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Morse Theory and Formation Control

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***Abstract*— Formation shape control for a collection of point agents is concerned with devising decentralized control laws which will ensure that the formation will move so that certain inter-agent distances assume prescribed values. A number of algorithms based on steepest descent of an error function have been suggested for various problems, and all display the existence of incorrect equilibria, though often the equilibria are saddle points or unstable. This paper introduces Morse theory as a tool for analyzing the number of such equilibria. A key conclusion is that for two-dimensional rigid formations of point agents, there will always be incorrect equilibria associated with any steepest descent law.**

***Index Terms*— Formation control, autonomous formations, shape control, Morse Theory**

I. INTRODUCTION

Consider a set of point agents moving in $\mathbb{R}^2$, which will be the ambient space throughout this paper. When in addition, a number of interagent distances are specified the set of agents is regarded as a *formation*. When a suitable set of distances is specified, the formation will be rigid, in the sense that the only continuous motions of which it will be capable are translations and rotations. For a formation of n agents in R^2, at least $2n-3$ distance specifications are required, and they must be well distributed; the formation is then termed rigid. Surprisingly at first glance, there can be rigid formations with the same distance set which are not congruent. [Consider for example a formation of four agents 1,2,3,4 with specification of $d_{12}, d_{23}, d_{31}, d_{14}, d_{24}$. One can think of the formation as defined by two triangles 123, 124, with common side 12; then one may have agents 3 and 4 on the same or opposite sides of 12. Thus two noncongruent formations are consistent with the distance data, and are related by moving one of 3 and 4 to its mirror image point through side 12. Such ambiguity can be eliminated by adding a further distance specification, d_{23}. If two formations with the same distance set are necessarily congruent, i.e. differ by translation, rotation or reflection of the whole formation, the formation is termed globally rigid. For an introduction to all these concepts, see e.g. [3]

Formations of specified shape may be useful for sensing and localizing objects, and formations of fixed shape can be contemplated for moving massive objects placed upon them. One problem of interest is *formation shape control*. This is the task of specifying control laws for the individual agents, generally using measurements just of neighboring agents, which will cause a formation to take up a prescribed shape. Literature on formation shape control via distances includes important early work by Olfati-Saber and Murray [13], work of Francis and coworkers, e.g. [14], [10], [7] and the present author and coworkers, e.g. [1], [6], [2], [5]. Almost all the control algorithms currently advanced have certain common features:

1) A distance may be controlled by one agent or two agents of the pair for which the distance is defined?
2) An agent's motion is defined using the errors in the distances to its neighbors, whose relative positions must also be sensed?
3) Let d_{ij}^*, d_{ij} denote respectively the desired distance between agents i, j and the actual distance (at some time t) between the same agents. An error function of the form $V = \sum_{ij} f_{ij}(d_{ij}, d_{ij}^*)$ is formed, where typically f_{ij} is zero when $d_{ij} = d_{ij}^*$ and is otherwise positive, and usually monotonic in $|d_{ij} - d_{ij}^*|$. Typically, f_{ij} will be the same for all ij pairs. A control law for each agent is then derived on the basis of moving V along a steepest descent trajectory. Error functions with the sum decomposition shown automatically lead to control laws where any one agent i only needs information concerning its neighbors?
4) Let the position of agent i be denoted by (x_i, y_i). Clearly, V can be expressed as a function of $x_1, y_1, x_2, \ldots, y_n$ and equilibrium points of the motion correspond to critical points of V, i.e. points where $\frac{\partial V}{\partial x_i} = 0, \frac{\partial V}{\partial y_i} = 0 \forall i$. The equilibrium points include those where $d_{ij} = d_{ij}^*$, i.e the formation has a correct shape.
5) For formations with a minimum of three agents, where the desired formation is rigid or globally rigid (the key formations of interest), there are always equilibria which are incorrect. Some at least of these are saddle points or unstable, and so in practical terms are not a serious impediment to formation control.

Our focus in this paper is on the existence of multiple equilibria. In particular, we aim to show that their existence is *not* a consequence of the particular algorithm that is used, but a *virtually automatic consequence of seeking to use a steepest descent law*. Further, we seek to introduce methods for counting equilibria with different stability properties, i.e. according to the eigenvalue pattern of the Hessian of V.

We note that robot navigation problems have also been examined from this point of view. In [9], the problem is examined of having a single robot find its way using steepest descent of a so-called navigation function to an arbitrary designated point in the ambient space, given a set of obstacles

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or forbidden zones (modelled as open disks or balls) and a requirement for the robot never to move outside a nominated closed disk/ball. The target point is an equilibrium point of the motion, by design. However, once there are obstacles there is no single equilibrium point, *no matter what navigation function is chosen*; interest centers on characterizing the stability or otherwise of the incorrect equilibrium points, and on seeking to find navigation functions for which none of the incorrect equilibria are unstable.

The tool for doing this in the robot navigation paper and the tool for examining multiple equilibria here is Morse Theory, [11], [12], though the usage in the two papers is not similar. In the next section, we summarize some aspects of this theory. In order to apply Morse Theory to formations, one needs a careful characterization of the space or manifold in which formations can be considered to lie; for this purpose, we need to null out the effects of translation and rotation, since a formation shape is unaffected by this. A formation with n agents of fixed shape lives in a $2n-3$ dimensional space; Section 3 is concerned with pinning this notion down, in terms suitable for the application of Morse Theory. Section 4 then demonstrates that for any smooth control law based on steepest descent of an error function, multiple equilibria are to be expected; certain of them are not stable. Section 5 examines in detail certain laws, to check their conformity with the general theory of Section 4, and Section 6 contains concluding remarks.

II. Introduction to Morse Theory

Consider a smooth function $f:\mathbb{R}\to\mathbb{R}$ with the property that $||x||\to\infty$ implies $f(x)\to\infty$. Suppose that at the critical points of f, i.e. those x_0 where $f'(x_0)=0$, there holds $f''(x_0)\neq 0$. . Then a moment's sketching reveals that there must be at least one minimum, and the number of minima must be one greater than the number of maxima[1]. Thus if m_0, m_1 denote the number of minima and maxima of f:

$$\begin{aligned} m_0 &\geq 1 \\ m_1 - m_0 &= -1 \end{aligned} \tag{1}$$

The same holds if $f:(a,b)\to\mathbb{R}$ with $f'(a)<0, f'(b)>0$ and a,b finite. Again, suppose that f is a smooth function $S^1\to\mathbb{R}$ defined on the circle. It is easy to see that

$$\begin{aligned} m_0 &\geq 1 \\ m_1 - m_0 &= 0 \end{aligned} \tag{2}$$

Morse theory is a major generalization of this idea; the underlying space does not have to be $\mathbb{R}$ or $\mathbb{R}^n$ for some $n>1$, but can be any smooth manifold. Variations exist depending on whether the manifold is bounded or unbounded. Examples of bounded manifolds include the sphere S^2, circle S^1, and the space $\mathbb{RP}^n$, defined by all lines through the origin in $\mathbb{R}^{n+1}$; each such line intersects the sphere S^n in two points and the space can then be identified with that of the sphere after antipodal points are identified. Examples of unbounded manifolds include $\mathbb{R}^+$, $\mathbb{R}^n$, $\mathbb{C}^n$. Given a manifold, one can associate with it a number of topological invariants termed *Betti numbers*, see [11], [12], which form a sequence $\beta_0,\beta_1,\ldots$. For commonly occurring manifolds, Betti numbers are recorded. Let us summarize some properties:

1) The Betti numbers are all nonnegative integers
2) When the underlying manifold is n-dimensional, there holds $\beta_j=0, j>n$
3) Consider two manifolds A,B for which Betti numbers β_{iA},β_{iB} are known for all i. Consider the manifold $C=A\times B$. Polynomial multiplication yields the β_{iC}:
$$\sum_{i\geq 0}\beta_{iC}z^i=\sum_{i\geq 0}\beta_{iA}z^i\sum_{i\geq 0}\beta_{iB}z^i \tag{3}$$
4) The Euler characteristic $\chi(A)$ of a manifold A is related to the Betti numbers by
$$\chi(A)=\sum_{j\geq 0}(-1)^j\beta_j \tag{4}$$
(Though we hardly make use of the Euler characteristic here, it often appears in conjunction with Morse Theory, and is obviously related to the Betti numbers.)
5) The manifolds $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{R}^+$ have $\beta_0=1,\beta_j=0$ for $j>0$; S^n has $\beta_0=1,\beta_n=1,\beta_j=0$ for $j\neq 0,n$. The manifold $\mathbb{CP}^n$ has $\beta_{2i}=1, i=0,1,\ldots,n$, else $\beta_j=0$.

Now consider a smooth function f defined on an n-dimensional manifold. If the manifold is not compact, the function must tend to ∞ for point sequences on the manifold progressively whose distance from some arbitrary but fixed point goes to infinity, and must have inwardly directed gradient at any finite boundary point. Suppose that the function has isolated critical points, i.e isolated points where its gradient is zero, and suppose that every critical point is hyperbolic, i.e. the associated Hessian matrix is nonsingular. Call the index of a critical point the number of negative eigenvalues of the Hessian. A minimum then corresponds to a point with index 0, and a maximum to a point with index n. Critical points with indices in $[1,n-1]$ are saddle points. Let m_i denote the number of critical points of index i.

The key result of Morse Theory is a collection of equality/inequality relations between the m_i and the β_i, as follows:

$$\begin{aligned} m_0 &\geq \beta_0 \\ m_1-m_0 &\geq \beta_1-\beta_0 \\ &\vdots \\ m_j-m_{j-1}+m_{j-2}-\ldots &\geq \beta_j-\beta_{j-1}+\beta_{j-2}-\ldots \\ &\vdots \\ m_n-m_{n-1}+m_{n-2}-\ldots &= \beta_n-\beta_{n-1}+\beta_{n-2}-\ldots \end{aligned} \tag{5}$$

The two examples given above, involving $\mathbb{R}$ and S^1, are easily seen to be special cases of these relations, using the Betti numbers recorded above for the two manifolds.

[1] If $f(x)\to-\infty$ as $||x||\to\infty$, a different relationships results

III. A MANIFOLD CHARACTERIZATION OF THE SPACE OF FORMATIONS

It seems obvious that a formation of n agents in $\mathbb{R}^2$ should be regarded as lying in $\mathbb{R}^{2n}$. Yet this observation, while true, is unhelpful. In this space, no critical point of a function V designed to measure errors between a desired formation shape and actual formation shape (in a way that draws no distinction between formations differing just by rotation or translation) could be an *isolated* critical point. There is in fact a continuum of critical points. Accordingly, the Morse equality/inequality relations cannot be used. We need to factor out the translation and rotation, and come up with a characterization of formation shape using $2n-3$ coordinates.

The first approach to such a problem appears due to Kendall, summarized (with earlier references) in [8]. The relevance of this characterization to formation control is emphasised by Belabbas, [4]. Kendall's main result quoted from [4] is as follows:

***Theorem** 1:* The space of formations of n agents in the plane, excluding formations in which all agents are collocated, can be represented as $\mathbb{CP}^{n-2} \times \mathbb{R}^+$

Remark: The appearance of $\mathbb{R}^+$ takes care of a scaling of the formation size. Provided not all inter-agent distances are zero, one can define a positive scaling parameter for a formation to be (for example) the square root of the sum of the squares of the interagent distances. The $\mathbb{CP}^{n-2}$ captures the $2n-2$ remaining coordinates. (It is standard that $\mathbb{CP}^n$ has (real) dimension $2n$: paralleling the definition of $\mathbb{RP}^n$, the manifold $\mathbb{CP}^n$ is the set obtained using $n+1$ complex numbers $z_1, z_2, \ldots z_{n+1}$ (equivalent to $2n+2$ real numbers) and making the identification

$$(z_1, z_2, \ldots, z_{n+1}) \simeq re^{j\phi}(z_1, z_2, \ldots, z_{n+1}) \tag{6}$$

for arbitrary $r \in \mathbb{R}^+, \phi \in [0, 2\pi)$. The identification process suppresses or factors out two real parameters.

Proof. Regard the positions of individual agents in $\mathbb{R}^2$ as being given by complex numbers, $z_i = x_i + jy_i$. By translational invariance, place agent 1 at the origin. Then the formation is described by $(z_2, z_3, \ldots, z_n)$ except that the formation with these coordinates is the same as the formation with coordinates $e^{j\phi}(z_2, z_3, \ldots, z_n)$ by rotational invariance. This is equivalent to coordinatizing by identifying $(z_2, z_3, \ldots, z_n)$ with the set $re^{j\phi}(z_2, z_3, \ldots, z_n)$ where $r \in \mathbb{R}^+$ and $\phi \in [0, 2\pi)$ (i.e. factoring out angle *and* scale, which defines $\mathbb{CP}^{n-2}$), and then re-introducing the scaling parameter r.

Example 1:. Consider three-agent formations. For $z_2, z_3 \in \mathbb{C}$ and not both zero, consider (z_2, z_3). If $z_2 \neq 0$, then in $\mathbb{CP}^1$ this is equivalent to $(1, z_3/z_2)$. This is a coordinatization of a formation with one agent at the origin, one on the x-axis at unit distance from the origin and one at an arbitrary point in R^2. If scaling is allowed (i.e. an additional coordinate $r \in \mathbb{R}^+$ is introduced), it covers all formations where agents 1 and 2 do not coincide. If $z_2 = 0$, then (z_2, z_3) is equivalent to $(0, 1)$, which says that the formation is equivalent to one with agents 1,2 collocated at the origin and agent 3 on the x-axis at unit distance from the origin. After scaling, all formations with agents 1,2 collocated are captured. Note that the case $z_2 \neq 0$ does include the possibility that $z_3 = 0$, i.e. agents 1 and 3 are collocated.

Example 2. Consider four-agent formations. As before, agent 1 is shifted to the origin. For $z_2, z_3, z_4 \in \mathbb{C}$ not all zero, consider (z_2, z_3, z_4). If $z_2 \neq 0$, then this is equivalent to $(1, z_3/z_2, z_4/z_2)$ and to within scaling this defines a formation with one agent at the origin, one on the x-axis at unit distance from the origin, and the remaining two agents at essentially arbitrary points. If $z_2 = 0, z_3 \neq 0$, then (z_2, z_3, z_4) is equivalent to $(0, 1, z_4/z_3)$. To within scaling this covers formations with agents 1 and 2 at the origin, agent 3 on the x-axis at unit distance from the origin and agent 4 at an arbitrary point. Finally, if $z_2 = z_3 = 0$, then z_4 is necessarily nonzero and (z_2, z_3, z_4) is equivalent to $(0, 0, 1)$, i.e. to within scaling a formation with agents 1 to 3 at the origin and agent 4 on the x-axis at unit distance from the origin.

Remark: the coordinatization here does *not* identify two formations which differ from one another by reflection. Hence, other than for formations all of whose agents are collinear, formations (with this coordinatization) occur in pairs differing from one another by simply reflection (though they are congruent). With the coordination of say example 2 above, when $z_2 \neq 0$, the two formations apart from the scaling are represented by $(1, z_3/z_2, z_4/z_2)$ and $(1, -z_3/z_2, -z_4/z_2)$.

IV. APPLYING MORSE THEORY TO FORMATION SPACE CHARACTERIZATION

We now have the ingredients to draw very general conclusions: we have characterized the $2n-3$-dimensional manifold which describes n-agent formations in which not all agents are collocated. Now we have:

***Theorem** 2:* The Betti numbers of the manifold $\mathbb{CP}^{n-2} \times \mathbb{R}^+$ are $\beta_{2i} = 1, i = 0, 1, \ldots, n-2$, else $\beta_j = 0$.

Proof:. The nonzero Betti numbers of $\mathbb{CP}^{n-2}$ are standard and are given by $\beta_{2i} = 1, i = 0, 1, \ldots, n-2$. The manifold $\mathbb{R}^+$ has one nonzero Betti number, viz. $\beta_0 = 1$. The product formula (3) then shows that the Betti numbers of $\mathbb{CP}^{n-2} \times \mathbb{R}^+$ are identical with those of $\mathbb{CP}^{n-2}$, and so are as claimed.

Now suppose that a control law for the formation has been found with the property that it is a steepest descent law for a function V, with the function possessing a minimum when the formation has the specified interagent distances. The function V may originally be defined in terms of Cartesian coordinates of the agents, but it must have the (reasonable) property that it is invariant under translation and rotation of the formation; thus it can be regarded as being defined using the coordinatization from $\mathbb{CP}^{n-2} \times \mathbb{R}^+$. It is further supposed that equilibria are isolated (in the event that the desired formation is rigid, and V reflects all the relevant distances, this property will hold). Since $\mathbb{CP}^{n-2} \times \mathbb{R}^+$ is not compact, we require V to go to ∞ when any interagent distance goes to ∞ and to have a local maximum when all interagent distances go to zero. Finally it is supposed that all critical points of V are hyperbolic, i.e. the associated Hessian is nonsingular. (A lemma in Morse theory demonstrates the existence of generic small perturbations of any smooth V for which the hyperbolic property does not hold such that the perturbed function has the hyperbolic property for its critical points, [11], see p. 47.) All the functions associated with the control laws in the cited references have this property.

With m_i as before denoting the number of critical points of index i, the Morse relations yield

$$\begin{aligned} m_0 &\geq 1 \\ m_1 - m_0 &\geq -1 \\ m_2 - m_1 + m_0 &\geq 2 \\ &\vdots \\ m_{2n-3} - m_{2n-4} + \dots &= -(n-1) \end{aligned} \tag{7}$$

Remark: If V is expressed as a function of the $2n$ coordinates of the agent positions, it may be well defined even if all agents are collocated. However, whether this is a critical point of V is irrelevant, since in reducing the description to $2n-3$ coordinates, such a formation has been excluded from consideration. So there can be no contribution to any of the m_i.

Now nearly all functions V that can be contemplated will be left invariant not just by rotation or translation of a formation, but also by reflection. This means that given any critical point corresponding to a formation where the agents are not collinear, there will be a second critical point with the same index obtained by reflection.

For a globally rigid formation, as noted in the introduction, specification of the distance set determines the formation up to congruence. Thus, noting this allows reflection, for a globally rigid formation we will necessarily have at least *two* equilibria which are minima of V.

For a rigid but not globally rigid formation with four or more agents, there are always noncongruent formations achieving the required distance set. Factoring out rotation and translation but not reflection, this means that there will be at least four equilibria which are minima.

We now have the following key result.

***Theorem** 3:* Consider a formation of $n \geq 3$ agents in $\mathbb{R}^2$ with an associated set of desired interagent distances corresponding to a rigid formation in which agents are not collinear. Suppose that control is to be achieved by steepest descent of a smooth function V which achieves minima when the formation distances are correct, and which is a Morse function, i.e. at its critical points, the associated Hessian is nonsingular. Then there are necessarily stationary points of V, equivalently configurations of the agents, which correspond to equilibrium points of the motion at which distances are not correct.

Proof: By the remarks prior to the theorem, we have $m_0 \geq 2$. By the second inequality, we see that $m_1 \geq 1$. This proves the result.

Remark: For the purposes of proving the theorem, it is enough to prove that any $m_i, i > 1$ is nonzero. The theorem is not claiming that there are necessarily *stable* critical points other than those where the formation has the correct shape.

Remark: It is easy to identify further nonzero critical indices: consider the succesive inequalities in which the first terms are m_{2i-1}, m_{2i} for some i. When these are added, there results $m_{2i} \geq 1$.

V. Three-agent formations

Three-agent formations have been the subject of detailed study, see e.g. [1], [5], [6]. We will explore some of the results of these papers using Morse Theory. We begin with [1]. The control law in this paper is one where agent i seeks to correct its distance from agent $(i+1)\mathrm{mod}3$. The correct formation is assumed to be a triangle of nonzero area, with defined side lengths. The actual function V is:

$$V = (d_{12} - d_{12}^*)^2 + (d_{23} - d_{23}^*)^2 + (d_{13} - d_{13}^*)^2 \tag{8}$$

If this is expressed in terms of the position coordinates of the different agents, then there will arise terms like $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$, which is not smooth when $x_1 = x_2, y_1 = y_2$ (and there are two other similarly defined conditions defining nonsmooth points). Each such condition corresponds to two agents being collocated (three being collocated is disallowed). In principle, the nonsmoothness rules out the use of Morse Theory (when we use the coordinatization provided by $\mathbb{CP}^1 \times \mathbb{R}^+$); however, we can imagine an infinitesimal perturbation to smooth V with associated infinitesimal perturbation of critical points, and with no change to their index. Thus we can still proceed, since V meets the other earlier listed requirements.

In [1], it was asserted that (modulo rotation and translation):

1) There are two minima of V corresponding to correct formation shapes which are reflections of one another; there are no other equilibria when the three agents are not collinear.
2) There are however three further stationary points of V. At these three stationary points, the three agents are collinear. One stationary point corresponds to each ordering of the three agents (i.e. which agent is between the other two). The three stationary points were identified as having index 1. (It turns out that if the three agents are initially collinear, then they will remain collinear, and tend to one of these three stationary points. The slightest displacement from collinearity however will result in the agents tending to their correct triangular formation.)

The first of these conclusions is correct. The second is misleading, in that though true it is incomplete. We shall now re-examine the situation with regard to equilibria associated with collinear agents.

Let us suppose that the agents are located on the x-axis, agent 1 is the origin, and agents 2,3 are at x_2, x_3. The function V with these agent position restrictions is

$$V_2 = (|x_2| - d_{12}^*)^2 + (|x_2 - x_3| - d_{23}^*)^2 + (|x_3| - d_{13}^*)^2 \tag{9}$$

We use the suffix 2 to denote that V_2 is a restriction of V: temporarily forget the coordinatization arrangements for three-agent formations, and simply regard this function as a function of two real variables, x_2, x_3 defined over all of $\mathbb{R}^2$. Let us consider the critical points of V_2 from this viewpoint. Later, we shall relate them to the formation coordinatization and critical points of V.

One can first search for extrema of V_2, assuming $|x_2|, |x_3|, |x_2 - x_3|$ are all nonzero. The following sign patterns for x_2, x_3 and $x_2 - x_3$ are all consistent: $[+,+,+],[+,+,-],[+,-,+],[-,+,+],[-,-,+],[-,-,-]$. In each region defined by these sign patterns, V is quadratic, and its stationary points and their index (in the one-dimensional space, remember) are easily determined. There is indeed a single minimum in each of these regions. The minima occur in three pairs, with the two members of one pair differing just by the replacing (x_2, x_3) by $(-x_2, -x_3)$.

Next, we can search for the existence of critical points of V_2 assuming one only of $|x_2|, |x_3|, |x_2 - x_3|$ is zero. This was not done in [1]. Suppose we search at $|x_2| = 0$. We shall show that V_2 has two saddle points lying on the line $|x_2| = 0$. Observe that with $|x_2| = 0$, V_2 becomes

$$V_2 = d_{12}^{*2} + (|x_3| - d_{23}^*)^2 + (|x_3|) - d_{13}^*)^2 \tag{10}$$

and so along the line $|x_2| = 0$, V_2 has two minima, at

$$|x_3| = \frac{d_{23}^* + d_{13}^*}{2} \tag{11}$$

(and they are nonzero, and so distinct). In the vicinity of the point $(x_2, x_3) = (0, x_3)$ through neglect of quadratic terms in x_2, it is easy to see that V_2 can be approximated by:

$$-2d_{12}^*|x_2| - 2\text{sgn}x_3(|x_3| - d_{23}^*)x_2 + w(|x_3|) \tag{12}$$

Thus in the vicinity of $|x_3| = \frac{d_{23}^*+d_{13}^*}{2}$, it can be approximated by

$$V_2 = [-2d_{12}^* - \text{sgn}x_2\text{sgn}x_3(d_{13}^* - d_{23}^*]|x_2| + w(\frac{d_{23}^* + d_{13}^*}{2}) \tag{13}$$

By the triangle inequality applying to the triple $(d_{12}^*, d_{23}^*, d_{13}^*)$, it is easily seen that

$$[-2d_{12}^* - \text{sgn}x_2\text{sgn}x_3(d_{13}^* - d_{23}^*] < 0 \tag{14}$$

for all sign possibilities of x_2, x_3. Accordingly, the points $(0, \pm\frac{d_{23}^*+d_{13}^*}{2})$ are saddle points for V_2; movement in the x_3 direction increases V_2 and movement in the x_2 direction decreases V_2.

There are two similar saddle pairs corresponding to $|x_3| = 0, |x_2 - x_3| = 0$.

For completeness, we note that it is easily checked that the point $(x_2, x_3) = (0,0)$ is a (local) maximum point for V_2. Consequently, we have identified the following critical points for V_2: 6 minima, 6 saddles and one maximum. We can check that these numbers are consistent with the Morse relations. The function V_2 is a function defined over the manifold $\mathbb{R}^2$, for which $\beta_0 = 1, \beta_i = 0, i \neq 0$. Accordingly, there must hold

$$\begin{aligned} m_0 &\geq 1 \\ m_1 - m_0 &\geq -1 \\ m_2 - m_1 + m_0 &= 1 \end{aligned} \tag{15}$$

Verification is straightforward.

Now we need to interpret these results back in the original space. As shown in [1] a critical point for V_2 is also a critical point for V, but with index one higher, since although collinear formations form an invariant set, the slightest perturbation away from collinearity produces a local instability. We also note that of the 6 minima for V_2, which occur in three pairs, if we view the associated formations corresponding to one pair as living in a three dimensional ambient space, agents being collinearly placed on the x-axis, the two formations differ through rotation by π radians (and therefore in $\mathbb{CP}^1 \times \mathbb{R}^+$ are identified). Consequently, the fact that $m_0 = 6$ for V_2 implies $m_1 = 3$ for V. Next, the six saddles for V_2 also occur in pairs with the two members of each pair to be identified in $\mathbb{CP}(1) \times \mathbb{R}^+$, due to the associated formations again differing simply through rotation by π radians. Accordingly, $m_1 = 6$ for V_2 implies $m_2 = 3$ for V. Last, the critical point of V_2 which is the maximum, with $x_2 = x_3 = 0$, corresponds to a formation with all agents collocated, which is excluded. Therefore, $m_2 = 1$ for V_2 implies $m_3 = 0$ for V.

To summarize then, for V, we have $(m_0, m_1, m_2, m_3) = (2, 3, 3, 0)$. If we specialize (7) to the case $n = 3$, we have

$$\begin{aligned} m_0 &\geq 1 \\ m_1 - m_0 &\geq -1 \\ m_2 - m_1 + m_0 &\geq 2 \\ m_3 - m_2 + m_1 - m_0 &= -2 \end{aligned} \tag{16}$$

and satisfaction of these relations is evident.

Next, we examine an alternative V (meeting the earlier listed requirements) advanced in [6] for a 3-agent formation when the desired formation is a triangle of nonzero area:

$$V = (d_{12}^2 - d_{12}^{*2})^2 + (d_{23}^2 - d_{23}^{*2})^2 + (d_{13}^2 - d_{13}^{*2})^2 \tag{17}$$

Similarly to the analysis of [1], it is concluded in [6] that there are two correct equilibria (which are reflections of one another) and these are the only equilibria corresponding to formations where the agents are not collinear. There is rather less analysis than in [1] of the nature of equilibria with collinear agents. We consider that now. For this purpose, adopt a coordinatization where agent 1 is at the origin, and agents 2,3 are on the x-axis at x_2, x_3 respectively. Introduce the following restriction of V to collinear formations:

$$V_2(x_2, x_3) = (x_2^2 - d_{12}^{*2})^2 + ((x_2 - x_3)^2 - d_{23}^{*2})^2 + (x_3^2 - d_{13}^{*2})^2 \tag{18}$$

Critical points of V_2, regarded as being defined over the manifold $\mathbb{R}^2$ are defined by setting its two derivatives to zero. They are precisely the solutions of the following equation:

$$x_2(x_2^2 - d_{12}^{*2}) = x_3(x_3^2 - d_{13}^{*2}) = -(x_2 - x_3)((x_2 - x_3)^2 - d_{23}^{*2}) \tag{19}$$

The associated Hessian matrix is

$$H = \begin{bmatrix} 3x_2^2 - d_{12}^{*2} & 0 \\ 0 & 3x_3^2 - d_{13}^{*2} \end{bmatrix} + (3(x_2 - x_3)^2 - d_{23}^{*2}) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{20}$$

It is *not* straightforward to solve the equations for the critical points and then determine the number of negative eigenvalues of the Hessian (yielding the index of the critical point). We can however make some progress. Observe that

1) (x_2,x_3)=(0,0) is a critical point and the associated index is 2 (i.e. it is a maximum)
2) Since V_2 diverges to ∞ when $|x_2|$ or $|x_3|$ tends to infinity, there is necessarily a minimum, i.e. $m_0 \geq 1$.
3) If (x_2,x_3) is a critical point, so is $(-x_2,-x_3)$. This means that for V_2, there are integers n_0,n_1 and n_2 such that $m_0 = 2n_0, m_1 = 2n_1, m_2 = 2n_2 + 1$.
4) By virtue of the Morse relations, and the Betti numbers for $\mathbb{R}^2$, we have:

$$\begin{aligned} 2n_0 &\geq 1 \\ 2n_1 - 2n_0 &\geq -1 \\ 2n_2 + 1 - 2n_1 + 2n_0 &= 1 \end{aligned} \tag{21}$$

It follows easily that $n_0 \geq 1, n_1 \geq n_0, n_2 = n_1 - n_0$

We now connect these conclusions to the critical points of V. The critical point of V_2 at $(0,0)$ has no counterpart critical point for V, since this corresponds to a formation with three agents collocated. All other critical points of V_2 occur in pairs (coordinates being reflected through the origin) and become single critical points of V, as they differ through formation rotation by π radians. Also, the index of a critical point of V_2 is increased by 1 in consideration of the instability associated with collinearity of a formation (as described in some detail in [6]). Accordingly, the quantities n_0, n_1, n_2 for V_2 become m_1, m_2, m_3 for V, and so $m_1 \geq 1, m_2 \geq m_1, m_3 = m_2 - m_1$. In addition, as already noted, $m_0 = 2$. It is straightforward to verify consistency with (16).

VI. Conclusions and Future Work

In this paper, we have shown using the tool of Morse Theory that multiple equilibria, including incorrect equilibria, are a consequence of any formation shape control algorithm which relies on a steepest descent notion of an error function. The argument is made more technically difficult by the requirement to find a coordinatization of formations which identifies all those formations that actually differ by translation and/or rotation. For n agent formations, this is of dimension $2n-3$; the Betti numbers of the relevant manifold are known, and this is the key to using Morse theory.

For the case $n = 3$, it is known that incorrect equilibria are all saddle points. Accordingly, given the presence of noise or unmodelled dynamics in any real system, they may be little impediment to use of the relevant algorithms. For $n > 3$, whether all incorrect equilibria are saddle points (or maxima) remains unknown. Some investigation for the case $n = 4$ with a formation modelled by a complete graph can be found in [2], [10].

We have yet to complete our investigation of four agent networks, but it is not inconceivable that Morse Theory could yield an answer to this question.

Another direction of research is the characterization of formations in a 3-dimensional ambient space, and the derivation of the relevant Betti numbers to assist in discussions of equilibria of formation control algorithms. One might seek a characterization of the relevant manifold in the form $A \times \mathbb{R}^+$, reflecting a scaling parameter, and a manifold A accounting for invariance under scaling and rotation, but in three dimensions, rather than 2.

The results here may also have application to the construction of algorithms for sensor network localization. In a sensor network, a collection of intersensor distances are unknown, but their relative positions are not. Disregarding the use of anchors to fix a sensor network in an absolute coordinate frame, one approach to sensor network localization can be obtained using formation control ideas: the *estimates* of sensor positions at a particular time t correspond to the position coordinates of the agents of a formation at that time, the measured intersensor distances correspond to prescribed formation distances d^*_{ij} between nominated agent pairs, and the task of adjusting the sensor position estimates parallels that of moving the formation. The sensor position estimates should be adjusted so that the intersensor distances as obtained from the position estimates agree with those obtained by measurement, and this task is equivalent to causing the formation to take up its prescribed shape.

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