Controlling Rectangular Formations

Soura Dasgupta, Brian D. O. Anderson, Changbin Yu and Tyler H. Summers

***Abstract*— We consider formation shape control of four point agents in the plane. Control laws based on specified interagent distances are used. For a complete graph, specification of all interagent distances determines the formation shape uniquely. Krick, Broucke and Francis showed that for a standard control law, there may exist equilibrium formations with incorrect interagent distances. This paper studies such equilibria and derives two main results. That each such incorrect equilibrium is attached to at most one desired formation. That all such equilibria are unstable if the desired formation is a rectangle.**

I. Introduction

A key problem in autonomous vehicle formations is their shape control. In this paper we analyze the stability properties of a control law, now almost standard, for a class of simple four-agent formation in which all interagent distances are to be preserved. These simple formations are considered because we do not know how to handle more complicated formations, and believe that studying a simple formulation may help provide the tools for addressing general formations. As is now standard we regard each formation as a graph. Each agent is treated as a node. If the distance between two agents is specified then the graph has an edge between the two corresponding nodes. The formations we consider have associated graphs that are K_4, the complete graph on 4 vertices.

Our control objective is to force the agents to cooperatively and autonomously achieve a specified desired formation shape, with each agent working to change its associated interagent distances to the correct values. Related work includes [1] that proposes the use of graph rigidity theory (see e.g. [2]) for modeling information architectures; [3] which also uses graph rigidity theory and proposes gradient control laws based on structural potential functions which are generated from the graph; [4] which using graph rigidity theory discusses application to formations of non-holonomic robots.

It has recently been observed that there is a fundamental distinction between formations where distances are maintained by both agents of a pair and where they are maintained by only one agent of a pair [5], [6], [7]. Triangular formations, modelled by directed graphs rather than undirected graphs so that only one agent of a pair is responsible for maintaining the interagent distance, have been studied in detail, see [8], [9], [10]. It has been pointed out in [11] that for a triangular formation, stability properties are essentially independent of whether the control structure is unidirectional (single agent controls a distance) or bidirectional (two agents control a distance), so that the triangle results almost all apply to the bidirectional case. The results in this paper are for formations where bidirectional control is used.

In [12] Krick et al provide a complete analysis showing that the desired formation shape is *locally* asymptotically stable under a gradient control law, provided that the information architecture is rigid (see [12] for a definition of rigidity). However, the global stability properties of the desired formation shape remain a challenging open problem. Krick et al also consider by way of example the four-agent formation considered here, with a complete graph information architecture (i.e. all interagent distances are actively controlled). A simulation shows that the formation appears to converge to an incorrect shape (i.e. to a formation with interagent distances not all the same as those in the desired shape), and they conclude that the desired shape is not globally asymptotically stable.

In [13] and [14], we have shown that the incorrect stationary point discovered by [12] is in fact a saddle point, and for all practical purposes, unlikely to be attained. We should note that [13] has a fundamental mistake that renders its principal results incorrect. It purported but failed to present a proof of the instability of equilibria corresponding to an incorrect shape when the incorrect shapes contained certain acute angles. In [14], we corrected [13], provided insights into the structure of false equllibria, and showed that should such an equllibrium represent a rectangle, then it must be unstable.

We derive here two results about the false equilibria: That at most one true formation can be associated with a given false stationary point. That should the true formation be a rectangle then all false equilibria are necessarily unstable.

Section II, contains preliminaries. Section III primarily reviews such results of [14] as are used in this paper. Section IV introduces a matrix that is very useful for subsequent analysis, and also shows that a given false stationary point can only result from at most one desired formation. Section V proves that when the desired formation is a rectangle, all false stationary points are unstable. Section VI concludes.

II. Equations of Motion and Examples

We first present equations of motion for the four-agent formation shape control problem. We then show that the incorrect equlibrium example of [12] and is an unstable saddle.

A. Equations of Motion

Let $p = [p_1, p_2, p_3, p_4]^T \in \Re^8$ be a vector of the four agent positions in the plane. Following [12], we use a single

Dasgupta is with the University of Iowa, Iowa City, IA 52242, USA, dasgupta@engineering.uiowa.edu

Anderson is with the Australian National University and National ICT Australia, ACT 0200, Australia, brian.anderson@ anu.edu.au

Yu is with the Australian National University, ACT 0200, Australia, brad.yu@ anu.edu.au.

Summers is with ETH Zurich, Switzeland, tsummers@control.ee.ethz.ch

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integrator agent model to describe the motion of each agent $\dot{p}_i = u_i$ where u_i is the control input to be specified. Let $\bar{d} = [\bar{d}_{12}, \bar{d}_{13}, \bar{d}_{14}, \bar{d}_{23}, \bar{d}_{24}, \bar{d}_{34}]^T$ be a vector of desired interagent distances that define the formation shape. Let $d = [d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}]^T$, sometimes written as $d(p)$, denote instantaneous interagent distances, which are to be actively controlled to obtain $\bar{d}$. We assume that the entries of $\bar{d}$ correspond to a realizable shape.

Evidently,

$$\begin{aligned} d^2(p) &= [||p_1 - p_2||^2, ||p_1 - p_3||^2, ||p_1 - p_4||^2, \\ &\quad ||p_2 - p_3||^2, ||p_2 - p_4||^2, ||p_3 - p_4||^2]^T \end{aligned} \quad (1)$$

We define also the error function

$$e(p) = d^2(p) - \bar{d}^2 = [e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}]^T \quad (2)$$

The desired formation shape is a three-dimensional manifold in $\Re^8$, non-empty for a realizable $\bar{d}$ and given by

$$P_d = \{p \in \Re^8 | d^2(p) = \bar{d}^2\}. \quad (3)$$

Now consider the potential function

$$V(p) = \frac{1}{2}||e(p)||^2 = \frac{1}{2}(e_{12}^2 + e_{13}^2 + e_{14}^2 + e_{23}^2 + e_{24}^2 + e_{34}^2).$$

The function quantifies the total interagent distance error between the current formation and the desired formation $\bar{d}$. Note that $V \geq 0$ and $V = 0$ if and only if $e(p) = 0$, that is if and only if the formation is in the desired shape. Thus, V is a suitable potential function from which to derive a gradient control law. Accordingly, let the control input be given by $u = -\nabla V(p)$. Then the closed-loop system is given by $\dot{p} = -\nabla V(p) = -[J_e(p)]^T e(p)$, where $J_e(p)$ is the Jacobian of the error function $e(p)$ (also known as the *rigidity matrix*). With $\otimes$ the Kronecker product, this can be expressed in the following form

$$\dot{p} = -(E(p) \otimes I_2)p \quad (4)$$

where the matrix $E(p)$ is given by

$$E(p) = \quad (5)$$

$$\begin{bmatrix} e_{12}+e_{13}+e_{14} & -e_{12} & -e_{13} & -e_{14} \\ -e_{12} & e_{12}+e_{23}+e_{24} & -e_{23} & -e_{24} \\ -e_{13} & -e_{23} & e_{13}+e_{23}+e_{34} & -e_{34} \\ -e_{14} & -e_{24} & -e_{34} & e_{14}+e_{24}+e_{34} \end{bmatrix}.$$

B. Equilibrium points and their stability

Following the work of [12]-[14], convergence occurs to one of the stationary points of (4). Among these those corresponding to the true formation form a manifold. This is so because by definition the true formation is specified by interagent distances. If a given formation meets these distance specifications then so would any that is obtained by rotating and translating it. In [12] this manifold is shown to be locally attractive. Moreover, all initializations sufficiently close to it, lead to exponential convergence to a point on this manifold.

As demonstrated in [12] there are potentially other stationary points that do not meet the requisite distance specifications. These would be variously called incorrect/false equilibria. Each set of of these too comprises a manifold as if a particular $p = p^* = [p_1^{*T}, p_2^{*T}, p_3^{*T}, p_4^{*T}]^T$ is a stationary point of (4) then so is any obtained by a translation and rotation, i.e. any that for arbitrary orthogonal $\Omega \in \Re^{2\times 2}$ and $\omega \in \Re^2$ obeys,

$$p = [\Omega p_1^{*T} + \omega^T, \Omega p_2^{*T} + \omega^T, \Omega p_3^{*T} + \omega^T, \Omega p_4^{*T} + \omega^T]^T \quad (6)$$

Such manifolds are locally attractive only if the negative of the Jacobian of the right side of (4), which is the same as, $H_V(p)$, the Hessian of V, given below, is positive semidefinite, [13].

$$H_V(p) = 2R^T(p)R(p) + E \otimes I_2, \quad (7)$$

were $R(p)$ is the rigidity matrix of the K_4 graph corresponding to p, i.e. obeys:

$$R = \begin{bmatrix} p_1^T - p_2^T & p_2^T - p_1^T & 0 & 0 \\ p_1^T - p_3^T & 0 & p_3^T - p_1^T & 0 \\ p_1^T - p_4^T & 0 & 0 & p_4^T - p_1^T \\ 0 & p_2^T - p_3^T & p_3^T - p_2^T & 0 \\ 0 & p_2^T - p_4^T & 0 & p_4^T - p_2^T \\ 0 & 0 & p_3^T - p_4^T & p_4^T - p_3^T \end{bmatrix}. \quad (8)$$

It is readily seen that $H_V(p)$ is congruent to:

$$H = 2\begin{bmatrix} R_x^T \\ R_y^T \end{bmatrix} \begin{bmatrix} R_x & R_y \end{bmatrix} + \text{ diag } \{E, E\}, \quad (9)$$

where with $p_i = [x_i, y_i]^T$, R_x and R_y are obtained by, replacing p_i in (8) by x_i and y_i, respectively.

In the sequel we will call a false stationary point $p = p^*$ *unstable/locally unstable* if $H(p^*)$ is not positive semidefinite. The corresponding distances will be called d_{ij}^*. As argued in [13], the linearization of (4) around such a p^* has at least one real positive pole, and is consequently unstable. Such unstable equilibria, if at all attained, can rarely be maintained in the presence of noise or other inaccuracies, and should all such be unstable, then for all practical purposes global convergence to the true formation is assured.

A crucial point to be exploited in the sequel is that the stability of an equilibrium point is independent of rotation and translation of the formation shape. Similarly one can independently rotate and translate the desired formation without altering the stationary points or their stability as such transformations leave $\bar{d}$ unaltered.

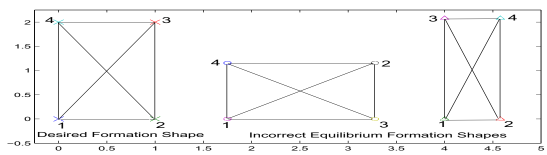

Fig. 1. The desired formation is rectangular and there are two possible "twisted rectangles" that are incorrect equilibrium shapes. Note that each formation has two different pairs of agents on the diagonals of the rectangle: (a) 13 and 24, (b) 12 and 34, and (c) 14 and 23.

C. The Krick Example

In [12], Krick et al study a four-agent example with K_4 information architecture where the formation appears to converge to an incorrect equilibrium formation shape. The example is as follows. Suppose the desired formation is a 1×2 rectangle given by $\bar{d}^2 = [1, 5, 4, 4, 5, 1]^T$. It is easy to verify that when we use the desired distances specified by $\bar{d}$, any formation with distance set $d^{*2} = [\frac{11}{3}, \frac{7}{3}, \frac{4}{3}, \frac{4}{3}, \frac{7}{3}, \frac{11}{3}]^T$ is also an equilibrium with incorrect interagent distances. The incorrect equilibrium is a "twisted rectangle" with the term twisted referring to a change in agent ordering, as illustrated in the middle formation of Figure 1. Krick et al concluded that the desired shape is not globally attractive since the control law appears to cause convergence to an incorrect equilibrium shape. However, as depicted through simulation in Figure 2, convergence is only apparent, occurring along the ridge of a saddle. There is initial convergence along a ridge to the saddle point representing a false equlibrium. The formation stays that way between about 1.5 to 2.5s, until numerical errors drive the formation away to the correct formation.The curves are the errors for the four agents from the true formation. Indeed, three eigenvalues of the Hessian evaluated at the incorrect shape are negative, [14], i.e. the shape is a saddle and thus unstable. Much of this paper is concerned with examining this sort of phenomenon.

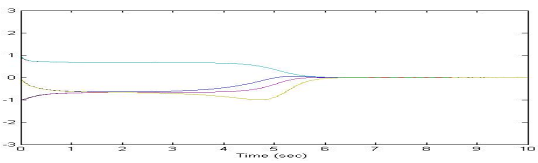

Fig. 2. Illustration of saddle point. The curves are the errors for the four agents from the true formation.

III. Some known properties of incorrect equilibria

In this section, we recount certain results from [14]. The first result concerns the structure of $E(p)$ and has largely been given in [14]. The proof is omitted.

Theorem 3.1: Suppose $p = p^*$ is an incorrect equilibrium of (4). Then $E(p^*)$ cannot be positive semidefinite. Further if p_1^* to p_4^* are noncollinear, then for some real $\mu > 0$ and $z \in \Re^4$, there holds:

$$E(p^*) = -\mu z z^T. \tag{10}$$

Further:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ p_1^* & p_2^* & p_3^* & p_4^* \end{bmatrix} z = 0. \tag{11}$$

We next set up a standing assumption to be made in most of subsequent results.

Theorem 3.2: Suppose the desired formation has a non-zero area. Also assume that some $p = p^*$ is a stationary point of of (4), and p_1^* to p_4^* are collinear. Then this stationary point is unstable.

Proof: As stability is invariant under translation and rotation of the incorrect formation one can without loss of generality assume it to be aligned with the x-axis, i.e. $R_y(p^*) = 0$. Then because of Theorem 3.1, $2R_y^T(p^*)R_y(p^*) + E(p^*)$ and hence $H(p^*)$ cannot be positive semidefinite. ■

As we are primarily interested in the stability of incorrect formations, in light of Theorem 3.2, in most of the subsequent results we make the following assumption.

Assumption 3.1: Both the desired formation and the formation representing a false stationary point have non-zero area.

The remaining results are all from [14].

Lemma 3.1: If there is an incorrect equilibrium with quadrilateral convex hull, with agent pairs 12, 34 diagonally opposite. Let $\bar{d}_{ij}$ and d_{ij}^* denote the distance sets of the correct and incorrect equilibrium respectively. Then there holds: $d_{12}^* > \bar{d}_{12}$, $d_{34}^* > \bar{d}_{34}$, $d_{13}^* < \bar{d}_{13}$, $d_{14}^* < \bar{d}_{14}$, $d_{23}^* < \bar{d}_{23}$, and $d_{24}^* < \bar{d}_{24}$.

A corresponding lemma applies to triangles.

Lemma 3.2: If there is an incorrect equilibrium shape with agent 2 lying in the convex hull of agents 1,3 and 4. Then there holds: $d_{12}^* < \bar{d}_{12}$, $d_{23}^* < \bar{d}_{23}$, $d_{24}^* < \bar{d}_{24}$, $d_{13}^* > \bar{d}_{13}$, $d_{14}^* > \bar{d}_{14}$, and $d_{34}^* > \bar{d}_{34}$.

Next we provide a twisting property for incorrect equilibria.

Theorem 3.3: Consider a four agent formation with a correct and an incorrect equilibrium which are both convex quadrilaterals. Then a pair of vertices that are diagonally opposite in the true formation are not diagonally opposite in the incorrect formation. Similarly suppose both formations are triangles of non-zero area. Then the two triangles cannot have the same agent in the convex hull of the remaining agents.

Finally, a result on incorrect formations that are rectangular:

Theorem 3.4: Suppose an incorrect stationary point represents a rectangle of nonzero area. Then it cannot be stable.

IV. Each false stationary point corresponds to a unique desired formation

Much of our analysis revolves around the rank-1 characterization of $E(p^*)$ provided in Theorem 3.1. Key to this approach is the vector $z = [z_1, z_2, z_3, z_4]^T$ in (10). This vector is specified by the p_i^* to within a scaling constant. Further, because of (5) and (10), there holds:

$$\bar{d}_{ij}^2 = d_{ij}^{*2} - \mu z_i z_j. \tag{12}$$

Given a set of $\bar{d}_{ij}$ and z, not every value of μ leads to d_{ij}^* that represent a valid formation in the plane. Whether a given set of interagent distances reflect a planar formation is provided by the Cayley-Menger matrix, [15]. However, for our purposes the Cayley-Menger matrix does not offer the requisite analytical facility. Instead we consider a matrix, proposed in [15] that does. Specifically for a given $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and some vector d of interagent distances, define:

$$F_i(d) = \begin{bmatrix} d_{ij}^2 & \frac{d_{ij}^2 + d_{ik}^2 - d_{kj}^2}{2} & \frac{d_{ij}^2 + d_{il}^2 - d_{lj}^2}{2} \\ \frac{d_{ij}^2 + d_{ik}^2 - d_{kj}^2}{2} & d_{ik}^2 & \frac{d_{il}^2 + d_{ik}^2 - d_{kl}^2}{2} \\ \frac{d_{ij}^2 + d_{il}^2 - d_{lj}^2}{2} & \frac{d_{il}^2 + d_{ik}^2 - d_{kl}^2}{2} & d_{il}^2 \end{bmatrix} = A_i^T A_i, \tag{13}$$

where $A_i = \begin{bmatrix} p_j - p_i & p_k - p_i & p_l - p_i \end{bmatrix}$. Also define:

$$G_i(z) = \begin{bmatrix} z_i z_j & \frac{z_i z_j + z_i z_k - z_k z_j}{2} & \frac{z_i z_j + z_i z_l - z_l z_j}{2} \\ \frac{z_i z_j + z_i z_k - z_k z_j}{2} & z_i z_k & \frac{z_i z_l + z_i z_k - z_k z_l}{2} \\ \frac{z_i z_j + z_i z_l - z_l z_j}{2} & \frac{z_i z_l + z_i z_k - z_k z_l}{2} & z_i z_l \end{bmatrix} \tag{14}$$

We have a key Theorem stated without proof.

Theorem 4.1: Suppose p^* is an incorrect stationary point, and $\bar{d}$ and d^* are the vectors of interagent distances in the desired formation and the formation represented by p^*. Suppose Assumption 3.1 holds, $\mu > 0$, z are as defined in Theorem 3.1, $F_i(\bar{d})$ and $F_i(d^*)$ are as defined in (13) and $G_i(z)$ is as in (14). Then with $\{i, j, k, l\} = \{1, 2, 3, 4\}$ for all i, the following hold:
(a)

$$F_i(\bar{d}) = F_i(d^*) - \mu G_i(z). \tag{15}$$

(b) Both $F_i(\bar{d})$ and $F_i(d^*)$ are positive semidefinite and have rank equal to two. (c) The vector $[z_j, z_k, z_l]^T$ is in the null space of $F_i(d^*)$. (d) Suppose η is in the null space of $F_i(\bar{d})$. Then there hold:

$$[z_j, z_k, z_l] G_i(z) [z_j, z_k, z_l]^T \leq 0. \tag{16}$$

$$\eta^T G_i(z) \eta \geq 0. \tag{17}$$

$$[z_j, z_k, z_l] G_i(z) \eta = 0. \tag{18}$$

(e) The matrices $F_i(\bar{d})$ and $F_i(d^*)$ do not share a nontrivial null vector. (f) The matrix $G_i(z)$ is not negative semidefinite.

Now we have the result implied by the title of this section that can be proved using (a,b,e,f) of the above Theorem.

Theorem 4.2: Suppose p^* is an incorrect stationary point, and Assumption 3.1 holds. Then there is at most one desired formation that gives rise to this incorrect stationary point.

V. Rectangular desired formations

In this section we prove that when the desired formation is a rectangle, all incorrect stationary points are locally unstable. Our approach to proving this is to first characterize the μ, z combinations that indirectly define the false stationary points. We show that these belong to three distinct categories. The first that can occur only if *the desired formation is a square*, and corresponds to an incorrect stationary point that defines a formation where two of the p_i^* are identical, and the formation is an isosceles right-angled triangle. The second corresponds to a isosceles trapezoid; the third to a rectangle.

Throughout we will assume that the desired formation is a rectangle with the same orientation as in Fig. 1(a). Further, we will assume that the horizontal and vertical sides have lengths whose squares are a and b, respectively. Our approach will extensively use F_i and G_i with $i = 1$ and j, k, l 2,3 and 4 respectively. It is easy to see that in this case with $u_1 = [1, 1, 0]^T$ and $u_2 = [0, 1, 1]$

$$F_1(\bar{d}) = a u_1 u_1^T + b u_2 u_2^T. \tag{19}$$

It is also evident that a null vector of this matrix is $[1, -1, 1]^T$.

A. Characterizing z

We begin with a Lemma.

Lemma 5.1: Suppose Assumption 3.1 holds and the desired formation is a rectangle with the same orientation as in Fig. 1(a), with the horizontal and vertical sides having lengths whose squares are a and b, respectively. Consider z as in Theorem 3.1, with $z_1 = 1$. Then:

$$(z_2 + z_4)\left[z_2 + z_4 - \left(z_2^2 + z_3^2 + z_4^2 + z_2 z_4\right)\right] = 0$$

Proof: Because of Theorem 4.1 and the fact that $[1, -1, 1]^T$ is a null vector of $F_1(\bar{d})$, $[1, -1, 1]^T G_1(z)[z_2, z_3, z_4]^T = 0$. As $z_1 = 1$, direct verification proves that this implies:

$$z_2^2 + z_4^2 + z_2 z_4 - \frac{z_2(z_3^2 + z_4^2)}{2} + \frac{z_3(z_2^2 + z_4^2)}{2} - \frac{z_4(z_3^2 + z_2^2)}{2} = 0$$

From (11) we have that $1 + z_2 + z_3 + z_4 = 0$. Substituting, the result is proved as

$$\frac{(z_2 + z_4)^2}{2} - \frac{(z_2 + z_4)(z_3^2 + z_2^2 + z_4^2 + z_2 z_4)}{2} = 0$$

■

The next Lemma establishes a structural property of z.

Lemma 5.2: Suppose Assumption 3.1 holds and the desired formation is a rectangle with the same orientation as in Fig. 1(a), with the horizontal and vertical sides having lengths whose squares are a and b, respectively. Then to within a scaling constant, z in Theorem 3.1 is one of the following:

$$z = [1, z_2, -1, -z_2]^T, \tag{20}$$

$$z = [0, z_2, 0, -z_2]^T. \tag{21}$$

Proof: Note, $[1, -1, 1]^T G_1(z)[z_2, z_3, z_4]^T = 0$ holds from Theorem 4.1 . First assume that $z_1 = 0$. In this case (14) holds. Further because of (11), $z_3 = -z_2 - z_4$. Direct verification shows that $[1, -1, 1]^T G_1(z)[z_2, z_3, z_4]^T$ equals $(z_2 + z_4)(2z_2^2 + 2z_4^2 + 3z_2 z_4) \geq (z_2^2 + z_4^2)/2$. Thus $z_2 = -z_4$. Then because of (11), (21) holds.

Now suppose $z_1 \neq 0$. Then to within a scaling of z one can assume that $z_1 = 1$. Because of Lemma 5.1, either $z_2 = -z_4$, in which case (20) holds because of (11) or

$$z_2 + z_4 - \left(z_2^2 + z_3^2 + z_4^2 + z_2 z_4\right) = 0. \tag{22}$$

Because of (11) some manipulation shows that the left hand-side of (22) equals $-(z_2^2 + z_4^2)/2 - (z_3 + 1)^2 - (z_3^2 + 1)/2 < 0$, violating (22). Thus (20) is true. ■

In either case for some β,

$$z_2 = -z_4 = \beta. \tag{23}$$

B. Implications of (21)

We now show that under (21) the desired formation must be square, and the false stationary point is an isosceles right triangle and moreover, unstable.

Lemma 5.3: Under the conditions of Lemma 5.2, suppose (21) and (23) hold. Then: (a) The false stationary point represents an isosceles right angled triangle with $d_{31}^{*2} = 2a$, and $d_{21}^{*2} = d_{23}^{*2} = a$; (b) $p_2^* = p_4^*$, (c) the desired formation is a square i.e. $a = b$; and (d) $\mu\beta^2 = 2a$.

Proof: First, (11) proves (b) as because the stationary point is false $z_2 \neq 0$. As $z_1 = z_3 = 0$, because of (12), for all i there hold, $d_{i1}^{*2} = \bar{d}_{i1}^2$ and $d_{i3}^{*2} = \bar{d}_{i3}^2$. Consequently, $d_{31}^{*2} = \bar{d}_{31}^2 = \bar{d}_{21}^2 + \bar{d}_{32}^2 = d_{21}^{*2} + d_{32}^{*2}$. Thus the false stationary point represents a right angled triangle with the segment joining p_1^* and p_3^* as the hypotenuse. Further, because of (b), $b = \bar{d}_{41}^2 = d_{41}^{*2} = d_{21}^{*2} = \bar{d}_{21}^2 = a$. Thus (a) and (c) hold. Further because of (12) and (b) $0 = d_{24}^{*2} = \bar{d}_{24}^2 - \mu\beta^2 = a + b - \mu\beta^2$. Thus (d) follows because of (c). ■

We now show that this stationary point represented by (21) cannot be stable.

Theorem 5.1: Under the conditions of Lemma 5.3, the false stationary point is unstable.

Proof: It suffices to show that $2R_x^T R_x - \mu zz^T$ is not positive semidefinite, where R_x is defined in Section (II). Because of Lemma 5.3 to within a translation and rotation it is enough to consider, $p_1^* = 0$, $p_2^* = p_4^* = [\sqrt{a}, 0]^T$ and $p_3^* = \sqrt{a}[1, \pm 1]^T$. Then using the value of μ proved in Lemma 5.3 one obtains the indefinite matrix:

$$2R_x^T R_x - \mu zz^T = 2a \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}.$$

■

C. Implications of (20)

We first show that under (20) the false stationary point is an isosceles trapezoid.

Lemma 5.4: Under the conditions of Lemma 5.2, suppose (20) and (23) hold. Then: (a) $(\beta^2 - 1)(d_{24}^{*2} - \mu) = 0$. (b) To within a rotation and a translation for some nonzero y, x_2, x_3 and x_4, with $x_3 = \beta(x_2 - x_4)$, there hold, $p_1^* = 0$, $p_2^* = [x_2, y]^T$, $p_3^* = [x_3, 0]^T$ and $p_4^* = [x_4, y]^T$. (c) The false stationary point represents an isosceles trapezoid with $d_{31}^{*2} = \beta^2 d_{24}^{*2}$, $d_{21}^{*2} = d_{43}^{*2}$ and $d_{41}^{*2} = d_{23}^{*2}$.

Proof: To within a rotation and translation one can always choose p_1^* and p_3^* as in (b). Then the rest of (b) follows as from (11):

$$\begin{bmatrix} \beta & -1 & -\beta \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ x_3 & 0 \\ x_4 & y_4 \end{bmatrix} = 0.$$

Then the false stationary point represents at least a trapezoid. Further as $y_2 = y_4$, $d_{31}^{*2} = x_3^2 = \beta^2(x_2 - x_4)^2 = \beta^2 d_{24}^{*2}$. The remaining equalities in (c) follow because from (12) one has, $\bar{d}_{12}^2 = d_{12}^{*2} - \beta\mu = \bar{d}_{34}^2 = d_{34}^{*2} - \beta\mu$ and $\bar{d}_{14}^2 = d_{14}^{*2} + \beta\mu = \bar{d}_{32}^2 = d_{32}^{*2} + \beta\mu$. Thus the figure is an isosceles trapezoid.

To prove (a) observe because of (12) $\bar{d}_{31}^2 = d_{31}^{*2} + \mu = \beta^2 d_{24}^{*2} + \mu$ and $\bar{d}_{24}^2 = d_{24}^{*2} + \mu\beta^2$. Then (a) follows because $\bar{d}_{31}^2 = \bar{d}_{24}^2$. ■

Observe (a) in particular presents two possibilities. One has $\beta^2 = 1$ and/or $d_{24}^{*2} = \mu$. The first possibility in fact forces the false stationary point to represent a rectangle. The next lemma is preparatory to proving this fact.

Lemma 5.5: Under the conditions of Lemma 5.4 suppose $\beta = 1$. Then $\mu = 2b/3$. If $\beta = -1$ then $\mu = 2a/3$.

Proof: Recall, (19) holds. When $\beta = 1$, using (14) we obtain that:

$$G_1(z) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & -\frac{3}{2} \\ \frac{1}{2} & -\frac{3}{2} & -1 \end{bmatrix} = \frac{1}{2}u_1 u_1^T - \frac{3}{2}u_2 u_2^T + \frac{1}{2}u_3 u_3^T.$$

Thus from (a) of Theorem 4.1

$$F_1(d^*) = \left(a + \frac{\mu}{2}\right) u_1 u_1^T + \left(b - \frac{3\mu}{2}\right) u_2 u_2^T + \frac{\mu}{2} u_3 u_3^T.$$

As $\mu > 0$, the result follows from (b) of Theorem 4.1. The case of $\beta = -1$ is similarly handled. ■

We now show that when $\beta^2 = 1$ the incorrect equilibrium is a rectangle. Because of Theorem 3.4 this must thus be unstable. This Lemma and subsequent development rely on the following equalities that follow because of (12) and (20).

$$\bar{d}_{12}^2 = d_{12}^{*2} - \mu z_1 z_2 = d_{12}^{*2} - \mu\beta \tag{24}$$

$$\bar{d}_{31}^2 = d_{31}^{*2} - \mu z_1 z_3 = d_{31}^{*2} + \mu \tag{25}$$

$$\bar{d}_{14}^2 = d_{14}^{*2} - \mu z_1 z_4 = d_{14}^{*2} + \mu\beta \tag{26}$$

$$\bar{d}_{23}^2 = d_{23}^{*2} - \mu z_2 z_3 = d_{23}^{*2} + \mu\beta \tag{27}$$

$$\bar{d}_{24}^2 = d_{24}^{*2} - \mu z_2 z_4 = d_{24}^{*2} + \mu\beta^2 \tag{28}$$

$$\bar{d}_{34}^2 = d_{34}^{*2} - \mu z_3 z_4 = d_{34}^{*2} - \mu\beta \tag{29}$$

Lemma 5.6: Under the conditions of Lemma 5.5 the false stationary point represents a rectangle.

Proof: Because of Lemma 5.5 $\mu = 2b/3$ when $\beta = 1$. In such a case (24) to (29) imply,

$$d_{12}^{*2} = d_{34}^{*2} = a + \frac{2b}{3}, d_{31}^{*2} = d_{24}^{*2} = a + \frac{b}{3} \text{ and } d_{14}^{*2} = d_{32}^{*2} = \frac{b}{3}.$$

Recall that the false stationary point represents a trapezoid. Further from Theorem 3.3 neither the pairs p_1^* and p_3^* and p_2^* and p_4^* are not diagonally opposite. Thus, the opposite sides are equal and the diagonals obey:

$$d_{12}^{*2} = d_{34}^{*2} = d_{31}^{*2} + d_{32}^{*2} = d_{31}^{*2} + d_{14}^{*2} = d_{41}^{*2} + d_{42}^{*2} = d_{24}^{*2} + d_{32}^{*2}.$$

The case $\beta = -1$ is similarly handled. ■

There remains (20) under $d_{24}^{*2} = \mu$. We fold this and the other cases into the following main result of this section.

Theorem 5.2: Suppose the desired formation is a rectangle with the same orientation as in Fig. 1(a), with the horizontal and vertical sides having lengths whose squares are a and b, respectively. Then every incorrect stationary point of (4) is unstable.

Proof: Because of Theorems 3.2, 5.1, and 3.4, and Lemmas 5.2 to 5.6 we only need to consider the case where under Assumption 3.1, $d_{24}^{*2} = \mu$ and (20) hold. To prove the theorem it suffices to show that $2R_y^T R_y - \mu zz^T$ is not positive semidefinite. From (b) of Lemma 5.1, the definition of R_y, and he fact that $d_{24}^{*2} = \mu$ it follows that

$$2R_y^T R_y - \mu z z^T = 2y^2 \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} - d_{24}^{*2} z z^T. \tag{30}$$

To establish a contradiction, suppose $B = 2R_y^T R_y - \mu z z^T$ is positive semidefinite. Then using (20) and (23) one must have:

$$\begin{aligned} z^T B z &= 2y^2 z^T \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} z - d_{24}^{*2} (z^T z)^2 \\ &= 8y^2(1+\beta^2) - 4d_{24}^{*2}(1+\beta^2)^2 \geq 0 \\ &\Rightarrow 2y^2(1+\beta^2) - d_{24}^{*2}(1+\beta^2)^2 \geq 0. \end{aligned}$$

Now observe from (28) (20) and $d_{24}^{*2} = \mu$ that $a+b = \bar{d}_{24}^2 = d_{24}^{*2} - \mu z_2 z_4 = d_{24}^{*2}(1+\beta^2)$. Thus $2y^2 \geq a+b$. Using (b) of Lemma 5.1, (24), (26), (27) and (29) we have a contradiction as under Assumption 3.1 :

$$\begin{aligned} 4y^2 &< 4y^2 + x_2^2 + x_4^2 + (x_2 - x_3)^2 + (x_4 - x_3)^2 \\ &= \bar{d}_{12}^2 + \bar{d}_{14}^2 + \bar{d}_{32}^2 + \bar{d}_{34}^2 = 2(a+b). \end{aligned}$$

■

VI. Conclusions and future work

Our ultimate goal is to show how to control an arbitrary formation to a shape that is uniquely specified, i.e. specified up to congruence, by a sufficiently large number of distance constraints (in effect, the associated graph must be what is known as *globally rigid*). A K_4 graph is the second simplest such graph, after the triangle, and one might reasonably expect that a general theory would need to properly encompass this special case. In this paper we have shown that for such a K_4 graph each incorrect equilibrium can result from at most one desired formation. We have also shown that rectangular desired formations have incorrect equilibria that are necessarily unstable. The most significant open problem is to show whether or not there can ever exist an incorrect attractive equilibrium for arbitrary K_4 formations. We believe that there are certain avenues that must be explored *en route* to such a result. First, while we have shown that each incorrect equilibrium is tied to at most one desired formation, the following reverse question should be answered: How many false equilibria are associated with a given desired formation? One possible tool for addressing such a question is Morse theory, [16]. An initial application of Morse theory in this context is presented in [17]. Ultimately to examine whether stable false equilibria exist at all, we need to determine whether the underlying Hessian is positive semidefinite. We believe that this would involve the sizing of the parameter μ. The matrix F_i introduced in Section IV may hold the key to this task.

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References

[1] T. Eren, P.N. Belhumeur, B.D.O. Anderson, and A.S. Morse. A framework for maintaining formations based on rigidity. In *Proceedings of the 15th IFAC World Congress, Barcelona, Spain*, pages 2752–2757, 2002.

[2] J.E. Graver, B. Servatius, and H. Servatius. *Combinatorial rigidity*. American Mathematical Society, 1993.

[3] R. Olfati-Saber and RM Murray. Graph rigidity and distributed formation stabilization of multi-vehicle systems. In *Proceedings of the 41nd IEEE Conference on Decision and Control, Las Vegas, NV*, pages 2965–2971, 2002.

[4] J. Baillieul and A. Suri. Information patterns and Hedging Brockett's theorem in controlling vehicle formations. In *Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, HI*, pages 556–563, 2003.

[5] J.M. Hendrickx, B.D.O. Anderson, J.C. Delvenne, and V.D. Blondel. Directed graphs for the analysis of rigidity and persistence in autonomous agent systems. *International Journal of Robust and Nonlinear Control*, 17(10-11):960–981, 2007.

[6] C. Yu, J.M. Hendrickx, B. Fidan, B.D.O. Anderson, and V.D. Blondel. Three and higher dimensional autonomous formations: Rigidity, persistence and structural persistence. *Automatica*, 43(3):387–402, 2007.

[7] C. Yu, B.D.O. Anderson, S. Dasgupta, and B. Fidan. Control of Minimally Persistent Formations in the Plane. *SIAM Journal on Control and Optimization*, 48(1):206–233, 2009.

[8] B.D.O. Anderson, C. Yu, S. Dasgupta, and A.S. Morse. Control of a three coleaders persistent formation in the plane. *Systems and Control Letters*, 56(9-10):573–578, Sep-Oct 2007.

[9] M. Cao, A.S. Morse, C. Yu, B.D.O. Anderson, and S Dasgupta. Controlling a triangular formation of mobile autonomous agents. In *Proc. 46th IEEE Conference on Decision and Control,New Orleans, LA*, pages 3603–3608, December 2007.

[10] M. Cao, C. Yu, A.S. Morse, B.D.O. Anderson, and S. Dasgupta. Generalized controller for directed triangle formations. In *Proc. of the IFAC World Congress, Seoul, Korea*, pages 6590–6595, July 2008.

[11] F. Dorfler and B. Francis. Geometric analysis of the formation problem for autonomous robots. *IEEE Transactions on Automatic Control*, pages 2379 – 2384, 2010.

[12] L. Krick, M.E. Broucke, and B.A. Francis. Stabilization of Infinitesimally Rigid Formations of Multi-Robot Networks. *International Journal of Control*, 82(3):423–439, 2009.

[13] T. Summers, C. Yu, B.D.O. Anderson, and S. Dasgupta. Formation shape control: Global asymptotic stability of a four-agent formation. In *Proc. of the 48th IEEE CDC, Shanghai, China*, Dec 2009.

[14] B.D.O. Anderson, C. Yu, S. Dasgupta, and T.H. Summers. Controlling four agent formations. In *Proceedings of 2nd IFAC Workshop on Distributed Estimation and Control in Networked Systems, (NecSys 10), Annency, France*, pages 139–144, September 2010.

[15] M.J. Sipple and H. A. Scheraga. Solution of the embedding problem and decomposition of symmetric matrices. *Proc. Natl. Acad. of Sci., USA*, 82:2197–2201, 1985.

[16] J.W. Milnor. *Morse Theory*. Princeton University Press, 1963.

[17] B.D.O. Anderson. Morse Theory and Formation Control. In *Proceedings of19th Mediterranean Conference on Control and Automation, Corfu, Greece, to appear*, 2011.