

# Extending Small Gain and Passivity Theory For Large-Scale System Interconnections

Wynita M. Griggs, S. Shravan K. Sajja, Brian D. O. Anderson and Robert N. Shorten

**Abstract**—In this paper, we use classical Nyquist arguments to derive stability results for large-scale interconnections of “mixed” linear, time-invariant (LTI) systems. We compare our results with P. J. Moylan and D. J. Hill, *Stability criteria for large-scale systems*, *IEEE Transactions on Automatic Control*, vol. 23, no. 2, 1978, pp. 143-149. Our results suggest that, if one relaxes the assumptions on the subsystems in an interconnection from assumptions of passivity or small gain to assumptions of “mixedness,” then the Moylan- and Hill-like conditions on the interconnection matrix become more stringent. We also explore the stability of large-scale, time-varying interconnections of strictly positive real systems. We determine a condition that guarantees the existence of a Lyapunov function for the interconnected system.

## I. INTRODUCTION

The study of “mixed” systems [1], [2] is inspired by situations in which high frequency dynamics neglected for modelling purposes might destroy the passivity properties of an otherwise passive system. These unmodelled dynamics will always be present in a real system. As such, the passivity theorem alone may not be adequate to show that the stability of the system interconnection is guaranteed [3]. The book [4], see also [5] and [6], described tools for establishing the stability of adaptive systems of the type examined in [3]; that is, where passivity-type properties hold only for low frequency signals.

A “mixed” LTI system, as defined in [7], is a system that combines notions of passivity and small gain behaviour in a certain manner. Roughly speaking, “mixed” systems exhibit small gain behaviours over frequency bands where passivity behaviour is violated. Hence, “mixed” systems formalise a notion that engineers have intuitively held for a long time: that keeping feedback-loop gain small at those frequencies where passivity is violated will avoid destabilisation of high frequency dynamics. A test for determining whether multi-input, multi-output (MIMO), LTI systems are “mixed” was introduced in [7].

Independently, the study of the stability of large-scale interconnections of systems is of increasing importance. Some works on this topic include [8]–[11].

In this paper, we apply classical Nyquist techniques to give stability results for interconnections of “mixed” LTI

systems; see Sections III and IV. Previous work in this direction appeared in [12], [13]. Our work goes beyond [13] in a number of ways. First, we present more detailed Nyquist arguments, based essentially on a Lyapunov argument. Secondly and most importantly, we utilise the techniques to obtain new sufficient conditions for the stability of large-scale interconnections of “mixed” systems. Our large-scale interconnection results suggest that, as one relaxes the assumptions on the transfer function matrices of the systems, eg: from assumptions of passivity to assumptions of “mixedness,” the Moylan- and Hill-like conditions on the interconnection matrix [8] become more severe.

This paper also corrects an error in Theorems 1, 6, 3 and 9 of [1], [2], [14] and [15], respectively. Determining bounded input, bounded output (BIBO) and finite-gain stability of interconnections of “mixed” LTI systems in a dissipative systems framework was the concern of these works. Roughly speaking, a system that produces a bounded output for any bounded input is said to be BIBO stable. The issue with the aforementioned results, however, is that the system output was assumed to be bounded *a priori*. In effect, the works indicate the existence of a bound on the output in terms of the input; but where BIBO stability is already assumed. Our present treatment of “mixed” LTI system interconnections via Nyquist techniques provides an approach for deriving the originally desired BIBO stability results.

Finally, in Section V, we explore the stability of large-scale, time-varying interconnections of single-input, single-output (SISO), strictly proper, strictly positive real (SPR) systems. We derive a condition that guarantees the existence of a Lyapunov function for the interconnected system. Particularly, we show that, by replacing passivity with SPRness as an assumption on the subsystems in a time-varying interconnection, the classic result [8, Thm. 4] extends in such a way that the condition on the interconnection matrix  $H$  now becomes that there exists a diagonal matrix  $Q > 0$  such that  $H(t)^T Q + QH(t) \geq 0$  for all time  $t \geq 0$ . This follows from the Kalman-Yakubovich-Popov (KYP) lemma. We conclude the paper in Section VI with a summary of our results and some directions for future research.

## Notation

The notation  $\Re[s]$  will be used to denote the real part of a complex number  $s$ . The conjugate of a complex number  $s = a + jb$ , where  $a, b$  are real and  $j^2 = -1$ , will be denoted by  $\bar{s} := a - jb$ . For a nonsingular matrix  $A$ ,  $A^{-*} := (A^{-1})^* = (A^*)^{-1}$ , where  $A^*$  denotes the conjugate transpose of  $A$ . The largest and smallest singular values of a matrix  $A$  will

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W. Griggs, S. Sajja and R. Shorten are with the Hamilton Institute, National University of Ireland, Maynooth, Co. Kildare, Ireland.

Corresponding author: Wynita M. Griggs. Phone: +353-(0)1-7086100. Fax: +353-(0)1-7086269. Email: wynita.griggs@nuim.ie

B. Anderson is with: (i) Research School of Engineering, ANU College of Engineering and Computer Science, RSISE Building 115, Australian National University, Canberra ACT 0200, Australia; and (ii) National ICT Australia Limited, Locked Bag 8001, Canberra ACT 2601, Australia.

be denoted by  $\bar{\sigma}(A)$  and  $\underline{\sigma}(A)$ , respectively. For a transfer function matrix  $G$  of a LTI system,  $G^*(j\omega) := [G(j\omega)]^*$ .  $\mathcal{R}$  denotes the set of proper, real-rational transfer function matrices.  $\mathcal{L}_\infty$  is a Banach space of matrix- (or scalar-) valued functions that are essentially bounded on  $j\mathbb{R}$  with norm  $\|G\|_\infty := \text{ess sup}_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega))$ .  $\mathcal{RL}_\infty := \mathcal{R} \cap \mathcal{L}_\infty$  consists of all proper, real-rational transfer function matrices with no poles on the imaginary axis. The Hardy space  $\mathcal{H}_\infty$  is the closed subspace of  $\mathcal{L}_\infty$  with functions that are analytic and bounded in the open right-half plane (RHP). In other words,  $\mathcal{H}_\infty$  is the space of transfer function matrices of stable, LTI, continuous-time systems.  $\mathcal{RH}_\infty := \mathcal{R} \cap \mathcal{H}_\infty$  consists of all proper, real-rational transfer function matrices with no poles in the closed RHP.

## II. DEFINITIONS AND PRELIMINARY RESULTS

Before presenting the main results of the paper, we establish several definitions. Consider a causal system with square transfer function matrix  $M \in \mathcal{RH}_\infty$ . Suppose that  $a, b, c, d \in \mathbb{R}$ .

**Definition 1:** [7] A causal system with square transfer function matrix  $M \in \mathcal{RH}_\infty$  is said to be input and output strictly passive over a frequency interval  $[a, b]$ ,  $(-\infty, c]$ ,  $[d, \infty)$  or  $(-\infty, \infty)$  if there exist  $k, l > 0$  such that

$$-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI \geq 0$$

for all  $\omega \in [a, b]$ ,  $(-\infty, c]$ ,  $[d, \infty)$  or  $(-\infty, \infty)$ , respectively.

We will say that the system is input strictly passive over a frequency interval if Definition 1 is satisfied with  $k = 0$ ; output strictly passive over a frequency interval if the definition is satisfied with  $l = 0$ ; and passive over a frequency interval if it is satisfied with  $k = l = 0$ . Note that any  $M(j\omega)$  satisfying Definition 1 over the frequency interval  $(-\infty, c]$ ,  $[d, \infty)$  or  $(-\infty, \infty)$  must be such that  $\lim_{\omega \rightarrow \pm\infty} \lambda_i[M^*(j\omega) + M(j\omega)] = c_{p_i} > 0$  for all  $i$ , where  $\lambda_i \in \mathbb{R}$  denotes the  $i$ th eigenvalue of the Hermitian matrix  $M^*(j\omega) + M(j\omega)$ . Then  $\lim_{\omega \rightarrow \pm\infty} \det[M^*(j\omega) + M(j\omega)] \neq 0$ .

**Definition 2:** [7] Define system gain over a frequency interval  $[a, b]$ ,  $(-\infty, c]$ ,  $[d, \infty)$  or  $(-\infty, \infty)$  as

$$\varepsilon := \inf\{\bar{\varepsilon} \in \mathbb{R}_+ : -M^*(j\omega)M(j\omega) + \bar{\varepsilon}^2 I \geq 0 \text{ for all } \omega \in [a, b], (-\infty, c], [d, \infty) \text{ or } (-\infty, \infty), \text{ respectively}\}.$$

The causal system with transfer function matrix  $M \in \mathcal{RH}_\infty$  is said to have a gain of less than one over the frequency interval  $[a, b]$ ,  $(-\infty, c]$ ,  $[d, \infty)$  or  $(-\infty, \infty)$ , respectively, if  $\varepsilon < 1$ .

For any system satisfying Definition 2 with gain of less than one over the frequency interval  $(-\infty, c]$ ,  $[d, \infty)$  or  $(-\infty, \infty)$ , it must hold that  $\lim_{\omega \rightarrow \pm\infty} \lambda_i[-M^*(j\omega)M(j\omega) + I] = c_{s_i} > 0$  for all  $i$ , where  $\lambda_i \in \mathbb{R}$  denotes the  $i$ th eigenvalue of the Hermitian matrix  $-M^*(j\omega)M(j\omega) + I$ . Then  $\lim_{\omega \rightarrow \pm\infty} \det[-M^*(j\omega)M(j\omega) + I] \neq 0$ . We now define a “mixed” system similarly to [7].

**Definition 3:** A causal system with square transfer function matrix  $M \in \mathcal{RH}_\infty$  is said to be “mixed” if, for each frequency  $\omega \in \mathbb{R} \cup \{\pm\infty\}$ : either (i)  $-kM^*(j\omega)M(j\omega) +$

$M^*(j\omega) + M(j\omega) - lI \geq 0$ ; and/or (ii)  $-M^*(j\omega)M(j\omega) + \varepsilon^2 I \geq 0$ . The constants  $k, l > 0$  and  $\varepsilon < 1$  are independent of  $\omega$ .

An example of a “mixed” system is a system with transfer function

$$m(s) = \frac{3}{(s+1)(s+2)}$$

and Nyquist diagram as depicted in Fig. 1. From the Nyquist diagram, it is clear that there exists a frequency  $\Omega$  such that, over the frequency band  $[-\Omega, \Omega]$ , Property (i) of Definition 3 holds and, over the frequency bands  $[-\infty, -\Omega]$  and  $[\Omega, \infty]$ , Property (ii) of the definition is satisfied. (Note that  $\Re[m(j\omega)] = \frac{1}{2}[m^*(j\omega) + m(j\omega)]$  and  $|m(j\omega)|^2 = m^*(j\omega)m(j\omega)$ .)

We will also require the following preliminary results.

**Lemma 4:** Suppose that  $G_1 \in \mathcal{RL}_\infty$  and  $G_2 \in \mathcal{RL}_\infty$ . Suppose further that, at some  $\omega \in \mathbb{R} \cup \{\pm\infty\}$ ,  $G_1^*(j\omega) + G_1(j\omega) > 0$  and  $G_2^*(j\omega) + G_2(j\omega) \geq 0$ . Then  $\det[I + G_1(j\omega)G_2(j\omega)] \neq 0$ .

Refer to [16] for the proof. Letting  $G_1 = I$  and setting  $G := G_2$  in the above lemma statement gives the following corollary. (Alternatively, we can set  $G := G_1$  and let  $G_2 = I$  to obtain a version of the corollary containing a strict inequality.)

**Corollary 5:** Suppose that  $G \in \mathcal{RL}_\infty$  and that, at some  $\omega \in \mathbb{R} \cup \{\pm\infty\}$ ,  $G^*(j\omega) + G(j\omega) \geq 0$ . Then  $\det[I + G(j\omega)] \neq 0$ .

Versions of the next corollary can be found in [17, Lem. 7 of Sec. VI.10] and [18, Thm. 2.3.4].

**Corollary 6:** Suppose that  $G \in \mathcal{RL}_\infty$  and that, at some  $\omega \in \mathbb{R} \cup \{\pm\infty\}$ ,  $G^*(j\omega) + G(j\omega) \geq 0$ . Let  $S(j\omega) := (G(j\omega) - I)(I + G(j\omega))^{-1}$ . Then  $-S^*(j\omega)S(j\omega) + I \geq 0$ .

See [16] for the proof. An extension to Lemma 4 is given below.

**Lemma 7:** Suppose that  $G_1 \in \mathcal{RL}_\infty$  and  $G_2 \in \mathcal{RL}_\infty$ . Suppose further that, at some  $\omega \in \mathbb{R} \cup \{\pm\infty\}$ ,  $G_1^*(j\omega) +$

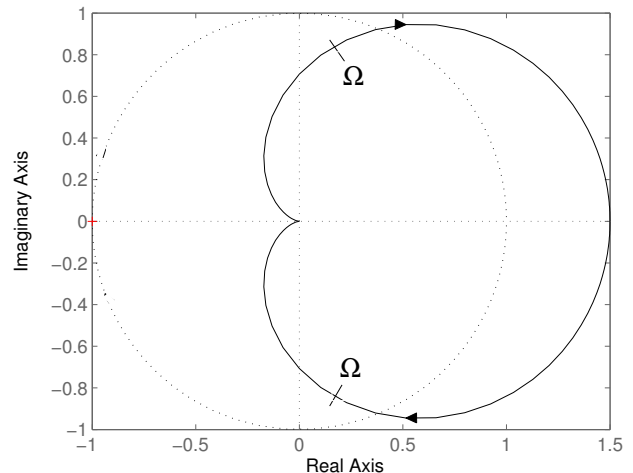


Fig. 1. Nyquist diagram of  $m(s)$ .

$G_1(j\omega) > G_1^*(j\omega)KG_1(j\omega)$  and  $G_2^*(j\omega) + G_2(j\omega) \geq -K$ , where  $K$  is a real-symmetric, positive semidefinite matrix. Then  $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)] \neq 0$  for any  $\kappa \geq 1$ , where  $\kappa \in \mathbb{R}$ .

For the proof, refer to [16]. Lastly, since our aim is to deduce stability of interconnections of “mixed” systems using arguments based on classical Nyquist techniques, we state a MIMO version of the Nyquist stability theorem.

**Theorem 8:** [19, Thm. 5.8] [20, Remark 4 of Sec. 4.9.2] Consider the feedback interconnection of systems depicted in Fig. 2. Suppose that  $G_1 \in \mathcal{RH}_\infty$ ,  $G_2 \in \mathcal{RH}_\infty$  and that the system interconnection is well-posed. Then the feedback-loop is stable if and only if the Nyquist plot of  $\det[I + G_1(j\omega)G_2(j\omega)]$  for  $-\infty \leq \omega \leq \infty$  does not make any encirclements of the origin.

In the above theorem, well-posedness and stability are defined in the sense of [19, Sec. 5.2 and Sec. 5.3]. Note, also, the following observations concerning the Nyquist plot of  $\det[I + G_1(j\omega)G_2(j\omega)]$ .

**Observation 1:** The Nyquist plot of  $\det[I + G_1(j\omega)G_2(j\omega)]$  belongs to the family of Nyquist plots of  $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)]$ , where  $\kappa \in [1, \infty)$ .

**Observation 2:** Each Nyquist plot of  $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)]$  is symmetric about the real axis of the complex plane, where  $\kappa \in [1, \infty)$ .<sup>1</sup>

**Observation 3:** As  $\kappa$  and  $\omega$  vary continuously, the point in the complex plane on which the Nyquist plot of  $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)]$  lies varies continuously.

**Observation 4:** As  $\kappa \rightarrow \infty$ ,  $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)] \rightarrow 1$ .

**Observation 5:** Suppose that  $\kappa$  is very large such that  $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)]$  is almost equal to 1 for all  $\omega \in \mathbb{R} \cup \{\pm\infty\}$ . Then suppose that  $\kappa$  is continuously decreased towards 1. Suppose that the Nyquist plot of  $\det[I + G_1(j\omega)G_2(j\omega)]$  encircles the origin at least once. Then there must exist at least one  $\kappa_0$  and one  $\omega_0$  for which  $\det[I + \frac{1}{\kappa_0}G_1(j\omega_0)G_2(j\omega_0)] = 0$ .

We have thus established the following corollary.

**Corollary 9:** Adopt the hypotheses of Theorem 8. Then a sufficient condition for the Nyquist plot of  $\det[I + G_1(j\omega)G_2(j\omega)]$  to make no encirclements of the origin

<sup>1</sup>Proof outline:  $\det[I + \frac{1}{\kappa}G_1(-j\omega)G_2(-j\omega)] = \det[(I + \frac{1}{\kappa}G_2^*(j\omega)G_1^*(j\omega))^T] = \det[I + \frac{1}{\kappa}G_2^*(j\omega)G_1^*(j\omega)]$  (from [21, Eqn. 6.1.4]) =  $\det[(I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega))^*] = \det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)]$  (from [21, Ex. 6.1.6]).

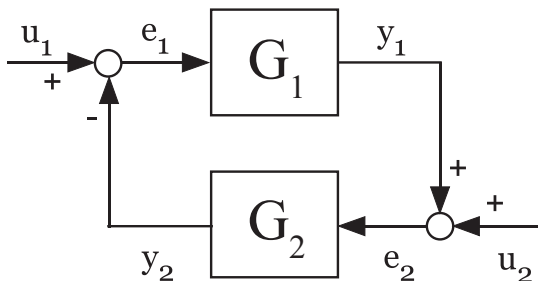


Fig. 2. A negative feedback interconnection.

is that, for all  $\kappa \in [1, \infty)$  and all  $\omega \in \mathbb{R} \cup \{\pm\infty\}$ ,  $\det[I + \frac{1}{\kappa}G_1(j\omega)G_2(j\omega)] \neq 0$ .

Subsequently, we will present scenarios in which this sufficient condition is satisfied and thus the stability of the negative feedback-loop is guaranteed.

### III. SIMPLE FEEDBACK-LOOP

We first give a rapid proof of a stability result for simple feedback interconnections of systems with “mixed” small gain and passivity properties. A result of this manner appeared in [13]. We utilise the Nyquist discussion presented above. As stated in the introduction, our purposes for doing so are twofold: first, we correct an error in Theorems 1, 6 and 3, respectively, of [1], [2] and [14] (in these, the system output signals were assumed to be bounded *a priori*); secondly, the technique paves the way to obtaining new sufficient conditions for the stability of large-scale interconnections of “mixed” systems, which we present in Section IV.

**Theorem 10:** Suppose that  $M_1 \in \mathcal{RH}_\infty$  and  $M_2 \in \mathcal{RH}_\infty$  denote the transfer function matrices of “mixed” subsystems interconnected as depicted in Fig. 3 and that this interconnection is well-posed. Suppose that there exist two distinct sets of frequency bands: (a) a set denoted by  $\Omega_p$  that consists of frequency intervals over which both  $M_1(j\omega)$  and  $M_2(j\omega)$  have associated with them Property (i) as given in Definition 3; and (b) a set denoted by  $\Omega_s$  that consists of frequency intervals over which both  $M_1(j\omega)$  and  $M_2(j\omega)$  have associated with them Property (ii) as given in Definition 3. Furthermore, suppose that  $\Omega_p \cup \Omega_s = \mathbb{R} \cup \{\pm\infty\}$ . Then the negative feedback-loop is stable.

**Proof:** Our aim is to show that, for all  $\kappa \in [1, \infty)$  and all  $\omega \in \mathbb{R} \cup \{\pm\infty\}$ ,  $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$ . From the previous section, this is a sufficient condition for stability. We split our proof into two parts: (i) first, we show that  $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$  for all  $\kappa \in [1, \infty)$  and all  $\omega \in \Omega_s$ ; and (ii) then, we show that  $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$  for all  $\kappa \in [1, \infty)$  and all  $\omega \in \Omega_p$ .

Part (i):  $\forall \omega \in \Omega_s$ . From Property (ii) of Definition 3, for  $i = 1, 2$ , there exists an  $\varepsilon_i < 1$  such that  $-M_i^*(j\omega)M_i(j\omega) + \varepsilon_i^2 I \geq 0$ . This implies that, for  $i = 1, 2$ ,  $\bar{\sigma}(M_i(j\omega)) < 1$ , which implies that  $\bar{\sigma}(M_1(j\omega)M_2(j\omega)) < 1$  since  $\bar{\sigma}(M_1(j\omega)M_2(j\omega)) \leq \bar{\sigma}(M_1(j\omega))\bar{\sigma}(M_2(j\omega))$ . Now

$$0 < 1 - \bar{\sigma}(M_1(j\omega)M_2(j\omega)) \leq \underline{\sigma}(I + M_1(j\omega)M_2(j\omega))$$

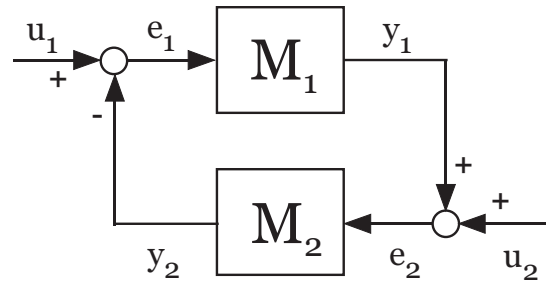


Fig. 3. A negative feedback interconnection of “mixed” systems.

from [19, Sec. 2.8] and so  $\underline{\sigma}(I + M_1(j\omega)M_2(j\omega)) \neq 0$  which is equivalent to  $\det[I + M_1(j\omega)M_2(j\omega)] \neq 0$ . Furthermore,  $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$  for any  $\kappa > 1$ . This is because  $\bar{\sigma}(M_1(j\omega)M_2(j\omega)) < 1$  is equivalent to  $\frac{1}{\kappa}\bar{\sigma}(M_1(j\omega)M_2(j\omega)) < \frac{1}{\kappa}$  (which is  $< 1$ ) for any  $\kappa > 1$ , and so  $\bar{\sigma}(\frac{1}{\kappa}M_1(j\omega)M_2(j\omega)) < 1$  for any  $\kappa > 1$ . Then

$$0 < 1 - \bar{\sigma}\left(\frac{1}{\kappa}M_1(j\omega)M_2(j\omega)\right) \leq \underline{\sigma}\left(I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)\right)$$

for any  $\kappa > 1$  and from this the determinant inequality is immediate.

Part (ii):  $\forall \omega \in \Omega_p$ . From Property (i) of Definition 3, for  $i = 1, 2$ , there exist  $k_i, l_i > 0$  such that  $-k_i M_i^*(j\omega)M_i(j\omega) + M_i^*(j\omega) + M_i(j\omega) - l_i I \geq 0$ . This implies that, for  $i = 1, 2$ ,  $M_i^*(j\omega) + M_i(j\omega) > 0$ . Observe that  $M_i^*(j\omega) + M_i(j\omega) > 0$  if and only if  $\frac{1}{\sqrt{\kappa}}M_i^*(j\omega) + \frac{1}{\sqrt{\kappa}}M_i(j\omega) > 0$ , where  $\kappa > 0$ . Then, from Lemma 4,  $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$  for any  $\kappa > 0$  and hence for any  $\kappa \geq 1$ . ■

#### IV. LARGE-SCALE INTERCONNECTIONS

Building on the techniques of the previous section, we now present sufficient conditions for the stability of large-scale interconnections of systems with mixtures of small gain and passivity properties. Consider a linear interconnection of  $N$  “mixed” systems with square transfer function matrices denoted by  $M_i \in \mathcal{RH}_\infty$ ,  $i = 1, \dots, N$ . The interconnection is described by

$$e_i = u_i - \sum_{j=1}^N H_{ij}y_j,$$

where  $e_i$  is the input to subsystem  $i$ ,  $y_i = M_i e_i$  is the output of subsystem  $i$ ,  $u_i$  is an external input and  $H_{ij}$  is a matrix with real, constant entries. Writing

$$e := \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix}, y := \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \text{ and } u := \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix},$$

the interconnection description may be written more compactly as

$$e = u - Hy, \quad (1)$$

where  $H$  is a matrix with block entries  $H_{ij}$ . Let  $\tilde{M} := \text{diag}(M_1, \dots, M_N)$  such that  $y = \tilde{M}e$ . Eliminating  $y$  from (1), we have  $e = (I + H\tilde{M})^{-1}u$ . Then

$$y = \tilde{M}(I + H\tilde{M})^{-1}u. \quad (2)$$

This set-up is depicted in Fig. 4. We will assume that the interconnection is well-posed and impose the following extra conditions on the systems in the interconnection, similarly to Theorem 10. We require the existence of two distinct sets of frequency bands: (a) a set denoted by  $\Omega_p$  that consists of frequency intervals over which every  $M_i(j\omega)$  has Property (i) as given in Definition 3 associated with it; and (b) a set denoted by  $\Omega_s$  that consists of frequency intervals over which every  $M_i(j\omega)$  has Property (ii) as given in Definition 3 associated with it. Again, we also require that  $\Omega_p \cup \Omega_s = \mathbb{R} \cup \{\pm\infty\}$ . In the following,  $p_i, q_i \in \mathbb{R}$  for  $i = 1, \dots, N$ .

*Theorem 11:* An interconnection of “mixed” subsystems, with input  $u$  and output  $y$ , as described above, is stable if there exist positive definite matrices  $P := \text{diag}(p_1 I, \dots, p_N I)$  and  $Q := \text{diag}(q_1 I, \dots, q_N I)$  such that  $H^T Q + QH > 0$  and  $-H^T P H + P > 0$ .

For the proof, see [16]. Fixing  $P = Q = I$  in the above theorem statement gives the following result.

*Corollary 12:* An interconnection of “mixed” subsystems, with input  $u$  and output  $y$ , as described above, is stable if  $H^T + H > 0$  and  $-H^T H + I > 0$ .

Our next version of a large-scale interconnected “mixed” systems stability result involves a degree of relaxation on the requirements of the interconnection structure described by the matrix  $H$ , compared to the conditions on  $H$  specified in Theorem 11. This relaxation is achieved by taking into account the values of  $k_i$  and  $\varepsilon_i$  associated with each of the “mixed” subsystems in the interconnection, where  $\varepsilon_i$  denotes the gain of the  $i$ th “mixed” system over frequencies in  $\Omega_s$ , while  $k_i$  provides a measure of output strict passivity for the  $i$ th “mixed” system over frequencies in  $\Omega_p$ . Suppose that  $K := \text{diag}(k_1 I, \dots, k_N I)$  and  $E := \text{diag}(\varepsilon_1 I, \dots, \varepsilon_N I)$ , where  $k_i > 0$  and  $0 < \varepsilon_i < 1$  for  $i = 1, \dots, N$ .

*Theorem 13:* An interconnection of “mixed” subsystems, with input  $u$  and output  $y$ , as described above, is stable if there exist positive definite matrices  $P := \text{diag}(p_1 I, \dots, p_N I)$  and  $Q := \text{diag}(q_1 I, \dots, q_N I)$  such that  $H^T Q + QH + QK > 0$  and  $-H^T P E^2 H + P > 0$ .

Refer to [16] for the proof. Set  $P = Q = I$  in the above theorem statement to obtain the following corollary.

*Corollary 14:* An interconnection of “mixed” subsystems, with input  $u$  and output  $y$ , as described above, is stable if  $H^T + H + K > 0$  and  $-H^T E^2 H + I > 0$ .

Next, we compare our stability results for large-scale interconnections of “mixed” systems to the large-scale interconnection stability results of [8, Sec. IV and Sec. V] (for example, see [8, Thm. 4 and Thm. 5]). In [8], a sufficient condition for the stability of large-scale interconnections of passive systems is the existence of a positive definite diagonal matrix  $Q$  such that  $H^T Q + QH > 0$ . A necessary condition for this linear matrix inequality (LMI) to be feasible is that  $H$  has all eigenvalues with positive real parts [22, Thm. 1 of Sec. 13.1]. Similarly in [8], a sufficient condition for the stability of large-scale interconnections of systems with finite gain is the existence of a positive definite diagonal matrix  $P$

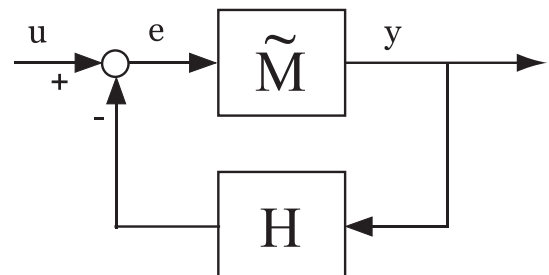


Fig. 4. A large-scale interconnection of “mixed” systems.

such that  $-H^T P E^2 H + P > 0$ .<sup>2</sup> A necessary condition for this LMI to be feasible is that all of the eigenvalues of  $EH$  lie inside the unit circle centred at the origin of the complex plane [23, Thm. 5.18]. Our results indicate that, as one loosens the suppositions on the subsystems in an interconnection, from passivity or finite gain, to “mixedness,” the [8]-like conditions for stability on the interconnection matrix itself become more stringent, ie: more restriction is imposed on the structure of the interconnection, ie: the matrix  $H$  has to “work harder” in order for stability to be guaranteed. For instance, in Theorems 11 and 13, the existence of solutions to a pair of LMIs, as opposed to a single LMI, is sufficient for stability; we illustrate this point further with the following example.

*Example 1:* Consider the example of an interconnected system from [8], depicted in Fig. 5, with interconnection matrix

$$H = \begin{bmatrix} 1 & 0 & -\gamma \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Assume that  $G_1$ ,  $G_2$  and  $G_3$  are passive and that  $-8 < \gamma < 1$ . According to [8], under these conditions, one should be able to find a positive definite diagonal matrix  $Q$  such that  $H^T Q + QH > 0$  which thus means that the interconnected system is stable. Using the Robust Control Toolbox (MATLAB R2009a) we verify that, for any  $-8 < \gamma < 1$ , finding a solution to the LMI  $H^T Q + QH > 0$  is indeed feasible.

Now suppose that we relax the suppositions on  $G_1$ ,  $G_2$  and  $G_3$  and assume that they are all “mixed” systems. For the same values of  $\gamma$ , we search for positive definite diagonal matrices  $P$  and  $Q$  that satisfy  $H^T Q + QH > 0$  and  $-H^T P H + P > 0$  simultaneously. We find that this LMI problem is not feasible for any  $-8 < \gamma < 1$ .  $\square$

We conclude the section with an example of an interconnection of “mixed” systems for which stability is guaranteed.

*Example 2:* Consider the interconnection of systems depicted in Fig. 6 and suppose that  $M_1$ ,  $M_2$  and  $M_3$  are “mixed,” with  $k_1 = k_2 = k_3 = 0.01$ . Let  $\gamma = 0.5$ . Then  $K = 0.01I$  and

$$H = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0.5 \\ -0.5 & -0.5 & 0 \end{bmatrix}.$$

<sup>2</sup>Note that, in [8], system gains  $\varepsilon_i$  appearing in  $E$  are not necessarily less than one.

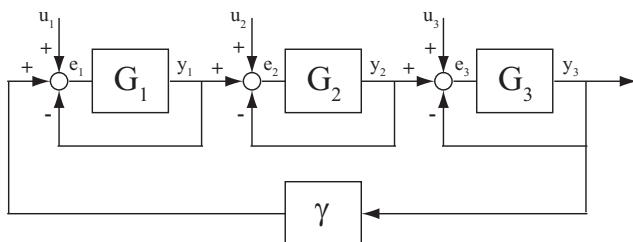


Fig. 5. Example 1.

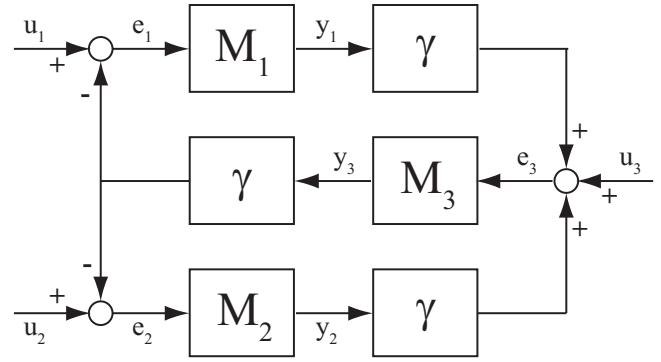


Fig. 6. Example 2.

Since the eigenvalues of  $H^T + H + K$  and  $I - H^T H$  are positive, the interconnection is stable by Corollaries 12 and 14.  $\square$

## V. TIME-VARYING INTERCONNECTIONS

The final contribution of this paper concerns obtaining a stability result for time-varying interconnections of SISO, LTI, strictly proper systems. Consider the  $N$  SISO, LTI systems

$$\begin{aligned} \dot{x}_i &= A_i x_i + b_i e_i, \\ y_i &= c_i^T x_i, \end{aligned}$$

$i = 1, \dots, N$ , where  $x_i(t) \in \mathbb{R}^{n_i \times 1}$ ,  $A_i \in \mathbb{R}^{n_i \times n_i}$ ,  $b_i \in \mathbb{R}^{n_i \times 1}$ ,  $c_i \in \mathbb{R}^{n_i \times 1}$  and  $A_i$  is Hurwitz, with transfer functions  $G_i(s) := c_i^T (sI - A_i)^{-1} b_i$ . Suppose that  $(A_i, b_i)$  is controllable and  $(c_i^T, A_i)$  is observable for  $i = 1, \dots, N$ . Define the vectors

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad e := \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix} \quad \text{and} \quad y := \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

and let

$$\dot{x} = Ax + Be, \tag{3}$$

$$y = C^T x, \tag{4}$$

where  $A := \text{diag}(A_1, \dots, A_N) \in \mathbb{R}^{(n_1 + \dots + n_N) \times (n_1 + \dots + n_N)}$ ,  $B := \text{diag}(b_1, \dots, b_N) \in \mathbb{R}^{(n_1 + \dots + n_N) \times N}$  and  $C := \text{diag}(c_1, \dots, c_N) \in \mathbb{R}^{(n_1 + \dots + n_N) \times N}$ . Then  $(A, B)$  is controllable,  $(C^T, A)$  is observable and  $A$  is Hurwitz. Denote the transfer function of this new system as  $G(s) := C^T (sI - A)^{-1} B = \text{diag}(G_1(s), \dots, G_N(s))$ .

Now, suppose that  $G(s)$  is SPR [18, Sec. 2.14], [24, Defn. 8.5], [25, Defn. 5.18]. Then  $QG(s)$  is SPR for any positive definite matrix  $Q := \text{diag}(q_1, \dots, q_N)$ , where  $q_i \in \mathbb{R}$  for  $i = 1, \dots, N$ , and the KYP lemma [18, Sec. 3.1.4], [24, Lem. 8.1], [25, Thm. 5.14] states that there exists a positive definite matrix  $P \in \mathbb{R}^{(n_1 + \dots + n_N) \times (n_1 + \dots + n_N)}$  such that

$$A^T P + PA < 0,$$

$$PB = CQ.$$

Let  $H(t)$  be some matrix with real entries that are bounded, continuous functions of time, that describes how the  $N$  subsystems are interconnected at time  $t \geq 0$ , as follows:

$$e = -H(t)y. \quad (5)$$

Substituting (5) and (4) into (3) gives

$$\dot{x} = [A - BH(t)C^T]x. \quad (6)$$

Define  $V(x) = x^T P x$  as a candidate Lyapunov function for (6). Then

$$\begin{aligned} \dot{V}(x,t) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T [A - BH(t)C^T]^T P x + x^T P [A - BH(t)C^T] x \\ &= x^T [A^T P + PA - PBH(t)C^T - CH(t)^T (PB)^T] x \\ &= x^T [A^T P + PA - CQH(t)C^T - CH(t)^T QC^T] x \\ &= x^T [A^T P + PA] x - x^T C [H^T(t)Q + QH(t)] C^T x. \end{aligned}$$

This derivative function is negative definite if  $H^T(t)Q + QH(t) \geq 0$  for all  $t \geq 0$ . Hence, we have the following result.

*Theorem 15:* The system described by (6) is uniformly asymptotically stable<sup>3</sup> if there exists a positive definite matrix  $Q := \text{diag}(q_1, \dots, q_N)$ , where  $q_i \in \mathbb{R}$  for  $i = 1, \dots, N$ , such that  $H^T(t)Q + QH(t) \geq 0$  for all  $t \geq 0$ .

## VI. CONCLUSIONS

The key contributions of this paper concern the derivation of sufficient conditions for the stability of large-scale: (i) time-invariant interconnections of “mixed” systems; and (ii) time-varying interconnections of SISO, strictly proper, SPR systems. Concerning the first contribution, we showed that, relaxing the assumptions on the systems in a large-scale interconnection, from suppositions of passivity or small gain, to assumptions of “mixedness,” results in the [8]-like conditions for stability on the interconnection structure itself becoming more stringent. Such a result has the potential to steer strategies for large-scale system design and is a direction for future research that the authors would like to pursue. In regards to the second contribution, the inspiration for studying time-varying interconnections emerges from applications concerning (for example) mobile vehicle networks, where agents, or vehicles, come in and out of range with each other (ie: links between the systems are created or broken over time). Extensions of these results for time-varying interconnections of MIMO, proper, SPR systems seem straightforward and will be published at a later date.

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<sup>3</sup>For a definition of uniform asymptotic stability, see [26, Chapt. 5] or [27, Thm. 2.5].

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