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On the Zero Properties of Linear Discrete-Time Systems with Multirate Outputs

Mohsen Zamani, *Student Member*, IEEE, and Brian D. O. Anderson, *Life Fellow*, IEEE.

***Abstract*— In this paper the zero properties of discrete-time linear systems with multirate outputs are studied. In the literature the zero properties of these systems are defined as those of their corresponding time-invariant blocked systems. Hence, the focus is on the zero properties of blocked systems resulting from blocking of linear systems with multirate outputs. In particular, we study the zero properties of tall blocked systems under a generic setting i.e. for generic parameter matrices; moreover, we only study the zero properties for choice of finite zeros. We demonstrate that tall blocked systems generically have no finite nonzero zeros. We then show when tall blocked systems can generically be zero-free at the origin i.e. $Z = 0$, and when they must have zeros at that point.**

I. INTRODUCTION

Discrete-time multirate linear systems have attracted attentions for some decades. The properties of these systems have been studied in such subdisciplines as systems and control [1], signal processing [2], communications [3] and econometric modeling [4]. The authors of this paper have become interested in studying of these systems due to their application in the econometric modeling [5],[6]. In econometric modeling, it is common to have some data which are collected monthly, while other data may be obtained quarterly or even annually [6], [4]. Furthermore, in econometric modeling of high dimensional time-series using factor models, one has very frequently to deal systems which have a larger number of outputs compared to their number of inputs i.e. tall systems [7], [8]. In the econometric context, the latent variables are modeled by systems with unobserved white noise inputs. In a single-rate setting i.e. monthly data only, authors in [8] have shown when the factor model is zero-free then the latent variables can be modeled as a singular autoregressive process whose parameters can be easily identified using Yule-Walker equations. A corresponding demonstration is still lacking for the multirate case. The results of this research enable us to understand better the properties of multirate factor models. However, this paper does not focus on the applications problem, but rather on the system theoretical issues involved with multirate systems with tall structure.

A technique termed blocking or lifting has been developed in signal processing and systems and control to deal with multirate linear systems [1], [2]. In systems and control literature, this method was initially introduced to transform linear discrete-time periodic systems to linear time-invariant systems, so that the well-developed tools in linear time-invariant systems can be extended for design and analysis of linear discrete-time periodic systems [9], [10], [11] and [12]. For example, the authors in [12] and [11] extended the notions of poles and zeros of linear time-invariant systems to linear periodic systems and the authors of [10] and [9] have defined zeros of multirate linear systems as those of their corresponding blocked systems.

In this paper, we assume that there exists an underlying system operating at the highest sample rate, which is linear time-invariant. However, because not all the outputs of this underlying system are actually measured at the same rate, we end up with a multirate linear system linking the inputs of the original system to those of its outputs which are measured. In particular, our main interest is to study the zero properties of tall multirate systems i.e. multirate systems with the number of outputs more than the number of inputs, which is more likely in econometric modeling. However, since zeros of the multirate systems are defined as zeros of their corresponding blocked systems [10], we provide a study on the zero properties of tall blocked systems resulting from blocking of linear multirate systems. To the authors best knowledge the pole properties of the blocked systems are well understood [12], [13]; whereas, it is little known about the zero properties of tall blocked systems.

References [14] and [15] have studied the zero properties of the blocked systems resulting from blocking of linear *time-invariant* systems. For instance, [14] uses the frequency domain approach and in particular the matrix fraction description (MFD) method to establish the relation between zero properties of blocked systems and zero properties of their corresponding unblocked systems. Meanwhile, authors in [15] have exploited the time domain approach to explore the zero properties of blocked systems. It has been noted in both references [14] and [15] that tall blocked systems are zero-free if and only if the related (time-invariant) unblocked systems are zero-free. Moreover, authors in [15] partially addresses the zero properties of tall blocked systems resulting from blocking of multirate linear systems but further study is still required.

The main objective of this paper is to investigate the zero properties of tall blocked systems for choice of finite zeros. Here, blocked systems are obtained from blocking of a multirate linear system with generic choice of parameter matrices i.e. A, B, etc. The results of this study reveal for

Proofs of some assertions in this paper are omitted due to page limitations; they are available from the first author upon request and will be given in a full length version of this paper.

M. Zamani and B. D. O. Anderson are with Research School of Engineering, Australian National University, Canberra, ACT 0200, Australia. B. D. O. Anderson is with Canberra Research Laboratory, National ICT Australia Ltd., PO Box 8001, Canberra, ACT 2601, Australia. {mohsen.zamani, brian.anderson}@anu.edu.au

almost all choices of parameter matrices what kind of zero properties tall blocked systems have. Note that there are already some results in the literature dealing with the zero properties for unblocked tall and generic linear time-invariant systems [16] and [17]. However, there has been a gap in the literature regarding the study of linear time-invariant systems formed by blocking of multirate linear system; this process results in a time-invariant system with relations among the parameters, i.e. one that is not fully generic.

Since the analysis of zero properties for tall blocked systems is quite complicated, we consider finite zeros in two cases, which are, 1) finite nonzero system zeros and 2) system zeros at zero. The next section of the paper is focused on the zero properties of tall blocked systems associated with finite nonzero zeros. It is explicitly established that tall blocked systems generically have no finite nonzero zeros. Moreover, in the subsequent Section III the zero properties of tall blocked systems are examined at $Z = Z_0 = 0$. It is shown when tall blocked systems can have a zero at $Z = Z_0 = 0$ and when they are zero-free at that point. Finally, Section IV offers concluding remarks.

II. Blocked systems with generic parameters-nonzero finite zeros

In this section, first the formulation of the problem under study is introduced. Then attention is given to the analysis of the zero properties of all blocked systems with generic parameters. In this section we only consider choice of finite nonzero zeros. Later, in the next section, choices of infinite zeros and zeros at the origin are explored.

The dynamics of an underlying system operating at the highest sample rate are defined by

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k) \\ y(k) &= Cx(k)+Du(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^p$ the output, and $u(k) \in \mathbb{R}^m$ the input. For this system, $y(k)$ exists for all k, and, separately, can be measured at every time k. However, we are also interested in the situation where $y(k)$ exists for all k, but not every entry is measured for all k. In particular, we consider the case where $y(k)$ has components that are observed at different rates. For simplicity, in this paper we consider a case where outputs are provided at two rates which we refer to as the fast rate and the slow rate.

Without loss of generality we decompose $y(k)$ as $y(k) = \begin{bmatrix} y^f(k) \\ y^s(k) \end{bmatrix}$ where $y^f(k) \in \mathbb{R}^{p_1}$ is observed at all k, the fast part, and $y^s(k) \in \mathbb{R}^{p_2}$ is observed at $k = 0, N, 2N, \cdots$, the slow part, also $p_1 > 0, p_2 > 0$ and $p_1+p_2 = p$. Accordingly, we decompose C and D as

$$C = \begin{bmatrix} C^f \\ C^s \end{bmatrix},\ D = \begin{bmatrix} D^f \\ D^s \end{bmatrix}.$$

Thus, the multirate linear system corresponding to what is measured has the following dynamics:

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k) && k=0,1,2,\ldots \\ y^f(k) &= C^f x(k)+D^f u(k) && k=0,1,2,\ldots \\ y^s(k) &= C^s x(k)+D^s u(k) && k=0,N,2N,\ldots \end{aligned} \tag{2}$$

We have actually N distinct alternative ways to block the system, depending on how fast signals are grouped with the slow signals. Even though these N different systems share some common zero properties their zero properties are not identical in the whole complex plane (see [12], pages 173-179).

For, $\tau \in \{1, 2, \cdots, N\}$, define

$$\begin{aligned} U_\tau(k) &\triangleq \begin{bmatrix} u(k+\tau) \\ u(k+\tau+1) \\ \vdots \\ u(k+\tau+N-1) \end{bmatrix}, \\ Y_\tau(k) &\triangleq \begin{bmatrix} y^f(k+\tau) \\ y^f(k+\tau+1) \\ \vdots \\ y^f(k+\tau+N-1) \\ y^s(k+N) \end{bmatrix}, \quad k=0,N,2N,\ldots \\ x_\tau(k) &\triangleq x(k+\tau). \end{aligned} \tag{3}$$

Then the blocked system $\sum_\tau$ is defined by

$$\begin{aligned} x_\tau(k+N) &= A_\tau x_\tau(k) + B_\tau U_\tau(k) \\ Y_\tau(k) &= C_\tau X_\tau(k) + D_\tau U_\tau(k), \end{aligned} \tag{4}$$

where,

$$\begin{aligned} &A_\tau \triangleq A^N, \\ &B_\tau \triangleq \begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & AB & B \end{bmatrix}, \\ &C_\tau \triangleq \begin{bmatrix} C^{f^T} & A^T C^{f^T} & \ldots & A^{(N-1)^T} C^{f^T} & A^{(N-\tau)^T} C^{s^T} \end{bmatrix}^T, \\ &D_\tau \triangleq \begin{bmatrix} D^f & 0 & \ldots & 0 \\ C^f B & D^f & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-2}B & C^f A^{N-3}B & \ldots & D^f \\ C^s A^{N-\tau-1}B & \ldots & D^s & * \end{bmatrix}, \end{aligned} \tag{5}$$

where "*" at the very right corner denotes $\tau-1$ zero matrices of size $p_2 \times m$ and when $N-\tau-1 < 0$, $CA^{-1}B$ is replaced by D^s and rest of the terms in the last row are replaced by zero matrices of size $p_2 \times m$.

Reference [12] defines a zero of (2) at time τ as a zero of its corresponding blocked system $\sum_\tau$ [1]. Hence, in the rest of this section we focus on the zero properties of the blocked system $\sum_\tau$, $\forall \tau \in \{1, 2, \ldots, N\}$.

Definition 2.1: The finite zeros of system $\sum_\tau$ are defined to be finite values of Z for which the rank of the following

[1]Zeros of the transfer function defined from (4) are identical with those defined here, provided the quadruple $\{A_\tau, B_\tau, C_\tau, D_\tau\}$is minimal.

system matrix falls below its normal rank

$$M_\tau(Z) = \begin{bmatrix} ZI - A_\tau & -B_\tau \\ C_\tau & D_\tau \end{bmatrix}.$$

Further, $V_\tau(Z) = C_\tau(ZI - A_\tau)^{-1}B_\tau + D_\tau$, $\tau \in \{1, 2, \ldots, N\}$, is said to have an infinite zero when $n + rank(D_\tau)$, $\tau \in \{1, 2, \ldots, N\}$, is less than the normal rank of $M_\tau(Z)$, $\tau \in \{1, 2, \ldots, N\}$, or equivalently the rank of D_τ, $\tau \in \{1, 2, \ldots, N\}$, is less than the normal rank of $V_\tau(Z)$, $\tau \in \{1, 2, \ldots, N\}$.

In addition to the above definition the following results from [14] and [18] are useful to the rest of this paper.

Lemma 2.2: The (A, B) is reachable if and only if the (A_τ, B_τ), $\forall\tau \in \{1, 2, \ldots, N\}$ is reachable.

The above lemma studies the reachability property of $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$ and the lemma below explores its transfer function.

Lemma 2.3: The transfer function $V_\tau(Z)$ associated with the blocked system (4) has the following property

$$V_{\tau+1}(Z) = \begin{bmatrix} 0 & I_{p_{1(N-1)}} & 0 \\ ZI_{p_1} & 0 & 0 \\ 0 & 0 & I_{p_2} \end{bmatrix} V_\tau(Z) \begin{bmatrix} 0 & Z^{-1}I_m \\ I_{m(N-1)} & 0 \end{bmatrix},$$

where $\tau \in \{1, 2 \ldots, N\}$.

The result of the above lemma is crucial for the study of the zero properties of $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$, for choice of finite nonzero zeros. The latter is the main focus for the remainder of this section. We treat the zero properties of $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$, under genericity and tallness assumptions. Given that $p_1, p_2 > 0$ and tallness is defined by $Np_1 + p_2 > Nm$, it proves convenient to consider partition the set of p_1, p_2 defining tallness into two subsets, as follows.

1) $p_1 > m$.
2) $p_1 \leq m$, $Nm < Np_1 + p_2$.

The first case is common, perhaps even overwhelmingly common in econometric modeling but the second case is important from a theoretical point of view, and possibly in other applications. Moreover, our results are able to cover both cases.

A. Case $p_1 > m$

According to Definition 2.1, the normal rank for the system matrix of $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$, plays an important role in the analysis of its zero properties; thus, we state the following result for the normal rank of $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$.

Lemma 2.4: For generic choice of the matrices $\{A, B, C^s, C^f, D^f, D^s\}$, $p_1 \geq m$, the system matrix of $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$, has normal rank of $n + Nm$.

Proof: Proof is omitted due to page limitation. ∎

In the situation where $p_1 > m$, obtaining a result on the absence of finite nonzero zeros is now rather trivial, since the blocked system contains a subsystem obtained by deleting some outputs which is provably zero-free.

Theorem 2.5: For a generic choice of the matrices $\{A, B, C^s, C^f, D^s, D^f\}$, $p_1 > m$, the system matrix of $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$, has full column rank for all finite nonzero Z.

Proof: Proof is omitted due to page limitation. ∎

B. Case $p_1 \leq m$, $Nm < Np_1 + p_2$

In the previous subsection the case $p_1 > m$ was treated where only considering the fast outputs alone generically leads to a zero-free blocked system, and the zero-free property is not disturbed by the presence of the further slow outputs. A different way in which the blocked system will be tall arises when $p_1 \leq m$ and $Nm < Np_1 + p_2$. The main result of this subsection is to show that $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$ with $p_1 \leq m$, $Nm < Np_1 + p_2$ generically has no finite nonzero zeros. In order to study the latter case we need to review properties of the Kronecker canonical form of a matrix pencil. Since the system matrix of $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$ is actually a matrix pencil, the Kronecker canonical form turns out to be a very useful tool to obtain insight into the zeros of (4) and the structure of the kernels associated with those zeros.

The main theorem on the Kronecker canonical form of the matrix pencil is obtained from [19].

Theorem 2.6: [19] Consider a matrix pencil $zR+S$. Then under the equivalence defined using pre- and postmultiplication by nonsingular constant matrices $\widetilde{P}$ and $\widetilde{Q}$, there is a canonical quasidiagonal form:

$$\widetilde{P}(zR+S)\widetilde{Q} = \text{diag}[L_{\epsilon_1}, \ldots, L_{\epsilon_r}, \tilde{L}_{\eta_1}, \ldots, \tilde{L}_{\eta_s}, zN-I, zI-K], \tag{6}$$

where:

1) L_μ is the $\mu \times (\mu + 1)$ bidiagonal pencil

$$\begin{bmatrix} z & -1 & 0 & \ldots & 0 & 0 \\ 0 & z & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & z & -1 \end{bmatrix}. \tag{7}$$

2) $\tilde{L}_\mu$ is the $(\mu + 1) \times \mu$ transposed bidiagonal pencil

$$\begin{bmatrix} -1 & 0 & \ldots & 0 & 0 \\ z & -1 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & z & -1 \\ 0 & 0 & \ldots & 0 & z \end{bmatrix}. \tag{8}$$

3) N is a nilpotent Jordan matrix.

4) K is in Jordan canonical form.

Furthermore, the possibility that $\mu = 0$ exists. The associated L_0 is deemed to have a column but not a row and $\tilde{L}_0$ is deemed to have a row but not a column, see [19].

The following corollary can be directly derived easily from the above theorem and provides detail about the vectors in the null space of the Kronecker canonical form. Because the matrices $\tilde{P}$ and $\tilde{Q}$ are nonsingular, it is trivial to translate these properties back to an arbitrary matrix pencil, including a system matrix.

Corollary 2.7: 1) For all z except for those which are eigenvalues of K, the kernel of the Kronecker canonical form has dimension equal to the number of matrices L_μ appearing in the form; likewise the co-kernel dimension is determined by the number of matrices $\tilde{L}_\mu$.

2) The vector $[1 \quad z \quad z^2 \ldots z^\mu]^T$ is the generator of the kernel of L_μ, a set of vectors $[0 \ldots 0 \ 1 \ z \ z^2 \ldots z^\mu \ 0 \ldots 0]^T$ are generators for the kernel of the whole canonical form which depend continuously on z, provided that z is not an eigenvalue of K; when z is an eigenvalue of K, the vectors form a subset of a set of generators.
3) When z equals an eigenvalue of K, the dimension of the kernel jumps by the geometric multiplicity of that eigenvalue, the rank of the pencil drops below the normal rank by that geometric multiplicity, and there is an additional vector or vectors in the kernel apart from those defined in point 2, which are of the form $[0\,0 \ldots v^T]^T$, where v is an eigenvector of K. Such a vector is orthogonal to all vectors in the kernel which are a linear combination of the generators listed in the previous point.
4) When z is an eigenvalue, say z_0 of K, the associated kernel of the matrix pencil can be generated by two types of vectors: those which are the limit of the generators defined by adding extra zeros to vectors such as $[1 \ z_0 \ z_0^2 \ldots, z_0^\mu]^T$ (these being the limits of the generators when $z \neq z_0$ but continuously approaches z_0), and those obtained by adjoining zeros to the eigenvector(s) of K with eigenvalue z_0, the latter set being orthogonal to the former set.

In the rest of this subsection, in order to provide insight about zero properties of $M_\tau(Z), \forall \tau \in \{1, 2, \ldots, N\}$, we first focus on the particular case of $M_1(Z)$. Later, we introduce the main result for the zero properties of $M_\tau(Z)$, $\forall \tau \in \{1, 2, \ldots, N\}$.

First we need to introduce some parameters. To this end, we argue first that the first $n + Np_1$ rows of $M_1(Z)$ are linearly independent. For the submatrix formed by these rows is the system matrix of the blocked system obtained by blocking the fast system defined by $\{A, B, C^f, D^f\}$, and accordingly has full row normal rank, since the unblocked system is generic and square or fat under the condition $p_1 \leq m$. Now define the square submatrix of $M_1(Z)$:

$$N(Z) \triangleq \begin{bmatrix} ZI - A_1 & -B_1 \\ \mathcal{C}_1 & \mathcal{D}_1 \end{bmatrix}, \tag{9}$$

such that normal rank $N(Z)$ =normal rank $M_1(Z)$, by including the first $n + Np_1$ rows of $M_1(Z)$ and followed by appropriate other rows of $M_1(Z)$ to meet the normal rank and squareness requirements. Hence there exists a permutation matrix P such that

$$PM_1(Z) = \begin{bmatrix} N(Z) \\ \mathcal{C}_2 \quad \mathcal{D}_2 \end{bmatrix} \tag{10}$$

where $\mathcal{C}_2$ and $\mathcal{D}_2$ capture those rows of C_1 and D_1 that are not included in $\mathcal{C}_1$ and $\mathcal{D}_1$, respectively.

The zero properties of $N(Z)$ ar studied in the following proposition.

Proposition 2.8: Let the matrix $N(Z)$ be the submatrix of $M_1(Z)$ formed via the procedure described. Then for generic values of the matrices A, B, etc. with $p_1 \leq m$ and $Np_1 + p_2 > Nm$, for any finite Z_0 for which the matrix $N(Z_0)$ has less rank than its normal rank, its rank is one less than its normal rank.

Proof: We distinguish two cases, $p_1 = m$, $p_1 < m$. In case $p_1 = m$, then $N(Z)$ is the system matrix for the system obtained by blocking the original system with slow outputs discarded. As such, the blocked system zeros are precisely the N-th powers of the unblocked system zeros [15]. For generic coefficient matrices, the unblocked system will have n distinct zeros; then the blocked system will have the same property. Further, the unblocked system will generically have a nonsingular direct feedthrough matrix, as will then the blocked system, so that $\mathcal{D}_1$ can be assumed to be nonsingular. It follows then that the zeros of the system with system matrix $N(Z)$ are identical with the eigenvalues of $\mathcal{A} - \mathcal{B}\mathcal{D}_1^{-1}\mathcal{C}_1$, which are then distinct, and since this matrix is $n \times n$, the eigenvector associated with each zero will be uniquely defined to within a scaling constant. It follows easily that there is a unique vector (to within scaling) in the kernel of $N(Z_0)$ where Z_0 is the zero of the blocked system.

We turn therefore to the case $p_1 < m$. We study the co-kernel of $N(Z_0)$. Let $Z_1, Z_2, \ldots,$ be a sequence of complex numbers such that (a) $Z_i \to Z_0$ and (b) rank $N(Z_i)$ equals the normal rank of $N(Z)$. From what has been described earlier using the Kronecker canonical form, we know that the sequence of co-kernels of $N(Z_i)$ converges, say to $\mathcal{K}$, with any vector in this limit also in the co-kernel of $N(Z_0)$. In addition, since $N(Z_0)$ has lower rank than the normal rank of $N(Z)$, the co-kernel, call it $\bar{\mathcal{K}}$, will be strictly greater than $\mathcal{K}$. Suppose its dimension is at least two more than that of $\mathcal{K}$. We shall show this situation is nongeneric.

Select two vectors w_1, w_2 which are in $\bar{\mathcal{K}}$ and which are orthogonal to $\mathcal{K}$. Then it is evident that there are two vectors call them v_1, v_2, constructed from linear combinations of w_1, w_2, which belong to $\bar{\mathcal{K}}$, which are still orthogonal to $\mathcal{K}$, and which for some pair $r < s$ have 1 and 0 in the r-th entry and 0 and 1 in the s-th entry respectively. Choose v_1, v_2 so that firstly, s is maximal, and secondly, for that s then r is maximal. It is not difficult to see that this means that v_1 has zero entries beyond the r-th and v_2 has zero entries beyond the s-th.

Now again we must consider two cases. Suppose firstly that s obeys $n + Np_1 + 1 \leq s \leq n + Nm$; in forming the product $v_2^T N(Z_0)$, the s-th entry of v_2 will be multiplying entries of $N(Z_0)$ defined using C^s, A, B, D^s. Consider an entry in the s-th row of $N(Z_0)$ and in the last m columns. Such an entry is an entry of D^s, and is independent of all other entries in $N(Z_0)$. Suppose this entry of D^s is continuously perturbed by a small amount. Then clearly v_1 remains in the co-kernel of $N(Z_0)$ but v_2 cannot.

The particular values of Z for which $N(Z)$ has rank less than its normal rank, i.e. the zeros of $N(Z)$, will depend continuously on the perturbation.

Accordingly, with a small enough perturbation, those not equal before perturbation to Z_0 will never change to Z_0, and it is therefore guaranteed that with a small enough nonzero perturbation, the co-kernel of $N(Z_0)$ is reduced by one in dimension, though never to zero. If the original (before perturbation) co-kernel $\bar{\mathcal{K}}$ had dimension greater than two in excess of the dimension of $\mathcal{K}$, and the excess after perturbation is still greater than one, the argument can be repeated. Eventually, the co-kernel of $N(Z_0)$ will have an excess dimension over $\mathcal{K}$ of 1, i.e. $N(Z_0)$ will have rank one less than the normal rank of $N(Z)$.

Now suppose that s obeys $s \le n + Np_1$. Then the last $N(m-p_1)$ entries of each of v_1, v_2 are zero. Remove these entries to define two linearly independent vectors $\tilde{v}_1, \tilde{v}_2$ of length $n + Np_1$, which evidently satisfy

$$\tilde{v}_i^T \begin{bmatrix} ZI_n - A^N & -A^{N-1}B & \dots & -B \\ C^f & D^f & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-1} & C^f A^{N-2}B & \dots & D^f \end{bmatrix} = 0,\ i = 1, 2. \tag{11}$$

The above equation contains a fat system matrix, corresponding to a blocked version of a fat time-invariant unblocked system. It can be concluded easily form the results provided in [15] that for generic values of the underlying matrices, there can be no Z_0 for which an equation such as (11) can even hold for a single nonzero $\tilde{v}_i$, let alone two linearly independent ones. This ends the proof. ■

The result of the previous proposition, although restricted to $\tau = 1$, enables us to establish the following main result applicable for any τ.

Theorem 2.9: Consider the system $\sum_\tau$, $\forall \tau \in \{1, 2, \dots, N\}$, with $p_1 \le m$, and $Np_1 + p_2 > Nm$. Then for generic values of the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ the system matrix $M_\tau(Z)$, $\forall \tau \in \{1, 2, \dots, N\}$, has rank equal to its normal rank for all finite nonzero values of Z_0, and accordingly Σ_τ has no finite nonzero zero.

Proof: We first focus on the case $\tau = 1$. Now, apart from the $p_2 - N(m - p_1)$ rows of the C^s, D^s, which do not enter the matrix $N(Z)$ defined by (9), choose generic values for the defining matrices, so that the conclusions of the preceding proposition are valid.

Let $Z_a, Z_b, \dots$ be the finite set of Z for which $N(Z)$ has less rank than its normal rank (the set may have less than n elements, but never has more), and let $w_a, w_b, \dots$ be vectors which are in the corresponding kernels (*not co-kernels*) and orthogonal to the subspace in the kernel obtained from the limit of the kernel of $N(Z)$ as $Z \to Z_a, Z_b, \dots$ etc. Now, due to the facts that $M_1(Z)$ and $N(Z)$ have the same normal rank and any existing vector in the kernel of $M_1(Z)$ is in the the kernel of $N(Z)$ one can conclude that the subspace in the kernel obtained from the limit of the kernel of $N(Z)$ as $Z \to Z_a, Z_b, \dots$ etc, coincides with the subspace in the kernel obtained from the limit of the kernel of $M_1(Z)$ as $Z \to$ zeros of $M_1(Z)$.

Now, to obtain a contradiction, we suppose that the system matrix $M_1(Z)$ is such that, for $Z_0 \ne 0$, $M_1(Z_0)$ has rank less than its normal rank, i.e. the dimension of its kernel increases. Since the kernel of $M_1(Z_0)$ is a subspace of the kernel of $N(Z_0)$, Z_0 must coincide to one of the values of $Z_a, Z_b, \dots$ and the rank of $M_1(Z_0)$ must be only one less than its normal rank; moreover, there must exist an associated nonzero w_1 in the kernel of $M_1(Z_0)$ which is orthogonal to the limit of the kernel of $M_1(Z)$ as $Z \to Z_0$. Then w_1 is necessarily in the kernel of $N(Z_0)$, orthogonal to the limit of the kernel of $N(Z)$ as $Z \to Z_0$ and thus w_1 in fact must coincide to within a nonzero multiplier with one of the vectors $w_a, w_b, \dots$.

Write this w_1 as

$$w_1 = \begin{bmatrix} x_1^T & u_1^T & u_2^T & \dots & u_N^T \end{bmatrix}^T, \tag{12}$$

and suppose the input sequence $u(i) = u_i$ is applied for $i = 1, 2 \dots, N$ to the original system, starting in initial state x_1 at time 1. Let $y^f(1), y^f(2), \dots$ denote the corresponding fast outputs and $y^s(N)$ the slow output at time N. Break this up into two subvectors, $y^{s1}(N), y^{s2}(N)$, where $y^{s1}(N)$ is associated with those rows of C^s, D^s which are included in $\mathcal{C}_1$, $\mathcal{D}_1$ and $y^{s2}(N)$ is related with the remaining rows of C^s and D^s. We have $N(Z_0)w_1 =$

$$\begin{bmatrix} Z_0 I_n - A^N & -A^{N-1}B & -A^{N-2}B & \dots & -B \\ C^f & D^f & 0 & \dots & 0 \\ C^f A & C^f B & D^f & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ C^f A^{N-1} & C^f A^{N-2}B & C^f A^{N-3}B & \dots & D^f \\ C^{s1} A^{N-1} & C^{s1} A^{N-2}B & C^{s1} A^{N-3}B & \dots & D^{s1} \end{bmatrix} w_1$$

$$= \begin{bmatrix} Z_0 x_1 - x(N+1) \\ y^f(1) \\ y^f(2) \\ \vdots \\ y^f(N) \\ y^{s1}(N) \end{bmatrix} = 0. \tag{13}$$

Now it must be true that $x_1 \ne 0$. For otherwise, we would have $N(Z)w_1 = 0$ for all Z, which would violate assumptions. Since also $Z_0 \ne 0$, there must hold $x(N+1) \ne 0$. Hence there cannot hold both $x(N) = 0$ and $u(N) = 0$. Consequently, we can always find C^{s2}, D^{s2} such that $y^{s2}(N) = C^{s2}x(N) + D^{s2}u(N) \ne 0$, i.e. the slow output value is necessarily nonzero, no matter whether $w_1 = w_a, w_b$, etc. Equivalently, the equation $[\mathcal{C}_2 \ \ \mathcal{D}_2]w_1 = 0$ cannot hold. Hence, if $M_1(Z)$ defines a system with a finite zero and it is nonzero, this is a nongeneric situation. Hence, $M_1(Z)$ generically has rank equal to its normal rank for all finite nonzero Z. Now, we show that the latter property holds for all $M_\tau(Z)$, $\tau \in \{1, 2, \dots, N\}$. First, note that the pair (A, B) is generically reachable so, according to Lemma 2.2 the pair (A_τ, B_τ), $\forall \tau \in \{1, 2, \dots, N\}$, is also reachable. Consider $Z_\zeta \in \mathbb{C} - \{0, \infty\}$, if Z_ζ does not coincide with the eigenvalues of A_τ then

$$rank(M_\tau(Z_\zeta)) = n + rank(V_\tau(Z_\zeta)). \quad (14)$$

Hence, using the result of Lemma 2.3, it is immediate that $rank(M_\tau(Z_\zeta)) = rank(M_{\tau+1}(Z_\zeta))$. If Z_ζ does coincide with an eigenvalue of A_τ then $rank(V_\tau(Z_\zeta))$ is ill-defined. However, since zeros of $M_\tau(Z)$, $\tau \in \{1, 2 \ldots, N\}$, are invariant under state feedback and pair (A_τ, B_τ) is reachable, one can easily find a state feedback to replace that eigenvalue [20] and then (14) is a well-defined equation and $rank(M_\tau(Z_\zeta)) = rank(M_{\tau+1}(Z_\zeta))$. Thus, we can conclude that all $M_\tau(Z)$, $\tau \in \{1, 2, \ldots, N\}$ generically have no finite nonzero zeros. This ends the proof. ■

III. Blocked systems with generic parameters-zeros at the origin

In the previous section the zero properties of tall blocked systems with generic parameters for the choice of finite nonzero zeros were studied. In this section the zero properties of the latter systems are investigated for choices of zeros at zero. As in the previous section, it is convenient to break up our examination of tall systems into the separate cases based on the relation between p_1 and m. However, the partitioning into different cases is slightly different.

A. Case $p_1 > m$

Theorem 3.1: For a generic choice of the matrices $\{A, B, C^s, C^f, D^s, D^f\}$, $p_1 > m$, the system matrix of $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$, has full column rank for $Z = Z_0 = 0$.

Proof: It was shown in [15] that $M^f(0)$, where the system matrix $M^f(0)$ can be formed by deleting rows of $M_1(0)$ which are related to C^s, has full column rank for $Z = 0$. Then, it becomes immediate that $M_\tau(0)$, $\forall\tau \in \{1, 2, \ldots, N\}$ has full column rank. ■

B. Case $p_1 < m$, $Nm < Np_1 + p_2$

As in the previous subsection, we study the zero properties of tall blocked system at the origin and infinity. The following theorem treats the former choice.

Theorem 3.2: Consider the system $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$, with $p_1 < m$ and $Np_1 + p_2 > Nm$. Then for generic values of the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ the system matrix $M_\tau(Z)$, $\tau \in \{1, 2, \ldots, N\}$ always has a finite zero at $Z = 0$.

Proof: The proof is omitted due to page limitation. ■

IV. Conclusion

The zero properties of tall discrete-time multirate linear system were addressed in this paper. The zero properties of multirate linear systems were defined as those of their corresponding blocked systems. The system matrix of tall blocked systems was investigated for generic choice of parameter matrices and finite zeros. It was specifically shown that the system matrix of tall blocked systems generically has no finite nonzero zeros. However, we specified situations where the system matrix of tall blocked systems always has zeros at $Z = 0$.

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