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RELATIONS BETWEEN REAL AND COMPLEX POLYNOMIALS FOR STABILITY AND APERIODICITY CONDITIONS

B. D. O. Anderson
Department of Electrical Engineering
University of Newcastle
New South Wales, 2308
Australia

E. I. Jury and F. L. C. Chaparro
Department of Electrical Engineering
University of California
Berkeley, California, 94720
U.S.A.

Abstract

Results are obtained connecting the zero distribution of a real polynomial and a complex polynomial of approximately half the degree of the real polynomial. The results are applied to the relating of real aperiodic polynomials with real and complex Hurwitz polynomials. "Unit circle" results are also outlined.

1. Introduction

A real monic polynomial of degree $2n$ and a complex monic polynomial of degree n are both described by $2n$ parameters; here, we show how from a real $2n$ degree polynomial which is Hurwitz, i.e. has all its zeros in $Re[z] < 0$, we can construct a complex n degree polynomial (with parameters in 1-1 relationship with those of the real polynomial) which is also Hurwitz. In fact, two different complex polynomials can be found, the result extends to odd degree real polynomials, and a converse can be stated. All this is the content of Theorem 1.

The results of Theorem 1 are probably known to many people, although we are unaware of the existence of a statement as complete as that given here. Further, we are unaware whether all the various proofs (suggested in Section 2 only in outline) are fully known.

In Section 3, Theorem 2 applies the results of Theorem 1 to aperiodic polynomials (those with zeros which are distinct, and negative real). We show how two previously stated criteria for aperiodicity can be immediately related, and we present a minor but new variation on one of these earlier stated criteria.

In Section 4, we offer concluding remarks, including a discussion of "unit circle" versions of the result.

2. Association of Real and Complex Polynomials

The main result relating the zero distribution of a real polynomial and a complex polynomial of approximately half the degree is as follows:

<u>Theorem 1</u> For real a_i, $i=0,1,\dots m$, consider the three polynomials

$$F_0(z) = \sum_{i=0}^{m} a_i z^i \qquad (1)$$

$$F_1(z) = \sum_{i=0}^{[m/2]} [a_{2i}(j)^i + a_{2i+1}(j)^{i+1}]z^i \qquad (2)$$

$$F_2(z) = \sum_{i=0}^{[m/2+1]} [a_{2i}(-j)^i - a_{2i-1}(-j)^{i+1}]z^i \qquad (3)$$

Here $[r]$ denotes the greatest integer less than or equal to the real number r, and in (2) and (3) one sets $a_k = 0$ if k is outside the range $[0, m]$. Suppose that $a_0 > 0$, $a_m > 0$, and $a_{2i} > 0$ or $a_{2i+1} > 0$ for $i = 1,2,\dots,[m/2]$. Then if any of the polynomials $F_k(z)$, $k = 0,1,2$ has its zeros in $Re[z] < 0$, the other two polynomials have this property.

<u>Remarks 1</u>. If $F_0(z)$ is Hurwitz, it is trivial to show that $a_m > 0$ implies $a_i > 0$ for all i. However, if $F_1(z)$ or $F_2(z)$ in (2) or (3) are Hurwitz, it does not necessarily follow that the sign conditions on the a_i of the theorem statement are fulfilled.

2. The theorem shows how to associate with any real Hurwitz polynomial two complex Hurwitz polynomials. Not every complex Hurwitz polynomial however has the form of $F_1(z)$ or $F_2(z)$ with the sign constraints on the a_i. Thus the mapping {real Hurwitz polynomials of degree m} $\to$ {complex Hurwitz polynomials of degree $[m/2]$ or $[m/2+1]$}, $F_0(z) \mapsto F_1(z)$ or $F_2(z)$, is not onto.

3. There is an obvious mapping {complex Hurwitz polynomials of degree n} $\to$ {real Hurwitz polynomials of degree $2n$}, viz. $f(z) \to f(z)f^*(z)$, where f^* is obtained from f by coefficient conjugation. This mapping (which again is not onto) is not related to that in the theorem.

4. If we write $F_0(z)$ in terms of its even and odd parts as

$$F_0(z) = G(z^2) + zH(z^2) \qquad (4)$$

then

$$F_1(z) = G(jz) + jH(jz)$$

$$F_2(z) = G(-jz) + zH(-jz) \qquad (5)$$

5. Some conclusions can be obtained from the fact that if $\sum_{i=0}^{m} b_i z^i$ is Hurwitz, so is $\sum_{i=0}^{m} b_{m-i} z^i$. (The zeros of the two polynomials are reciprocals). Thus, suppose $F_0(z)$ is known to be Hurwitz, and suppose the claims of Theorem 1 involving only $F_0(z)$ and $F_1(z)$ for m even have been established. With $m = 2n$, the following polynomials are then Hurwitz:

$$F_3(z) = \sum_{i=0}^{2n} a_{2n-i} z^i \qquad \text{(by coefficient reversal)}$$

$$F_4(z) = \sum_{i=0}^{n} [a_{2n-2i}(j)^i + a_{2n-2i-1}(j)^{i+1}]z^i \qquad \text{(by Theorem 1)}$$

$$F_5(z) = \sum_{i=0}^{n} [a_{2i}(j)^{n-i} + a_{2i-1}(j)^{n-i+1}]z^i \qquad \text{(by coefficient reversal)}$$

Observe that $F_5(z)$ is none other than $j^n F_2(z)$. This means that if the claims regarding $F_0(z)$ and $F_1(z)$ are established, part at least of those involving $F_2(z)$ follow easily. For m odd however, one cannot deduce claims concerning $F_2(z)$ from those involving $F_0(z)$ and $F_1(z)$ as the polynomial corresponding to $F_5(z)$ becomes a multiple of $F_1(z)$.

6. One proof of this result easily follows using Hurwitz determinants. The Hurwitz character of $F_1(z)$ can be described by positivity conditions on the even dimension leading principal minors of $m \times m$ matrix

$$\begin{bmatrix} a_m & a_{m-2} & a_{m-4} & \cdots \\ a_{m-1} & a_{m-3} & a_{m-5} & \cdots \\ 0 & a_m & a_{m-2} & \cdots \\ 0 & a_{m-1} & a_{m-3} & \cdots \\ 0 & 0 & a_m & \cdots \\ 0 & 0 & a_{m-1} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \qquad (a_k = 0 \quad \text{for} \quad k < 0)$$

for m even, and with minor variation when m is odd, as noted in [1, see pp. 248-250]. The same conditions together with the sign conditions on the a_i guarantee by the Lienard-Chipart criterion that $F_0(z)$ is Hurwitz, [1, p.221]. The idea is not hard to extend to $F_2(z)$.

7. A second proof will follow using the reduced Hermite test as described in, for example, [2]. Direct calculation will show that the Hermite matrices for $F_1(z)$ and $F_2(z)$ are essentially the same as the two reduced Hermite matrices for $F_0(z)$. The sign constraints on the a_i and positive definiteness of either reduced Hermite matrix imply $F_0(z)$ is Hurwitz, while positive definiteness of the Hermite matrix of $F_1(z)$ guarantees the Hurwitz property for that polynomial and likewise for $F_2(z)$. In this way, Theorem 1 follows.

8. A third way of establishing Theorem 1 is to count the zeros of $F_0(z)$, $F_1(z)$ and $F_2(z)$ in $\text{Re}[z] < 0$ by observing the change in argument of these polynomials as z moves around a contour comprising the imaginary axis, and a semicircle of arbitrarily large radius extending in the left ha half plane. The Cauchy index may be used to compute this change of argument, as is standard for this sort of problem, see [1, Chapter XV]; the relations (4) and (5) prove crucial in relating the zeros of $F_0(z)$, $F_1(z)$ and $F_2(z)$. Actually, the argument closely follows that of [1, pp. 221-225] used in establishing the Liénard-Chipart stability criterion from the Hurwitz criterion.

3. Aperiodicity Conditions

A polynomial $f_0(z)$ is termed aperiodic if all its zeros are distinct and negative real. Tests for aperiodicity appear in [3-5].

Here, we link up these tests and the theorem of Section 1.

Theorem 2 Suppose

$$f_0(z) = \sum_{i=0}^{n} \alpha_i z^i$$

and

$$F_0(z) = f_0(z^2) + z\,\frac{df_0(z^2)}{d(z^2)}$$

$$F_1(z) = f_0(jz) + j\,\frac{df_0(jz)}{d(jz)}$$

$$F_2(z) = f_0(-jz) + z\,\frac{df_0(-jz)}{d(-jz)}$$

Suppose also that $\alpha_i > 0$, $i = 0,1,\ldots,n$. Then aperiodicity of $f_0(z)$ implies and is implied by the Hurwitz nature of $F_0(z)$, $F_1(z)$ or $F_2(z)$.

Remarks 1. $F_0(z)$ is a real 2n-th degree polynomial, and $F_1(z)$ and $F_2(z)$ are complex n-th degree polynomials. Moreover, $F_0(z)$, $F_1(z)$ and $F_2(z)$ are related in the same way as they are in Theorem 1. Also, positivity of the α_i implies and is implied by positivity of a_0, a_{2n} and a_{2i} or a_{2i+1} for $i = 1,2,\ldots,(n-1)$, where $F_0(z) = \sum_{i=0}^{2n} a_i z^i$.

2. Aperiodicity of $f_0(z)$ and the Hurwitz property of $F_0(z)$ are shown to be equivalent in [3, 4]. A speedy proof is given below. Aperiodicity of $f_0(z)$ and the Hurwitz property of $F_1(z)$ are shown to be equivalent in [5]. Therefore, theorem 1 relates the results of [3, 4] and [5], and also introduces $F_2(z)$ into the aperiodicity picture.

3. To prove theorem 2, we recall an easily proved and fairly well known result, see [1, p.228]: the polynomial $f(z) = h(z^2) + zg(z^2)$ is Hurwitz if and only if the zeros of $h(z)$ and $g(z)$ are distinct, negative real, and interlace, and the highest coefficients of $h(z)$ and $g(z)$ are of like sign.

Identify $F_0(z)$ with $f(z)$, $f_0(z)$ with $h(z)$ and $\frac{df_0(z)}{dz}$ with $g(z)$. Then if $F_0(z)$ is Hurwitz, $f_0(z)$ is immediately aperiodic, while if $f_0(z)$ is aperiodic, it is evident that its zeros will interlace those of $\frac{df_0(z)}{dz}$; the other requirements of negative realness and sign identity of the highest coefficients can be checked, so that $F_0(z)$ is Hurwitz.

4. Conclusions

We have shown how any real Hurwitz polynomial defines two complex Hurwitz polynomials of approximately one half the degree, and we have described various ways (Hurwitz determinants, Hermite matrices, Cauchy index method of zero counting) which can be used to view this result. We have also shown the application of these ideas in a discussion of aperiodicity conditions.

One might now ask whether there are unit circle equivalents of these results. Let $w = \frac{z+1}{z-1}$ (thus also $z = \frac{w+1}{w-1}$) and let $G_0(w)$ be an arbitrary degree m real polynomial in w. One can construct the sequence

$$G_0(w) \mapsto F_0(z) \mapsto F_1(z) \mapsto G_1(w)$$

via

$$F_0(z) = (z-1)^m G\left(\frac{z+1}{z-1}\right), \quad G_1(w) = (w-1)^{[m/2]} F_1\left(\frac{w+1}{w-1}\right)$$

to obtain a polynomial $G_1(w)$ of degree $[m/2]$ which has all zeros inside $|z| = 1$ if and only if $G_0(w)$ has this property, given also the satisfaction of certain inequalities involving linear combinations of the coefficients of $G_0(w)$. Using results of [6] relating the Schur-Cohn matrix and Hermite matrix of two polynomials related by transformations of the above type, together with results of [7] describing a reduced Schur-Cohn criterion (analogous to the reduced Hermite criterion), one can even show that a reduced Schur-Cohn criterion matrix associated with $G_0(w)$ is (to within an inessential, and coefficient independent, transformation) the same as the Schur-Cohn matrix of $G_1(w)$. [Similar results hold if $F_2(z)$ is used in lieu of $F_1(z)$]. All this seems of little interest however, because the mapping $G_0(w) \rightarrow G_1(w)$ *is not* one that has a straightforward explicit description. Each coefficient of $G_1(w)$ will be a linear combination of (normally) all the coefficients of $G_0(w)$, weighted via various products of binomial coefficients, so that the aesthetically pleasing simplicity of the relation between $F_0(z)$ and $F_1(z)$ [see (1) and (2)] is lost.

References

[1] Gantmacher, F.R., The Theory of Matrices, Chelsea, New York, 1959.

[2] Anderson, B.D.O., "The Reduced Order Hermite Criterion with Application to Proof of the Liénard-Chipart Criterion", IEEE Transactions on Automatic Control, Vol AC-17, Oct. 1972, pp. 669-672.

[3] Fuller, A.T., "Conditions for Aperiodicity in Linear Systems", Brit. J. Appl. Phys., Vol 6, June 1965, pp. 195-198.

[4] Romanov, M.I., "Algebraic Criteria for Aperiodicity of Linear Systems", Dokl. Akad. Nank, SSSR, Vol 12f, No 2, 1959, pp. 291-294.

[5] Meerov, M.V., "Criteria for Aperiodic Systems", Isvest. Akad. Nauk, SSSR, OTN, No 12, 1945, p.1169.

[6] Jury, E.I., and Anderson, B.D.O., "On the Reduced Hermite and Reduced Schur-Cohn Matrix Relationships", International Journal of Control, to appear.

[7] Anderson, B.D.O., and Jury, E.I., "A Simplified Schur-Cohn Test", IEEE Transactions on Automatic Control, Vol AC-18, No 2, Apr. 1973, pp. 157-163.