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Stability and Performance Verification before switching in new controllers using closed-loop data

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***Abstract*— Consider an interconnection of an unknown or partially known linear plant and a known linear stabilizing controller, and assume that some knowledge of the closed-loop system is available. Suppose, on the basis of that knowledge, the use of a new controller appears attractive. We provide analysis and introduce novel tests utilizing a limited amount of experimental and possibly noisy data obtained from the existing closed-loop for verifying that the introduction of a proposed new controller will not only stabilize the plant but also provide assurance for the closed-loop performance before the insertion of a new controller. The importance of this capability arises in iterative identification and control algorithms, multiple-model adaptive control, and multi-controller adaptive switching.**

I. Introduction

Let $[P, C_0]$ denote a feedback interconnection of a plant P and a controller C_0, and assume that the transfer function of the plant is unknown while $C_0(s)$ is known. Let C_1 be a new controller designed to replace C_0. We present tools and techniques to verify whether C_1 in place of C_0 will be stabilizing and result in better closed-loop performance. The results are based on the knowledge of C_0 and C_1 and on data obtained from experiments on the closed-loop system $[P, C_0]$, but not directly on P, and exhibit tolerance of noise.

Many multi-controller adaptive switching algorithms do not explicitly rule out the possibility of placing a destabilising controller in the closed-loop [1], [2], [3], [4], [5]. Even if the new controller is ensured to be stabilising, performance verification with the new controller to satisfy the performance requirements and guarantee a better performing closed-loop is not straightforward.

In adaptive control, it is very frequently the case that a model is explicitly or implicitly constructed, and the performance of the model is compared with the actual system. When the actual system comprises a plant in combination with a controller, the model may for example comprise an estimate of the plant obtained by identification in combination with the same controller. Or the model may be a desired closed-loop transfer function, and the actual system comprises the closed loop formed by the actual plant and the controller. Thus a measure of quality for the model is usually an *a priori* agreement. The central difficulty is that a model may be a good model with one set of experimental conditions, but not with another set.

Consider for example two transfer functions $P_1(j\omega)$ and $P_2(j\omega)$ and suppose that $|P_1(j\omega) - P_2(j\omega)| < 0.01\ \forall\omega$. Then $P_2(j\omega)$ might be reasonably deemed to be a good model of $P_1(j\omega)$. Let $P_1(s) = 1/(s+1)$ and $P_2(s) = 1/(s+1)(.1s+1)$, and compare the open-loop responses and the closed-loop responses when connected to two constant gain controllers $C_1 = -1$ and $C_2 = -100$. With C_1 the closed-loop responses are very similar, and with C_2 they are very different. Evidently, $P_2(s)$ is a good approximation of $P_1(s)$ in open loop, and remains so with a small feedback gain, but it is a poor approximation with a large feedback gain.

Changing from having no controller in the loop to having a controller in the loop is another way of changing the experimental conditions, and there is then no guarantee that what was a good model will remain so. Similarly, if two closed-loop transfer functions are close, it does *not* imply that the two open-loop transfer functions are close. Even more crucially for adaptive control, if the closed-loop transfer functions are close with one controller, and the controller is changed, the two new closed-loop transfer functions may no longer be close. Such pitfalls are discussed in detail in [3].

Despite the rich volumes of literature on adaptive control algorithms, one cannot find an indication—at least to the point of verifying user-specific performance requirements—of research attempts to evaluate prospective closed-loop performance before a new controller C_1 is inserted into the loop. In fact, it is certainly not straightforward to even verify whether the new controller will maintain the closed-loop stability before it is switched in.

We review a technique for verifying the closed-loop stability [6], [7], [8], and extend it to treating performance, with the introduction of a new linear controller C_1 using a limited amount of noisy input-output experimental data obtained from the stable closed-loop $[P, C_0]$. Our stability verification results exploit phase information of the current closed-loop data, analogously to the Nyquist stability criterion, to ascertain closed-loop stability with the new controller. The tests offer robustness to noise and require gathering of information only on an easily-estimated frequency region whose size depends on the size of the controller change. This links the experimental effort to the size of the controller update. In particular, if the controller change has limited size then it is sufficient to obtain an estimate of the phase of the current closed-loop system up to a finite frequency which can be inferred from the closed-loop bandwidth.

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The structure of the paper is as follows. Section II collects the required definitions and notations from the relevant literature. We shall state the problem of interest in Section III before presenting the stability verification tests in Section IV and performance verification tools in Section V. A simulation example is illustrated in Section VI, and Section VII contains concluding remarks and future research directions.

II. Preliminaries

The notations are standard and borrowed from [9], [10]. The number $\mathrm{wno}(\cdot)$ denotes the winding number of the Nyquist diagram of a scalar transfer function, evaluated on a contour along the imaginary axis and indented to the right around any pure imaginary pole. The nearest integer function $\mathrm{nint}[\cdot]$ returns the integer closest to $[\cdot]$ with the additional rule that half-integers are always rounded to even numbers. We utilize the coprime factor representations of P and C, and assume that the transfer functions of P and C are all proper.

Consider the interconnection $[P,\ C]$ in Fig. 1.

***Definition** 1:* The interconnection $[P,\ C]$ is "well-posed" if the transfer function matrix mapping $\left[\begin{smallmatrix} r \\ w \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ exists. Put another way, $[P,\ C]$ is well-posed if $(I-CP)^{-1} \in \mathscr{R}$.

Given such well-posedness in Fig. 1 we have

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} P \\ I \end{bmatrix} (I-CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix} = H_{[P,C]} \begin{bmatrix} r \\ w \end{bmatrix}.$$

***Definition** 2:* The interconnection $[P,\ C]$ is said to be "internally stable" if it is well-posed and $H_{[P,C]} \in \mathscr{RH}_\infty$.

We shall define the (2,2) entry of $H_{[P,C]}$ as $S : w \mapsto u$, $S = (I-CP)^{-1}$ and the (1,1) entry of $H_{[P,C]}$ as $Q : r \mapsto y$, $Q = -P(I-CP)^{-1}C$. In the scalar case $Q + S = 1$.

***Lemma** 3 ([10], [11]):* Let P be scalar in the interconnection of Fig. 1. Let PM and GM denote the phase-margin and gain-margin of $[P, C]$, and let $M_r^S := \max_\omega |S(j\omega)| = \|S\|_\infty$ and $M_r^Q := \max_\omega |Q(j\omega)| = \|Q\|_\infty$ be the resonant peaks for $S(s)$ and $Q(s)$, respectively. Then:

i. $GM \geq \frac{M_r^S}{M_r^S - 1}$;
ii. $PM \geq 2 \arcsin\left(\frac{1}{2M_r^S}\right) \geq \frac{1}{2M_r^S}$;
iii. $GM \geq 1 + \frac{1}{M_r^Q}$;
iv. $PM \geq 2 \arcsin\left(\frac{1}{2M_r^Q}\right) \geq \frac{1}{2M_r^Q}$.

***Note** 4:* Let ω_g denote phase-crossover frequency (gain-margin frequency) and ω_p be gain-crossover frequency (phase-margin frequency). We have

$$|S(j\omega_p)| = |Q(j\omega_p)| = \frac{1}{\sqrt{2(1-cos(PM))}}$$

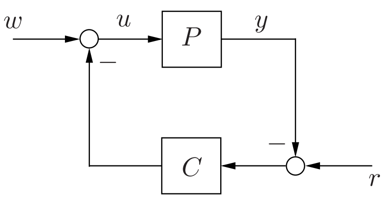


Fig. 1. Standard Feedback Configuration

***Remark** 5:* For good robustness, it is advised to have GM > 2 and PM $> 30°$. The following are direct consequence of the results in Lemma 3:

i. $M_r^S = 2$ guarantees $GM \geq 2$ and $PM \geq 29°$;
ii. $M_r^Q = 2$ guarantees $GM \geq 1.5$ and $PM \geq 29°$;
iii. For $GM \geq 2$ and $PM \geq 29°$, ensure $M_r^S > 2$.

***Definition** 6:* The ordered pair $\{N, M\}$, with $M, N \in \mathscr{RH}_\infty$, is a right-coprime factorization (*rcf*) of $P \in \mathscr{R}$ if M is invertible in $\mathscr{R}$, $P = NM^{-1}$, and N and M are right-coprime over $\mathscr{RH}_\infty$. Furthermore, the ordered pair $\{N, M\}$ is a normalized *rcf* of P if $\{N, M\}$ is a *rcf* of P and $M^*M + N^*N = I$.

***Definition** 7:* The ordered pair $\{\tilde{U}, \tilde{V}\}$, with $\tilde{U}, \tilde{V} \in \mathscr{RH}_\infty$, is a left-coprime factorization (*lcf*) of $C \in \mathscr{R}$ if $\tilde{V}$ is invertible in $\mathscr{R}$, $C = \tilde{V}^{-1}\tilde{U}$, and $\tilde{U}$ and $\tilde{V}$ are left-coprime over $\mathscr{RH}_\infty$. Furthermore, the ordered pair $\{\tilde{U}, \tilde{V}\}$ is a normalized *lcf* of C if $\{\tilde{U}, \tilde{V}\}$ is a *lcf* and $\tilde{V}\tilde{V}^* + \tilde{U}\tilde{U}^* = I$.

We utilize the normailized coprime factors in the sequel noticing that for some results the coprime factors do not need to be normalized.

***Definition** 8:* The graph symbol of P, G, and the inverse graph symbol of C, $\tilde{K}$, are defined by

$$G := \begin{bmatrix} N \\ M \end{bmatrix} \quad , \quad \tilde{K} := \begin{bmatrix} -\tilde{U} & \tilde{V} \end{bmatrix} \tag{1}$$

***Theorem** 9:* [10, Proposition 1.9] Let G and $\tilde{K}$ be defined as in (1). Then the following are equivalent:

i. $[P,\ C]$ is internally stable;
ii. $(\tilde{K}G)^{-1} \in \mathscr{RH}_\infty$;
iii. $\det(\tilde{K}G)(j\omega) \neq 0 \ \forall \omega$ and $\mathrm{wno} \det(\tilde{K}G) = 0$.

***Definition** 10:* The unwrapped phase of a scalar transfer function is denoted by unwarg and refers to the phase of the frequency response when it is in the form of a continuous function of the frequency. The unwrapped phase is computed from the phase frequency response by changing absolute jumps greater than π to their 2π complements, and ensures that all appropriate multiples of 2π are included.

***Definition** 11:* For a (not necessarily scalar) plant P, the "robust stability margin" $b_{[P,C]}$ is defined by

$$b_{[P,C]} := \begin{cases} \left\|H_{[P,C]}\right\|_\infty^{-1} \ , & \text{if } [P,C] \text{ is stable} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, larger $b_{[P,C]}$ corresponds to the smaller norm, but this norm cannot be smaller than 1 which means that for any P and C, $b_{[P,C]} \in [0,1]$; see [10].

***Note** 12:* For a scalar plant, large $b_{[P,C]}$ guarantees small sensitivity at frequencies where $|P(j\omega)|$ is large, and small complementary sensitivity at frequencies where $|P(j\omega)|$ is small. Also, both functions are well behaved around $|P(j\omega)| \approx 1$. The converse is also true.

***Note** 13 (P.38 [10]):* For $X \in \mathbb{C}^{p \times q}$ and $Z \in \mathbb{C}^{q \times p}$ the "point-wise version" of $b_{[P,C]}$ is defined by

$$\rho(X, Z, \omega) = 1/\bar{\sigma}\left(\begin{bmatrix} X \\ I \end{bmatrix}(I - ZX)^{-1}[-Z \quad I]\right)(j\omega) \tag{2}$$

***Definition** 14:* For stable $[P, C]$, $b_{[P,C]} = \inf_\omega \rho(P, C, \omega)$.

Note 15: Since $H_{[P,C]} = G(\tilde{K}G)^{-1}\tilde{K}$, one obtains

$$\begin{aligned}\rho(P,C,\omega) &= 1/\overline{\sigma}[(G(\tilde{K}G)^{-1}\tilde{K})(j\omega)] \\ &= 1/\overline{\sigma}[((\tilde{K}G)^{-1})(j\omega)] = \underline{\sigma}[(\tilde{K}G)(j\omega)] \quad (3)\end{aligned}$$

and if $[P,C]$ is stable $b_{[P,C]} = 1/\left\|(\tilde{K}G)^{-1}\right\|_{\infty}$.

Lemma 16 ([12]): Let P be scalar. If $[P,C] \in \mathscr{RH}_{\infty}$,

$$b_{[P,C]} = \left[(|C| + \frac{1}{|C|})|Q|\right]^{-1}, \quad (4)$$

Theorem 17 ([10]): Let P be scalar. If $[P,C] \in \mathscr{RH}_{\infty}$,

$$GM \geq \frac{(1 + b_{[P,C]})}{(1 - b_{[P,C]})} \text{ and } PM \geq 2\arcsin(b_{[P,C]}).$$

III. Problem Statement

Suppose the feedback control interconnection $[P, C_0]$ in Fig. 1, comprised of an *unknown* (or perhaps partially known) plant P and a stabilizing controller C_0, is internally stable. Further suppose that based on the data collected from the stable closed-loop of $[P, C_0]$ the use of a new controller C_1 appears attractive–a typical aspect of recursive identification/controller redesign. *How can one verify—without actual insertion in the closed-loop—if the introduction of the new controller C_1 will stabilize the plant and ensure a sensible level of closed-loop performance?*

We shall present our novel tests for verifying stability and performance with C_1 in advance of its insertion into the closed-loop using a limited amount of noisy input-output experimental data obtained from the stable loop $[P, C_0]$. The verification tests will offer robustness to noise. First we shall present the stability verification results.

IV. Stability Verification of $[P, C_1]$

The theorem below defines the *experimental setting* for the presentation of our stability verification tests.

Theorem 18 ([8]): Let $[P, C_0]$ be internally stable, $C_0 = \tilde{V}_0^{-1}\tilde{U}_0$ and $C_1 = \tilde{V}_1^{-1}\tilde{U}_1$ be left coprime factorizations over $\mathscr{RH}_{\infty}$. Consider Fig. 2 and define $T : r \mapsto z$ to be

$$T = [-\tilde{U}_1 \quad \tilde{V}_1] \begin{bmatrix} P(I - C_0P)^{-1} \\ (I - C_0P)^{-1} \end{bmatrix} \tilde{V}_0^{-1} \quad (5)$$

Then the following are equivalent:

a) $[P, C_1]$ is internally stable;
b) $T^{-1} \in \mathscr{RH}_{\infty}$;
c) $\det T(j\omega) \neq 0 \; \forall\omega$ AND $\text{wno}\det T = 0$;
d) $\det T(j\omega) \neq 0 \; \forall\omega$ AND $\text{unwarg}\det T(j\infty) = \text{unwarg}\det T(j0)$.

Fig. 2 depicts the so-called "observer-form implementation" of controller C_0. Simple manipulation shows that the controller equation can also be rewritten as $u = [-\tilde{U} \quad I + \tilde{V}]\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] - r$ which justifies the reference to the observer-form configuration; see [13], [4] for a similar implementation.

Note that P is unknown and hence one cannot explicitly construct T in closed form. However, the stable mapping $T : r \rightarrow z$ *can be studied in a safe experiment*, i.e. one in which no instability can occur, as shown in Fig. 2. Even though we do not have an explicit characterization of T, the reference signal r and the constructed output signal z (computed as a filtered version of the measured signals $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ via $\tilde{K}_1$) can be used experimentally to infer the required properties of T for verifying condition (d) in Theorem 18. For the development of our data-based stability tests, the following assumptions[1] are introduced.

Assumption 19: The factors $\tilde{V}_0$ and $\tilde{V}_1$ are such that $\tilde{V}_0(j\infty) = \tilde{V}_1(j\infty) = I$.

Assumption 20: The transfer functions PC_0 and PC_1 are strictly proper.

Notice that the transfer function T can be written as

$$T = \tilde{V}_1(I - C_1P)(I - C_0P)^{-1}\tilde{V}_0^{-1} \quad (6)$$

for which under Assumptions 19 and 20 we have

$$\det T(j\infty) = \frac{\det \tilde{V}_1(j\infty)}{\det \tilde{V}_0(j\infty)} \frac{\det(I - C_1P)(j\infty)}{\det(I - C_0P)(j\infty)} = 1\,. \quad (7)$$

Thus, $\det T(j\infty)$ is strictly positive and known and will be used as a datum for the verification of condition (d) in Theorem 18. Next, note the following easy-to-use test.

Theorem 21 ([8]): Let the suppositions of Theorem 18 and Assumptions 19 and 20 hold. Let e_i denote a reference signal where a step is applied at the $i-$th input while the other inputs are kept at 0. Perform n experiments with reference signal $r(t) = e_i(t)$, $i = 1, \ldots, n$ and let $\bar{z}_i$ be the steady state output of T recorded in each experiment. Define $\bar{Z} = [\bar{z}_1, \ldots, \bar{z}_n]$. Then

$$[P, C_1] \text{ is internally stable } \Rightarrow \det \bar{Z} > 0\,.$$

Thus if $\det \bar{Z} \leq 0$, stability of $[P, C_1]$ is falsified.

The experimental test in Theorem 21 is quite simple to carry out; it requires recording the steady state values of m step responses. However such an experiment can only be used to check a necessary stability condition.

Condition (d) in Theorem 18 can be verified in both its necessary and sufficient parts by using more sophisticated identification techniques to compute the *full* frequency response for T. However, this is not desirable on the grounds of complexity. In fact, one does not need to estimate the full

[1] Assumption 19 is without loss of generality and Assumption 20 captures a typical situation.

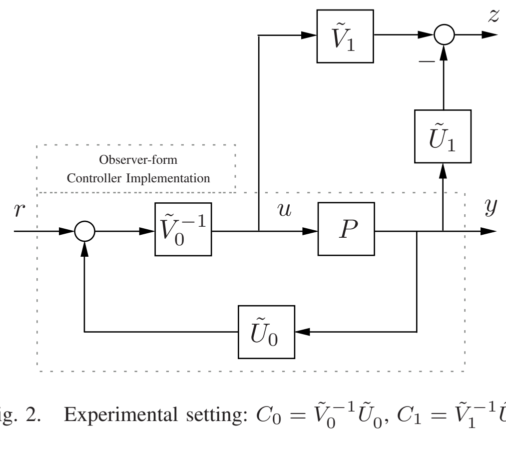


Fig. 2. Experimental setting: $C_0 = \tilde{V}_0^{-1}\tilde{U}_0$, $C_1 = \tilde{V}_1^{-1}\tilde{U}_1$.

frequency response of T, but instead its frequency response up to a certain finite frequency ω_0 will suffice. Moreover, experimental determination of a phase change known to be a multiple of 2π allows room for some error in the measurements. A mechanism to estimate ω_0 is advanced next by exploiting the structure of T.

***Lemma** 22 ([8]):* Let the suppositions of Theorem 18 hold. Then T can be expressed as

$$T = I + T', \tag{8}$$

$$T' = \begin{bmatrix} -(\tilde{U}_1 - \tilde{U}_0) & (\tilde{V}_1 - \tilde{V}_0) \end{bmatrix} \begin{bmatrix} P(I - C_0 P)^{-1} \\ (I - C_0 P)^{-1} \end{bmatrix} \tilde{V}_0^{-1} \tag{9}$$

The expression in (9) shows that T is the sum of a known term (i.e. I) and a term which, under Assumptions 19 and 20, is strictly proper. Thus measuring the frequency response of T up to a frequency, say ω_0, where the response of T' has nearly vanished is enough to characterize the full frequency response of T. This fact is utilized in Theorem 23.

***Theorem** 23 ([8]):* Suppose the hypothesis of Theorem 18 and Assumption 19 and 20 hold. Let $T' \in \mathscr{RH}_\infty^{n \times n}$, $T' = T - I$ as in (9), and $\omega_0 \in [0, \infty)$ be a frequency such that

$$\begin{cases} \rho(T'(j\omega)) < 1 & n = 1 \\ \rho(T'(j\omega)) < \sin(\dfrac{\pi}{n}) & n \geq 2 \end{cases} \quad \forall \omega \geq \omega_0 \tag{10}$$

Then the condition

$$\begin{cases} \det T(j\omega) \neq 0 \ \forall \omega \in [0, \omega_0) & \text{AND} \\ 2\pi \times \mathrm{nint}\left[\dfrac{\mathrm{unwarg} \det T(j\omega_0)}{2\pi}\right] = \mathrm{unwarg} \det T(j0) \end{cases} \tag{11}$$

is equivalent to condition (d) in Theorem 18.

***Note** 24:* Using the necessary and sufficient condition for the stability of $[P, C_1]$ in Theorem 23 requires estimation of the frequency response of the $T(j\omega)$ up to ω_0.

1. Recall that $\rho(T'(j\omega)) \leq \overline{\sigma}(T'(j\omega))$ and under Assumptions 19 and 20, $\overline{\sigma}(T'(j\omega))$ tends to zero as ω tends to infinity. Practically, one may have a rough estimate of the bandwidth of $[P, C_0]$ which can be used to obtain an estimate of ω_0 by assuming that $\overline{\sigma}(T'(j\omega))$ remains below the right-hand side of (10) over some known high-frequency region; a conservatively larger value makes the choice of ω_0 robust.
2. A small controller change certainly reduces the frequency ω_0 and, as a consequence, reduces the experimental effort. The structure of T' in (9) is such that $\rho(T'(j\omega))$ depends on the size of the controller change; e.g. for the SISO case one can choose $\tilde{V}_1 = \tilde{V}_0 = 1$, $\tilde{U}_0 = C_0$ and $\tilde{U}_1 = C_1$ resulting in $T' = (C_0 - C_1)P/(1 - C_0 P)$.
3. The value $\mathrm{nint}[\mathrm{unwarg} \det T(j\omega_0)/2\pi]$ is only used in condition (11); a rough estimate of $\mathrm{unwarg} \det T(j\omega_0)/2\pi$ is enough and hence the test can tolerate considerable estimation errors.
4. The estimate of the frequency response of $T(j\omega)$ up to ω_0 can be obtained via parametric [14] or non parametric [15] estimation methods. In practice, at each frequency one can use $\overline{\sigma}(T) \leq \|T\|_F = (\sum_{i,j} |T_{ij}|^2)^{\frac{1}{2}}$ along with $\rho(T(j\omega)) \leq \overline{\sigma}[T(j\omega)]$ to find an upper bound on the eigenvalues of $T(j\omega)$ in order to check (10). Alternatively the inequality $\overline{\sigma}[T(j\omega)] \leq \sqrt{n}\, \|T(j\omega)\|_1$ can be utilized.
5. The unwrapped phase can be obtained by phase unwrapping techniques of [16].

V. Performance Verification of $[P, C_1]$

We shall next build on the novel tools of previous section for projecting internal stability of $[P, C_1]$ to enable measuring performance aspects of $[P, C_1]$ in advance. For this, recall that P is unknown, C_0 is stabilizing and C_1 is verified to be stabilizing via the tools of previous section.

***Lemma** 25:* Consider the interconnection in Fig. 2 and Definition 8. The mapping $T : r \mapsto z$ in (5) and (6) can be expressed as

$$T = (\tilde{K}_1 G)(\tilde{K}_0 G)^{-1}. \tag{12}$$

Proof:

$$\begin{aligned} T &= \tilde{V}_1 (I - C_1 P)(I - C_0 P)^{-1} \tilde{V}_0^{-1} \\ &= \begin{bmatrix} -\tilde{U}_1 & \tilde{V}_1 \end{bmatrix} \begin{bmatrix} P(I - C_0 P)^{-1} \\ (I - C_0 P)^{-1} \end{bmatrix} \tilde{V}_0^{-1} \\ &= \tilde{K}_1 \underbrace{\left[G(\tilde{K}_0 G)^{-1} \right]}_{: r \to \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]} = (\tilde{K}_1 G)(\tilde{K}_0 G)^{-1} \end{aligned}$$

■

From (12), we have $(\tilde{K}_0 G)^{-1} = M^{-1}(I - C_0 P)^{-1} \tilde{V}_0^{-1}$. Since M is invertible, T can be expressed as

$$\begin{aligned} T &= (\tilde{K}_1 G)(\tilde{K}_0 G)^{-1} \\ &= [\tilde{V}_1 (I - C_1 P) M][M^{-1}(I - C_0 P)^{-1} \tilde{V}_0^{-1}] \\ &= \tilde{V}_1 S_1^{-1} S_0 \tilde{V}_0^{-1} \\ &= W_1^{-1} W_0, \end{aligned} \tag{13}$$

where $W_0 = M(\tilde{K}_0 G)^{-1} \tilde{V}_0 = (I - C_0 P)^{-1} \tilde{V}_0$ and $W_1 = M(\tilde{K}_1 G)^{-1} \tilde{V}_1 = (I - C_1 P)^{-1} \tilde{V}_1$.

***Theorem** 26:* Suppose the hypothesis of Theorem 18 holds, and consider the setting in Fig. 2. If the conditions in Theorem 18 are satisfied, $W_1 = W_0 T^{-1} \in \mathscr{RH}_\infty$.

Proof: From (13), one can express

$$T^{-1} := W_0^{-1} W_1. \tag{14}$$

Since $[P, C_0] \in \mathscr{RH}_\infty$, we have $(\tilde{K}_0 G)^{-1} \in \mathscr{RH}_\infty$. Also, we have $M \in \mathscr{RH}_\infty$. Thus $W_0 = M(\tilde{K}_0 G)^{-1} \in \mathscr{RH}_\infty$. Pre- and post-multiplying (14) by S_0 one obtains $W_0 T^{-1} := W_1$. We shall show that $W_1 \in \mathscr{RH}_\infty$. From the condition (b) in Theorem 18, $T^{-1} \in \mathscr{RH}_\infty$. Since $W_0 \in \mathscr{RH}_\infty$, we have $W_1 = W_0 T^{-1} \in \mathscr{RH}_\infty$. ■

The aforementioned results in Theorem 26 can be utilized to verify the performance of W_1 given the availability of performance aspects of W_0 and T^{-1}. Next, we present results for projecting some performance aspects with C_1 prior to its actual insertion into the closed-loop, assuming that the transfer function P is unknown.

***Theorem** 27:* Let $[P, C_0] \in \mathscr{RH}_\infty$ and $C_0 = \tilde{V}_0^{-1} \tilde{U}_0$ and $C_1 = \tilde{V}_1^{-1} \tilde{U}_1$ be left coprime factorizations over $\mathscr{RH}_\infty$. Consider Fig. 2 and define $T : r \mapsto z$ to be as in (5). Let

$b_{[P,C_0]}$ and $b_{[P,C_1]}$ be indices as in Definition 11. If $[P, C_1]$ is internally stable,

$$b_{[P,C_0]}\|T^{-1}\|_\infty^{-1} \le b_{[P,C_1]} \le b_{[P,C_0]}\|T\|_\infty. \quad (15)$$

Proof: See [11]. ■

***Remark** 28:* The results in Theorem 27 is a flag indicating that the sensitivity of closed-loop $[P, C_1]$ might be very bad if $\|T\|$ can become very small, but will not be very bad in the contrary case. The pointwise version of Theorem 27 is given below.

***Theorem** 29:* Suppose the hypothesis of Theorem 18 holds. Consider the setting in Fig. 2 and mapping $T : r \mapsto z$ in (5). Then $\forall\omega$,

$$\frac{1}{\bar{\sigma}[T^{-1}(j\omega)]}\rho(P, C_0, \omega) \le \rho(P, C_1, \omega) \le \bar{\sigma}[T(j\omega)]\rho(P, C_0, \omega). \quad (16)$$

Proof: See [11]. ■

***Theorem** 30:* Suppose P is scalar in Fig. 2 and the hypothesis of Theorem 27 holds. Then,

$$|T(j\omega)|\rho(P, C_0, \omega) = \rho(P, C_1, \omega) \quad \forall\omega. \quad (17)$$

Proof: From (12), we have

$$(T\tilde{K}_0 G)(j\omega) = (\tilde{K}_1 G)(j\omega) \quad \forall\omega. \quad (18)$$

which implies

$$\underline{\sigma}[(T\tilde{K}_0 G)(j\omega)] = \underline{\sigma}[(\tilde{K}_1 G)(j\omega)] \quad \forall\omega. \quad (19)$$

Since $T(j\omega)$ is a scalar here and $|T(j\omega)|$ can be factored out, via (3) we obtain

$$|T(j\omega)|\, \rho(P, C_0) = \rho(P, C_1). \quad (20)$$

■

***Corollary** 31:* Suppose P is scalar in Fig. 2 and the hypothesis of Theorem 27 hold. If $[P, C_1]$ is internally stable,

a. $GM_{[P,C_1]} \ge \frac{1+b_{[P,C_0]}\|T^{-1}\|_\infty^{-1}}{1-b_{[P,C_0]}\|T\|_\infty}$;

b. $PM_{[P,C_1]} \ge 2\arcsin(b_{[P,C_0]}\|T^{-1}\|_\infty^{-1})$.

Proof: The results follow from Theorems 17 and 27. ■

Before presenting experimental tests for projecting some performance measures of the closed-loop $[P, C_1]$ before inserting C_1 into the loop, let u_{ss} and z_{ss} denote the (sinusoidal) steady-state of the accessible signals $u(t)$ and $z(t)$, respectively. Recall that for a stable system $y = Gu$ the sinusoidal steady-state response can be expressed as $y_{ss}(t) = |G(j\omega)| \cos(\omega t + \angle G(j\omega))$.

In our framework, $[P, C_1]$ is internally stable,

$$W_1(j\omega) = W_0(j\omega)T^{-1}(j\omega), \quad \forall\omega$$

via Theorem 26. At each frequency, this relationship facilitates the computation of $|W_1(j\omega)|$ and $\angle W_1(j\omega)$. Note that verifying the performance of $[P, C_1]$ does not require the full frequency response of W_1, but some frequency-domain characteristics discussed earlier. The following procedures can be utilized to project the phase margin (PM) and gain margin (GM) of $[P, C_1]$, and infer other performance aspects.

***Algorithm** 32 (Projecting Phase Margin with C_1):* Suppose the hypothesis of Theorem 18 hold and consider the setting in Fig. 2.

Step 1: Ensure $[P, C_1]$ is internally stable via tools presented in Section IV;
Step 2: Set frequency $\omega = 0$.
Step 3: Insert a sinusoidal signal of frequency ω at reference input r while the other input $w = 0$;
Step 4: Compute u_{ss} and z_{ss} using the method discussed in the preceding paragraphs, and obtain $|W_1(j\omega)|$ at each frequency ω;
Step 5: If $|W_1(j\omega)| = 1$, set $\omega_p = \omega$ and go to Step 6; otherwise increase ω by ω_Δ (the step size of the frequency grid) and go to Step 3.
Step 6: Measure $\angle W_1$ at ω_p by computing the phase difference between u and z;
Step 7: Phase Margin of $[P, C_1]$ is $(180° - \angle W_1)$.

***Algorithm** 33 (Projecting Gain Margin with C_1):* Suppose the hypothesis of Theorem 18 hold and consider the setting in Fig. 2.

Step 1: Ensure $[P, C_1]$ is internally stable via tools presented in Section IV;
Step 2: Set frequency $\omega = 0$.
Step 3: Insert a sinusoidal signal of frequency ω at reference input r while the other input $w = 0$;
Step 4: Compute phase shift between u and z at steady state using the method discussed in the preceding paragraphs, and obtain $\angle W_1(j\omega)$ at each frequency ω;
Step 5: If $\angle W_1(j\omega) = -180°$, set $\omega_g = \omega$ and go to Step 6; otherwise increase ω by ω_Δ (the step size of the frequency grid) and go to Step 3.
Step 6: Measure $|W_1|$ at ω_g;
Step 7: Gain Margin of $[P, C_1]$ is $-20\log_{10}|W_1|$.

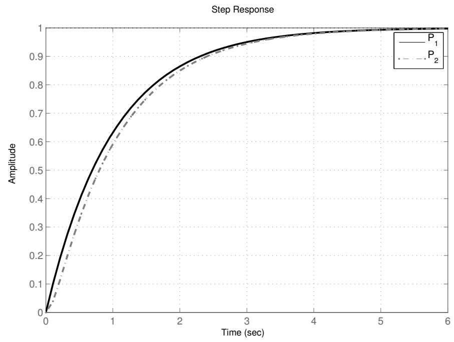
The procedures 32 and 33 for projecting the Phase and Gain Margins of $[P, C_1]$ require calculations over a limited frequency ranges, i.e. $\omega \in [0, \omega_p]$ and $\omega \in [0, \omega_g]$.

***Note** 34:* Similar Algorithms to those in 32 and 33 could be developed for determining the bandwidth, M_r and other frequency-domain characteristics of $[P, C_1]$ via properties of W_1. One can also utilise the indirect measurement results in Lemma 27 and Corollary 31 to determine lower bounds on PM and GM of $[P, C_1]$.

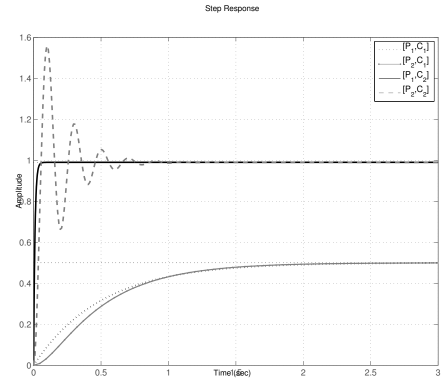
VI. Illustrative Example

Reconsider the example discussed in the introduction and let a second order plant $P_1(s) = 1/(s+1)(.1s+1)$ be approximated by $P_2(s) = 1/(s+1)$. Let $C_1 = -1$ and $C_2 = -100$ be candidate controllers which both would give good performance for the model P_2. This is illustrated in Fig. 3.

For illustration purposes and due to space limitation, let us consider a simple scenario. Consider Fig. 2 and assume $C_0 = 0$ and construct $T : r \mapsto z$ with the controller C_1 as a potential replacement. We shall denote this mapping as T_1. Similarly, let us denote T_2 the mapping resulting from C_2. The Nyquist diagrams of T_1 and T_2 are shown in Fig. 4.



(a) Plant and Model responses in Open Loop



(b) Plant and Model responses in Closed Loop

Fig. 3. Comparison of step responses

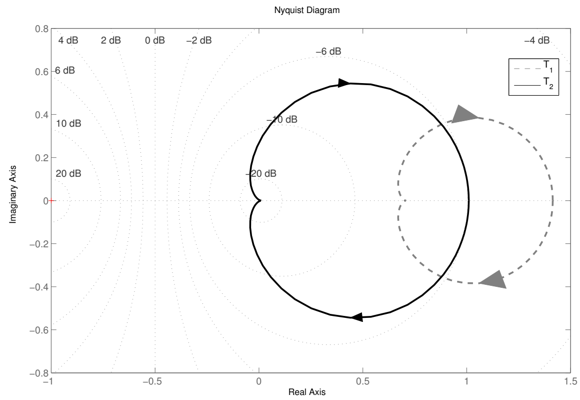


Fig. 4. Nyquist diagrams of T_1 and T_2

Note that the study of T in Fig. 4 with both the proposed controllers and the plant P_1 shows that $|T|$ gets very close to zero with C_2, implying that performance may be severely degraded.

VII. Conclusions

We have presented stability and performance validation tests for linear time-invariant systems which aim to project stability and some aspects of closed-loop performance with the introduction of a new controller C_1. The tests utilize a limited amount of experimental data obtained from the stable closed-loop interconnection $[P, C_0]$.

One of the stability verification results of Section IV uses step response properties of T to falsify the controller C_1, while Theorem 23 proposes a type of phase test analogous to the Nyquist criterion and utilized the noisy frequency response information of $T : r \mapsto z$ in Fig. 2 up to a finite frequency ω_0 to check if C_1 will stabilize the unknown plant P. It verifies that C_1 will be stabilizing (in place of C_0) if the Nyquist plot of T does not encircle the origin.

The performance verification results of Section V project performance of the closed-loop with C_1 before its insertion into the closed-loop and raise a red flag if, for example, at any frequencies T has small magnitude. Also, bounds on potential performance degradation have been developed.

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