rank(D_b)$ *is less than the normal rank of* $M_b(Z)$*, or equivalently the rank of* D_b *is less than the normal rank of* $V(Z)$.

3.2 Blocked systems and unblocked systems-the normal rank

As shown in the last subsection, the normal rank plays an important rule in characterization of zeros. Thus, in this subsection an important result regarding the relation between the normal rank of $V(Z)$ and the normal rank of $W(z)$ is given.

Theorem 5. $V(Z)$ *has normal rank* Nr *if and only if the normal rank of* $W(z)$ *is* r, $r \leq min\{m, p\}$.

Proof. The proof is omitted due to page limitation. □

The above theorem relates the normal rank of associated unblocked and blocked transfer functions. We can also relate the normal rank of associated system matrices to the respective transfer functions and to each other.

Lemma 6. *The normal rank of $M(z)$ is $n + r$ if and only if the normal rank of $W(z)$ is r, $r \leq min\{m, p\}$.*

Proof. The proof is omitted due to page limitation. □

Corollary 7. *The normal rank of $M(z)$ is $n+r$ if and only if the normal rank of $M_b(Z)$ is $Nr + n$, $r \leq min\{m, p\}$.*

Proof. The proof is immediate using the results of Lemma 6 and Theorem 5. □

3.3 Blocked systems and unblocked systems-zeros

In the last subsection the relation between the normal rank of $V(Z)$ and the normal rank of $W(z)$ was studied. In this subsection, the relation between zeros of blocked systems and those of their corresponding unblocked systems is investigated. As stated earlier, due to the complexity of analysis, we consider three cases separately, that is, 1) finite nonzero system zeros; 2) system zeros at infinity; and 3) system zeros at zero.

Theorem 8. *Consider the unblocked system (1) with transfer function $W(z)$ defined by (2) and the blocked system (4) with transfer function $V(Z)$ denoted by (6). Suppose that the quadruple $\{A, B, C, D\}$ is minimal and $W(z)$ has normal rank r, $r \leq min\{m, p\}$. Then $V(Z)$ has a finite zero at $Z = Z_0 = z_0^N \neq 0$ if $W(z)$ has a finite zero at $z_0 \neq 0$. Conversely, suppose $V(Z)$ has a finite zero at $Z = Z_0 \neq 0$, let z_0 be any $N-th$ root of Z_0. Then $W(z)$ has a finite zero at one or more of $z = z_0 \neq 0$ or $z = \omega z_0 \neq 0 \ldots z = \omega^{N-1} z_0 \neq 0$, where $\omega = \exp(\dfrac{2\pi j}{N})$.*

Proof. The proof is omitted due to page limitation. □

The above theorem treats the zero properties of the blocked system for choice of finite nonzero zeros; it is natural to ask what happens to zeros at infinity, and the following theorem deals with this case.

Theorem 9. *Consider the unblocked system (1) with transfer function $W(z)$ defined by (2) and the blocked system (4) with transfer function $V(Z)$ denoted by (6). Suppose that the quadruple $\{A, B, C, D\}$ is minimal and $W(z)$ has normal rank r, $r \leq min\{m, p\}$. Then $W(z)$ has a zero at $z = \infty$ if and only if $V(Z)$ has a zero at*

$Z = \infty$.

Proof. The proof is omitted due to page limitation. $\square$

So far the zero properties of blocked system (4) have been studied for choices of finite nonzero zeros and infinite zeros. In order to cover all possible choices in the remainder of this subsection, we examine the zero properties of blocked systems for zeros at the origin. In order to deal with this case we first need to review the following result from [11], obtained by specializing Lemma 1 of [11] to the case where the unblocked system is time-invariant.

Lemma 10. *[11]*
Let $\widetilde{A}_b = I_N \bigotimes A$, $\widetilde{B}_b = I_N \bigotimes B$, $\widetilde{C}_b = I_N \bigotimes C$ and $\widetilde{D}_b = I_N \bigotimes D$. Furthermore, define $E_Z \triangleq \begin{bmatrix} 0 & 1 & & 0 \\ & & & \\ 0 & & \ddots & \\ \vdots & & \ddots & 1 \\ Z & 0 & & 0 \end{bmatrix}$, *$E_Z \in \mathbb{C}^{n\times n}$ and $\widetilde{E}_Z = E_Z \bigotimes I_N$. Then there exist invertible matrices T_l and T_r and matrices X and Y such that for all $Z \in \mathbb{C}$*

$$\begin{bmatrix} I_{n(N-1)} & 0 & 0 \\ 0 & ZI - A_b & -B_b \\ 0 & C_b & D_b \end{bmatrix} = \begin{bmatrix} T_l & 0 \\ X & I \end{bmatrix} \begin{bmatrix} \widetilde{E}_Z - \widetilde{A}_b & -\widetilde{B}_b \\ \widetilde{C}_b & \widetilde{D}_b \end{bmatrix} \begin{bmatrix} T_r & Y \\ 0 & I \end{bmatrix}. \tag{17}$$

Theorem 11. *Consider the unblocked system (1) with transfer function $W(z)$ defined by (2) and the blocked system (4) with transfer function $V(Z)$ denoted by (6). Suppose that the quadruple $\{A, B, C, D\}$ is minimal and $W(z)$ has normal rank r, $r \le min\{m, p\}$. Then $W(z)$ has a zero at $z = 0$ if and only if $V(Z)$ has a zero at $Z = 0$.*

Proof. The proof is omitted due to page limitation. $\square$

4 Conclusions

The zero properties of the blocked system obtained from blocking of linear discrete time-invariant systems were studied in this paper. The zero properties of blocked systems were investigated for all possible choices of zeros. In particular, it was shown that the blocked system is zero-free if and only if the related unblocked system is zero-free.

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