Properties of Blocked Linear Systems ⋆

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Abstract: This paper presents a systematic study on the properties of blocked linear systems that are resulted from blocking discrete-time linear time invariant systems. The main idea is to explore the relationship between the blocked and the unblocked systems. Existing results are reviewed and a number of important new results are derived. Focus is given particularly on the zero properties of the blocked system as no such a study has been found in the literature.

1. INTRODUCTION

Blocking (or lifting) is an important technique that has been used in signal processing and multirate sampled-data systems (Chen and Francis [1991]; Meyer and Burrus [1975]).

In the literature, the blocking technique has most often been used to transform linear discrete-time periodic systems into linear time-invariant systems so that the well-established analysis and design tools in linear time-invariant systems can be extended to linear discrete-time periodic systems (Meyer and Burrus [1975]; Bolzern et al. [1986]; Grasselli and Longhi [1988]; Grasselli et al. [1995]; Colaneri and Kucera [1997]); Bittanti and Colaneri [2009]. For example, the notions of poles and zeros of linear time-invariant systems have been extended to linear periodic systems in Bolzern et al. [1986] and Grasselli and Longhi [1988]. The structural properties such as observability and reachability have been studied in Bittanti [1986], Grasselli and Longhi [1991], Gohberg et al. [1992] and Bittanti and Colaneri [2009]. The realization problem has been researched in Colaneri and Longhi [1995] and the related references listed in Bittanti and Colaneri [2009].

The blocking technique has also been used to block linear time-invariant systems. For example, in Chen and Francis [1991], linear time-invariant discrete-time systems have been blocked for the purpose of dealing with multi-rate sampled-data systems, while in Khargonecker et al. [1985], linear time-invariant discrete-time systems have been blocked for the design of periodic controllers.

In this paper, a systematic study will be presented on the properties of the blocked systems resulting from blocking linear time-invariant systems. There are several reasons for doing this research. First, the blocked systems of linear time-invariant systems are useful in multirate sampled-data systems and in controller design as shown by Chen and Francis [1991] and Khargonecker et al. [1985]. Second, as linear time-invariant systems can be viewed as a special class of linear periodic systems, the results developed for linear periodic systems will have their counterparts for linear time-invariant systems and most likely they will be nicer and will be easier to prove. The purpose here is to review some of these results and offer simpler proofs for them. Lastly, it is not clear how the zeros of the blocked system relate to the zeros of the unblocked linear time-invariant system although it is well understood how the poles of the blocked system relate to those of the unblocked time-invariant system (Khargonecker et al. [1985]).

The main idea of this study is to explore and reveal the relationship between the blocked and the unblocked linear time-invariant systems. In this way, some structural properties are reviewed and simpler proofs are offered. Moreover, a number of important new results are obtained. Particularly, a relationship between the normal ranks of the transfer functions of the blocked system and the unblocked system is discovered. More importantly, the relationship between the zeros of the blocked system and the unblocked system is established and nice results are derived.

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2. THE UNBLOCKED AND BLOCKED SYSTEMS

The unblocked system is defined by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $y_k \in \mathbb{R}^p$ the output, and $u_k \in \mathbb{R}^m$ the input, which may be white noise.

For the unblocked system, its transfer function is defined as

$$W(z) = [D + C(zI - A)^{-1}B], \tag{2}$$

where z is used as both a forward-shift operator and a complex number.

Throughout this paper, the following assumption, which is effectively just a full normal rank assumption, will be used.

Assumption 1: The dimension of the output vector is not smaller than the dimension of the input vector, i.e. $p \geq m$ and the normal rank of $W(z)$ is m.

Define

$$Y_k = \begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+N-1} \end{bmatrix}, \quad U_k = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+N-1} \end{bmatrix}, k = 0, N, 2N, \cdots \tag{3}$$

Then, the blocked system is defined by

$$\begin{aligned} x_{k+N} &= A_b x_k + B_b U_k \\ Y_k &= C_b x_k + D_b U_k \end{aligned} \tag{4}$$

where

$$A_b = A^N, \; B_b = [\, A^{N-1}B \quad A^{N-2}B \quad \cdots \quad B \,],$$
$$C_b = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix},$$
$$D_b = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \cdots & D \end{bmatrix}. \tag{5}$$

Define an operator Z such that it satisfies $Zx_k = x_{k+N}, ZY_k = Y_{k+N}, ZU_k = U_{k+N}$. Then the transfer function of the blocked system is given by

$$V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b. \tag{6}$$

In this paper the relationship between the unblocked system (1) and the blocked system (4) will be investigated.

3. EXISTING RESULTS

In this section, some existing results will be reviewed and proved.

3.1 Observability, reachability, and minimal realization

The concepts of observability, reachability, and minimal realization are defined as follows: The system (1) is said to be *reachable* if the matrix $[B \; AB \; ... \; A^{n-1}B]$ is of full row rank, and it is said to be *observable* if the matrix $[C' \;\; A'C' \;\; ... \;\; (A')^{n-1}C']'$ is of full column rank, where $'$ means transpose. Given a transfer function $W(z)$, the system (1) is said to be a *minimal realization* of $W(z)$ if the system (1) is reachable and observable.

The results obtained in Bittanti [1986], Grasselli and Longhi [1991], Gohberg et al. [1992] and Colaneri and Longhi [1995] for linear periodic systems, when specialized to linear time-invariant systems, lead to the following theorem.

Theorem 1. Consider the unblocked system (1) with transfer function $W(z)$ given by (2) and the blocked system (4) with transfer function $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, where A_b, B_b, C_b, D_b are defined by (5) . Then

- The blocked system (4) is reachable if and only if the unblocked system (1) is reachable.
- The blocked system (4) is observable if and only if the unblocked system (1) is observable.
- The blocked system (4) is a minimal realization of $V(Z)$ if and only if the unblocked system (1) is a minimal realization of $W(z)$.

Proof: Suppose the unblocked system(1) is not reachable, then there exists a nonzero vector w and a scalar λ such that $w^*A = \lambda w^*, w^*B = 0$, where * means conjugate and transpose. It follows that $w^*A^N = \lambda^N w^*, w^*A^jB = 0, j = 0, 1, \cdots, N-1$, i.e. the blocked system (4) is not reachable. Conversely, suppose that the blocked system (4) is not reachable, then there exists a nonzero vector w and a scalar μ such that

$$w^*A^N = \mu w^*, w^*\,[\, A^{N-1}B \quad A^{N-2}B \quad \cdots \quad B \,] = 0.$$

Since $w^*\,[\, A^{N-1}B \quad A^{N-2}B \quad \cdots \quad B \,] = 0$ implies that

$$w^*\,[\, B \quad AB \quad \cdots \quad A^{N-1}B \quad A^NB \quad A^{N+1}B, \cdots] = 0,$$

it follows that $w^*\,[\, B \quad AB \quad \cdots \quad A^{n-1}B \,] = 0$ and thus the unblocked system (1) is not reachable. This proves the first claim. The second claim follows similarly. The third claim is an immediate result of the first two claims. ¶

3.2 Transfer functions of the blocked and unblocked systems

In this subsection, the relationship between $W(z)$ and $V(Z)$ will be reviewed. The following results were provided in Khargonekar et al. [1985] and Bittanti and Colaneri [2009].

Theorem 2. Consider the unblocked system (1) with transfer function $W(z)$ and the blocked system (4) with transfer function $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, where A_b, B_b, C_b, D_b are defined by (5) . Then

$$V(Z)=\begin{bmatrix} V_1(Z) & Z^{-1}V_N(Z) & Z^{-1}V_{N-1}(Z) & \cdots & Z^{-1}V_2(Z) \\ V_2(Z) & V_1(Z) & Z^{-1}V_N(Z) & \ddots & Z^{-1}V_3(Z) \\ V_3(Z) & V_2(Z) & V_1(Z) & \ddots & Z^{-1}V_4(Z) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ V_N(Z) & V_{N-1}(Z) & V_{N-2}(Z) & \cdots & V_1(Z) \end{bmatrix}. \tag{7}$$

and

$$W(z)=V_1(z^N)+z^{-1}V_2(z^N)+\cdots+z^{-(N-1)}V_N(z^N). \tag{8}$$

where $V_1(Z)=D+C(ZI-A^N)^{-1}A^{N-1}B$ and $V_j(Z)=CA^{l-2}B+C(ZI-A^N)^{-1}A^{N+l-2}B, j=2,\cdots,N$.

4. NEW RESULTS

In this section, a number of new results will be provided on the properties of the blocked system (4). The matrix fraction description (MFD) of a transfer function will be used as the main tool to derive the main results.

Suppose the unblocked system (1) is a minimal realization of $W(z)$. Since the poles of the unblocked and blocked systems are the eigenvalues of A and A^N, it is obvious that Z_p is a pole of $V(Z)$ if and only if $W(z)$ has a pole at z_p with $z_p^N=Z_p$ for one or more of the Nth roots of Z_p Khargonecka et al. [1985]. As the relationship between the poles of the blocked and unblocked systems is now well understood, focus will bee given particularly on system zeros in this paper. We can conjecture that zeros of blocked and unblocked systems may be related in a like way to poles. Indeed, for square systems, since zeros of a system are poles of the inverse, this should be no surprise. The result however is less obvious for nonsquare systems, and zeros at infinity are also of interest.

A definition of system zeros is needed in this section, and the one defined in Anderson and Deistler [2009] is quoted below for convenience (combining as it does finite and infinite zeros in the one definition).

Definition 1. The finite zeros of the transfer function $W(z)$ with minimal realization $\{A,B,C,D\}$ are defined to be the finite values of z for which the rank of the following matrix falls below its normal rank

$$M(z)=\begin{bmatrix} zI-A & B \\ C & D\end{bmatrix}. \tag{9}$$

Further, $W(z)$ is said to have an infinite zero when $n+rk(D)$ is less than the normal rank of $M(z)$, or equivalently the rank of D is less than the normal rank of $W(z)$.

4.1 MFDs of the blocked and unblocked systems

Suppose the transfer function $W(z)$ of the unblocked system has a coprime matrix fraction description (MFD) as

$$W(z)=P^{-1}(z)Q(z) \tag{10}$$

where

$$P(z)=P_\mu+P_{\mu-1}z+\cdots+P_0z^\mu,$$
$$Q(z)=Q_\mu+Q_{\mu-1}z+\cdots+Q_0z^\mu. \tag{11}$$

where μ is defined so that P_0 and Q_0 are not both zero. By coprimeness, P_μ and Q_μ are not both zero.

Since the pair $(P(z),Q(z))$ is coprime, according to Kailath [1980] and Wolovich [1974], the *finite zeros* of $W(z)$ are those values of z such that the numerator matrix $Q(z)$ has rank less than its normal rank.

Using (10) and (11), it is easy to see that $y_k=W(z)u_k$ can be rewritten as

$$P_0y_{k+\mu}+P_1y_{k+\mu-1}+\cdots+P_{\mu-1}y_{k+1}+P_\mu y_k=Q_0u_{k+\mu}$$
$$+Q_1u_{k+\mu-1}+\cdots+Q_{\mu-1}u_{k+1}+Q_\mu u_k. \tag{12}$$

The above can be further rewritten as

$$P_0y_k+P_1y_{k-1}+\cdots+P_{\mu-1}y_{k-\mu+1}+P_\mu y_{k-\mu}=Q_0u_k$$
$$+Q_1u_{k-1}+\cdots+Q_{\mu-1}u_{k-\mu+1}+Q_\mu u_{k-\mu}. \tag{13}$$

Define

$$\tilde{P}(z)=P_0+P_1z^{-1}+\cdots+P_\mu z^{-\mu},$$
$$\tilde{Q}(z)=Q_0+Q_1z^{-1}+\cdots+Q_\mu z^{-\mu}. \tag{14}$$

It is easy to see that

$$\tilde{P}(z)=z^{-\mu}P(z),\ \ \tilde{Q}(z)=z^{-\mu}Q(z). \tag{15}$$

and

$$W(z)=\tilde{P}^{-1}(z)\tilde{Q}(z)=P^{-1}(z)Q(z). \tag{16}$$

In the following, a blocked version of (13) will be derived. For this purpose, let $\mu=\alpha N+\nu, 0\le\nu<N$, where $\alpha\in\{0,1,2,\cdots\}$. Then, we have

$$\mathcal{A}_0Y_k+\mathcal{A}_1Y_{k-N}+\cdots+\mathcal{A}_\alpha Y_{k-\alpha N}+\mathcal{A}_{\alpha+1}Y_{k-(\alpha+1)N}$$
$$=\mathcal{B}_0U_k+\mathcal{B}_1U_{k-N}+\cdots+\mathcal{B}_\alpha U_{k-\alpha N}+\mathcal{B}_{\alpha+1}U_{k-(\alpha+1)N} \tag{17}$$

where $k=0,N,2N,\cdots$, and

$$\mathcal{A}_0=\begin{bmatrix} P_0 & P_1 & \cdots & P_{N-1} \\ 0 & P_0 & \ddots & P_{N-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & P_0\end{bmatrix},$$
$$\mathcal{A}_1=\begin{bmatrix} P_N & P_{N+1} & \cdots & P_{2N-1} \\ P_{N-1} & P_N & \ddots & P_{2N-2} \\ \vdots & \ddots & \ddots & \vdots \\ P_1 & P_2 & \cdots & P_N\end{bmatrix},$$
$$\mathcal{A}_i=\begin{bmatrix} P_{iN} & P_{iN+1} & \cdots & P_{(i+1)N-1} \\ P_{iN-1} & P_{iN} & \ddots & P_{(i+1)N-2} \\ \vdots & \ddots & \ddots & \vdots \\ P_{(i-1)N+1} & P_{(i-1)N+2} & \cdots & P_{iN}\end{bmatrix} \tag{18}$$

where $i=2,\cdots,\alpha+1$ and $P_j=0$ if $j>\mu$. The matrices $\mathcal{B}_i, i=0,1,\cdots,\alpha+1$ are defined exactly the same way.

There are two cases. One is that $\nu = 0$, and the other is that $0 < \nu < N$. If $\nu = 0$, it is easy to see that

$$\mathcal{A}_{\alpha+1} = 0, \mathcal{A}_\alpha = \begin{bmatrix} P_\mu & 0 & \cdots & 0 \\ P_{\mu-1} & P_\mu & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ P_{(\alpha-1)N+1} & P_{(\alpha-1)N+2} & \cdots & P_\mu \end{bmatrix} \tag{19}$$

and

$$\begin{aligned} &\mathcal{A}_0 Y_k + \mathcal{A}_1 Y_{k-N} + \cdots + \mathcal{A}_\alpha Y_{k-\alpha N} \\ &= \mathcal{B}_0 U_k + \mathcal{B}_1 U_{k-N} + \cdots + \mathcal{B}_\alpha U_{k-\alpha N} \end{aligned} \tag{20}$$

where $k = 0, N, 2N, \cdots$.

If $0 < \nu < N$, it follows from the definition given by (18) that $\mathcal{A}_{\alpha+1}$ is not equal to zero when any matrix among $P_{\alpha N}, \cdots, P_\mu$ is nonzero. The blocked version of (13) will be (17).

Based on the notations introduced above, (17) can be rewritten as

$$\begin{aligned} &(\mathcal{A}_0 + \mathcal{A}_1 Z^{-1} + \cdots + \mathcal{A}_\alpha Z^{-\alpha} + \mathcal{A}_{\alpha+1} Z^{-(\alpha+1)}) Y_k \\ &= (\mathcal{B}_0 + \mathcal{B}_1 Z^{-1} + \cdots + \mathcal{B}_\alpha Z^{-\alpha} + \mathcal{B}_{\alpha+1} Z^{-(\alpha+1)}) U_k \end{aligned} \tag{21}$$

where $Z^{-1} U_k = U_{k-N}, k = 0, N, 2N, \cdots$.

The above equation is the blocked version of (13). It is obvious that

$$Y_k = V(Z) U_k, Z^{-1} U_k = U_{k-N} \tag{22}$$

where

$$V(Z) = \mathcal{A}^{-1}(Z) \mathcal{B}(Z) \tag{23}$$

with

$$\begin{aligned} \mathcal{A}(Z) &= \mathcal{A}_0 Z^{(\alpha+1)} + \mathcal{A}_1 Z^\alpha + \cdots + \mathcal{A}_\alpha Z + \mathcal{A}_{\alpha+1}, \\ \mathcal{B}(Z) &= \mathcal{B}_0 Z^{(\alpha+1)} + \mathcal{B}_1 Z^{(\alpha)} + \cdots + \mathcal{B}_\alpha Z + \mathcal{B}_{\alpha+1}. \end{aligned} \tag{24}$$

Define

$$\begin{aligned} \tilde{\mathcal{A}}(Z) &= \mathcal{A}_0 + \mathcal{A}_1 Z^{-1} + \cdots + \mathcal{A}_\alpha Z^{-\alpha} + \mathcal{A}_{\alpha+1} Z^{-(\alpha+1)}, \\ \tilde{\mathcal{B}}(Z) &= \mathcal{B}_0 + \mathcal{B}_1 Z^{-1} + \cdots + \mathcal{B}_\alpha Z^{-\alpha} + \mathcal{B}_{\alpha+1} Z^{-(\alpha+1)} \end{aligned} \tag{25}$$

Two lemmas are needed.

Lemma 1. Given N complex numbers $\lambda_i, i = 1, 2, \cdots, N$, which satisfy $\lambda_i \neq \lambda_j$ for $i \neq j$, and also N real $p \times m$ matrices $\Pi_i, i = 1, 2, \cdots, N$, which are all of full column rank, then the following matrix

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 & \cdots & \Pi_N \\ \lambda_1 \Pi_1 & \lambda_2 \Pi_2 & \cdots & \lambda_N \Pi_N \\ \vdots & \ddots & \ddots & \vdots \\ \lambda_1^{N-1} \Pi_1 & \lambda_2^{N-1} \Pi_2 & \cdots & \lambda_N^{N-1} \Pi_N \end{bmatrix} \tag{26}$$

is of full column rank.

Proof: Rewrite Π as

$$\Pi = \begin{bmatrix} I_p & I_p & \cdots & I_p \\ \lambda_1 I_p & \lambda_2 I_p & \cdots & \lambda_N I_p \\ \vdots & \ddots & \ddots & \vdots \\ \lambda_1^{N-1} I_p & \lambda_2^{N-1} I_p & \cdots & \lambda_N^{N-1} I_p \end{bmatrix} \begin{bmatrix} \Pi_1 & 0 & \cdots & 0 \\ 0 & \Pi_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \Pi_N \end{bmatrix}, \tag{27}$$

The first matrix on the right is a Kronecker product of a VanderMonde matrix with the identity matrix, and accordingly is nonsingular. Given the properties of the Πs, the conclusion follows immediately. ¶

The following lemma is also needed.

Lemma 2. For a nonzero complex number Z_0, let $z_i, i = 1, 2, \cdots, N$ be N distinct complex numbers such that $z_i^N = Z_0, i = 1, 2, \cdots, N$. Choose any $m \times m$ nonsingular matrix Ω and define

$$\begin{aligned} \Upsilon &= \begin{bmatrix} \Omega & \Omega & \cdots & \Omega \\ z_1^{-1}\Omega & z_2^{-1}\Omega & \cdots & z_N^{-1}\Omega \\ \vdots & \ddots & \ddots & \vdots \\ z_1^{-(N-1)}\Omega & z_2^{-(N-1)}\Omega & \cdots & z_N^{-(N-1)}\Omega \end{bmatrix}, \\ \Lambda &= \begin{bmatrix} \tilde{Q}(z_1)\Omega & \tilde{Q}(z_2)\Omega & \cdots & \tilde{Q}(z_N)\Omega \\ z_1^{-1}\tilde{Q}(z_1)\Omega & z_2^{-1}\tilde{Q}(z_2)\Omega & \cdots & z_N^{-1}\tilde{Q}(z_N)\Omega \\ \vdots & \ddots & \ddots & \vdots \\ z_1^{-(N-1)}\tilde{Q}(z_1)\Omega & z_2^{-(N-1)}\tilde{Q}(z_2)\Omega & \cdots & z_N^{-(N-1)}\tilde{Q}(z_N)\Omega \end{bmatrix}. \end{aligned} \tag{28}$$

Then

$$\tilde{\mathcal{B}}(Z_0)\Upsilon = \Lambda \tag{29}$$

with $\tilde{\mathcal{B}}(Z)$ from (25) and $\tilde{Q}(z)$ from (14).

Proof: Using the definitions of $\mathcal{B}_i, i = 0, 1, \cdots, \alpha + 1$, it is easy to check

$$\tilde{\mathcal{B}}(Z_0) \begin{bmatrix} \Omega \\ z_i^{-1}\Omega \\ \vdots \\ z_i^{-(N-1)}\Omega \end{bmatrix} = \begin{bmatrix} \tilde{Q}(z_i)\Omega \\ z_i^{-1}\tilde{Q}(z_i)\Omega \\ \vdots \\ z_i^{-(N-1)}\tilde{Q}(z_i)\Omega \end{bmatrix}. \tag{30}$$

Using the above equation, the conclusion of the lemma follows immediately. ¶

Remark 1. In Bittanti and Colaneri [2009], a concept of lifted polynomials has been proposed. A relation between the lifted polynomial and its original polynomial has been established. The relation is similar to (30) but is for a scalar original polynomial. In fact, (30) establishes a relation between matrix polynomials and their lifted polynomials. Hence, the relation (30) can be viewed as a generalized form of that obtained in Bittanti and Colaneri [2009]. It is instrumental for deriving the new results in this paper.

4.2 The normal ranks of the transfer functions of the blocked and unblocked systems

Regarding the relationship between the normal ranks of the transfer functions of the blocked and unblocked systems, the following result holds.

Theorem 3. The normal rank of $V(Z)$ is mN if and only if the normal rank of $W(z)$ is m.

Proof: Sufficiency. Since the normal rank of $W(z)$ is m, it follows from the fact that $W(z) = P^{-1}(z)Q(z)$ that the normal rank of $Q(z)$ is m. Then there exists a complex number $Z_0 \neq 0$ and $z_i, i = 1, 2, \cdots, N$ such that $det(\mathcal{A}(Z_0)) \neq 0$, $z_i^N = Z_0, i = 1, 2, \cdots, N$, $rk(Q(z_i)) = m, i = 1, 2, \cdots, N$. Now choose any $m \times m$ nonsingular matrix Ω and define Υ and Λ as in (28), then it follows from Lemma 2 that $\tilde{\mathcal{B}}(Z_0)\Upsilon = \Lambda$. Noting that$z_i \neq z_j$ for $i \neq j$ and that Ω is nonsingular, it follows from Lemma 1 that Υ and Λ are of full column rank. Since Υ is a square matrix, it must be nonsingular, which implies that $\tilde{\mathcal{B}}(Z_0)$ is of full column rank. Since (24) and (25) imply that $\tilde{\mathcal{B}}(Z_0)$ has the same rank as $\mathcal{B}(Z_0)$ for any $Z_0 \neq 0$, the matrix $\mathcal{B}(Z_0)$ must be of full column rank. This together with the fact that $det(\mathcal{A}(Z_0)) \neq 0$ implies that $V(Z_0) = \mathcal{A}^{-1}(Z_0)\mathcal{B}(Z_0)$ is of full column rank, which in turn proves that the normal rank of $V(Z)$ is mN.

Necessity. Since the normal rank of $V(Z)$ is mN, there exists a complex number $Z_0 \neq 0$ such that $det(\mathcal{A}(Z_0)) \neq 0$ and $\mathcal{B}(Z_0)$ is of full column rank. Since $Z_0 \neq 0$, it follows from (24) and (25) that $\tilde{\mathcal{B}}(Z_0)$ is of full column rank. Now let $z_i, i = 1, 2, \cdots, N$ be complex numbers such that $z_i^N = Z_0, i = 1, 2, \cdots, N$. Now choose any $m \times m$ nonsingular matrix Ω and define Υ and Λ as in (28), then it follows from Lemma 2 that $\tilde{\mathcal{B}}(Z_0)\Upsilon = \Lambda$. Noting that$z_i \neq z_j$ for $i \neq j$ and that Ω is nonsingular, it follows from Lemma 1 that Υ is nonsingular. This together with the fact that $\tilde{\mathcal{B}}(Z_0)$ is of full column rank implies that Λ is of full column rank. It follows from the definition of Λ that $\tilde{Q}(z_i)\Omega, i = 1, 2, \cdots, N$ are of full column rank. Noting that Ω is nonsingular, it follows that $\tilde{Q}(z_i), i = 1, 2, \cdots, N$ are of full column rank, which means that $Q(z_i), i = 1, 2, \cdots, N$ are of full column rank. This proves that the normal rank of $W(z)$ is m. ¶

4.3 Zeros of the blocked and unblocked systems

Although the relationship between poles of the blocked and unblocked systems is very simple, it is not clear whether such a simple relation still holds or not for system zeros. If such a simple relation holds also for zeros, how can it be proved? It turns out the relationship between zeros of the blocked and unblocked systems is highly nontrivial and is much harder to study.

Due to lack of space, some lemmas and theorems are provided without proof.

Lemma 3. Under Assumption 1, suppose that $W(z)$ has a coprime MFD defined by (10) and (11). Suppose also that $\tilde{P}(z)$ and $\tilde{Q}(z)$ are defined by (14). Then $W(z)$ has a finite zero at $z_0 \neq 0$ if and only if there exists a finite number $z_0 \neq 0$ such that $rk(\tilde{Q}(z_0)) < m$.

As the relationship between the zeros of unblocked and blocked systems is quite complicated, one needs to consider three cases separately, that is, 1) finite nonzero system zeros; 2) system zeros at infinity; and 3) system zeros at zero.

Lemma 4. Under Assumption 1, suppose also that $V(Z) = \mathcal{A}^{-1}(Z)\mathcal{B}(Z) = \tilde{\mathcal{A}}^{-1}(Z)\tilde{\mathcal{B}}(Z)$, where $\mathcal{A}(Z), \mathcal{B}(Z)$ and $\tilde{\mathcal{A}}(Z), \tilde{\mathcal{B}}(Z)$ are derived from a coprime MFD of $W(z)$ and are defined by (24) and (25). Suppose also that $V(Z)$ has a finite zero at $Z_0 \neq 0$. Then $rk(\tilde{\mathcal{B}}(Z_0)) < mN$.

Lemma 5. There exists a finite complex number $Z_0 \neq 0$ such that $rk(\tilde{\mathcal{B}}(Z_0)) < mN$ if and only if there is a finite complex number $z_0 \neq 0$ such that $rk(\tilde{Q}(z_0)) < m$. In this case, there holds $z_0^N = Z_0$.

Proof: Sufficiency. Since $rk(\tilde{Q}(z_0)) < m$, there exists a nonzero vector β such that

$$\tilde{Q}(z_0)\beta = 0. \tag{31}$$

Since $z_0 \neq 0$, we can define a nonzero vector as

$$\Psi = \begin{bmatrix} \beta' & z_0^{-1}\beta' & \cdots & z_0^{-(N-1)}\beta' \end{bmatrix}'. \tag{32}$$

Let $Z_0 = z_0^N$, then, it is easy to check that

$$\tilde{\mathcal{B}}(Z_0)\Psi = \begin{bmatrix} \tilde{Q}(z_0)\beta \\ z_0^{-1}\tilde{Q}(z_0)\beta \\ \vdots \\ z_0^{-(N-1)}\tilde{Q}(z_0)\beta \end{bmatrix} = 0, \tag{33}$$

which means that $rk(\tilde{\mathcal{B}}(Z_0)) < mN$.

Necessity. Suppose there exists a complex number $Z_0 \neq 0$ such that $rk(\tilde{\mathcal{B}}(Z_0)) < mN$. Since $Z_0 \neq 0$, there exist N distinct complex numbers $z_i, i = 1, 2, \cdots, N$ such that $z_i^N = Z_0, i = 1, 2, \cdots, N$. If there exists a complex number $z_{i_0}, i_0 \in \{1, 2, \cdots, N\}$ such that $rk(\tilde{Q}(z_{i_0})) < m$, the necessity is proved.

Now, assume that $\tilde{Q}(z_i), i = 1, 2, \cdots, N$ are all of full column rank. According to Lemmas 1 and 2 $\tilde{\mathcal{B}}(Z_0)$ would be of full column rank, which is a contradiction. This completes the proof. ¶

The first main result in this subsection is provided in the following theorem.

Theorem 4. Consider the unblocked system (1) with transfer function $W(z)$ and the blocked system (4) with transfer function $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, where A_b, B_b, C_b, D_b are defined by (5). Under Assumption 1 and suppose that (A, B, C, D) is minimal, then $V(Z)$ has a finite zero at $Z_0 \neq 0$ if and only if $W(z)$ has a finite zero at $z_0 \neq 0$ with $z_0^N = Z_0$ for one or more of the Nth roots of Z_0.

Proof: Necessity. Since the normal rank of $W(z)$ is m, it follows from Theorem 3 that the normal rank of $V(Z)$ is mN. For the finite zero $Z_0 \neq 0$ of $V(Z)$, it follows from Lemma 4 that $rk(\tilde{\mathcal{B}}(Z_0)) < mN$. This according to Lemma 5 and Lemma 3 proves the necessity.

Sufficiency. Suppose that z_0 is a zero of the unblocked system, then for some nonzero $[x_0' \; u_0']'$ there holds

$$\begin{bmatrix} z_0 I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0.$$

Then, $Ax_0 = z_0 x_0 - Bu_0$. Using this equation repeatedly, it follows that

$$\begin{aligned} A^i x_0 &= z_0^i x_0 - [\, A^{i-1}B \quad A^{i-2}B \quad \cdots \quad B \quad 0 \quad \cdots \quad 0\,] \\ &\quad \begin{bmatrix} u_0 \\ z_0 u_0 \\ \vdots \\ z_0^{N-1} u_0 \end{bmatrix}, 1 \le i \le N \\ CA^i x_0 &= -\, [\, CA^{i-1}B \quad CA^{i-2}B \quad \cdots \quad CB \quad D \quad \cdots \quad 0\,] \\ &\quad \begin{bmatrix} u_0 \\ z_0 u_0 \\ \vdots \\ z_0^{N-1} u_0 \end{bmatrix}, 1 \le i \le N-1. \end{aligned} \tag{34}$$

It is immediate that

$$\begin{bmatrix} z_0^N I - A_b & -B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \\ z_0 u_0 \\ \vdots \\ z_0^{N-1} u_0 \end{bmatrix} = 0.$$

Since the normal rank of $W(z)$ is m, it follows from Theorem 3 that the normal rank of $V(Z)$ is mN. This fact together with the above equation proves that $V(Z)$ has a finite zero at $Z_0 \neq 0$. ¶

For a zero at infinity we have the following result.

Theorem 5. Consider the unblocked system (1) with transfer function $W(z)$ and the blocked system (4) with transfer function $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, where A_b, B_b, C_b, D_b are defined by (5) . Under Assumption 1 and suppose that (A, B, C, D) is minimal, then $W(z)$ has a zero at $z = \infty$ if and only if $V(Z)$ has a zero at $Z = \infty$.

For the third case (zero at zero), the following result holds.

Theorem 6. Consider the unblocked system (1) with transfer function $W(z)$ and the blocked system (4) with transfer function $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, where A_b, B_b, C_b, D_b are defined by (5) . Under Assumption 1 and suppose that (A, B, C, D) is minimal, then $V(Z)$ has a zero at $Z = 0$ if and only if $W(z)$ has a zero at $z = 0$.

For the unblocked system (1) with transfer function $W(z)$ and the blocked system (4) with transfer function $V(Z) = D_b + C_b(ZI - A_b)^{-1}B_b$, where A_b, B_b, C_b, D_b are defined by (5), under the assumptions that the unblocked system (1) is minimal and the normal rank of $W(z)$ is m, the main results in this subsection lead to the following conclusions:

- The blocked system has a zero at $Z = 0$ if and only if the unblocked system has a zero at $z = 0$.
- The blocked system has a zero at $Z = \infty$ if and only if the unblocked system has a zero at $z = \infty$.
- The blocked system has a finite zero at $Z = Z_0 \neq 0$ if and only if the unblocked system has a finite zero at $z = z_0 \neq 0$ with $Z_0 = z_0^N$.

5. CONCLUSIONS

In this paper, the properties of the blocked system of a linear time invariant system have been studied through investigating the relationship between the blocked and unblocked systems. It has been shown that the transfer function of the blocked system is of full column normal rank if and only if the transfer function of the unblocked system is of full column normal rank. This new result has been found applicable to the study of the relationship between the zeros of the blocked and unblocked systems. With its help and under certain conditions, it has been demonstrated that there is a close relationship between the zeros of the blocked and unblocked systems. These results are appealing and important. One future topic is how to extend the obtained results to the blocked systems of linear periodic systems.

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