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Deterministic Gossiping with a Periodic Protocol

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***Abstract*— A sequence of allowable gossips between pairs of agents in a group is *complete* if the gossip graph which the sequence generates is a connected spanning subgraph of the graph of all allowable gossip pairs; such a sequence is *minimally complete* if there is no shorter sequence which is complete. An infinite sequence of gossips is *repetitively complete* with period T if each successive subsequence of length T within the gossip sequence is complete. Any such sequence converges exponentially fast. A repetitively complete gossip sequence is *periodic* with period T if each gossip in the sequence is repeated once every T time steps. The rate of convergence of a periodic gossiping process is determined by the Tth root of the second largest eigenvalue in magnitude of the stochastic matrix of the complete gossip subsequence. In the case when the graph of allowable gossips is a tree and the complete gossip subsequence is minimally complete, this eigenvalue is independent of the order in which the gossips occur within the complete gossip subsequences.**

I. INTRODUCTION

There has been considerable interest recently in developing algorithms for distributing information among the members of a group of sensors or mobile autonomous agents via local interactions. Notable among these are those algorithms intended to cause such a group to reach a consensus in a distributed manner [1], [2], [3], [4], [5]. In a typical consensus seeking process, the agents in a given group are all trying to agree on a specific value of some quantity. Each agent initially has only limited information available. The agents then try to reach a consensus by communicating what they know to their neighbors either just once or repeatedly, depending on the specific problem of interest. One particular consensus process which has received much attention is called "gossiping." In a typical gossiping process each agent has control over a real-valued scalar "gossiping" variable. What distinguishes gossiping from more general consensus seeking is that during a gossiping process each agent is allowed to gossip with at most one of its neighbors at one clock time. A pair of agents gossip by updating the current values of their gossip variables to new values which are both equal to the average of their current values. Generally not every pair of agents is allowed to gossip. The edges of a given, undirected "allowable gossip graph" specify which gossip pairs are allowable. The actual sequence of gossip pairs which occurs during a specific gossip sequence might be determined either probabilistically [6] or deterministically, depending on the problem of interest. It is the latter type of problem to which this paper is addressed.

Of particular interest is the rate at which a sequence of agent gossip variables converge to a common value. The convergence rate question for more general deterministic consensus problems has been studied recently in [7], [8]. In [6], [9], [10] the convergence rate question is addressed for gossiping algorithms in which the sequence of gossip pairs under consideration is determined probabilistically. A modified gossiping algorithm intended to speed up convergence is proposed in [11] without proof of correctness, but with convincing experimental results. The algorithm has recently been analyzed in [12]. Recent results concerning convergence rates appear in [13] for *periodic gossiping* and in [14], [15], [16] for deterministic aperiodic gossiping. This paper presents a more comprehensive treatment of the work in [13].

A typical gossiping process can be modeled as a discrete time linear system of the form $x(t+1) = M(t)x(t), \quad t = 0, 1, \ldots$ where x is a vector of agent gossip variables and each value of $M(t)$ is a specially structured doubly stochastic matrix {§II-A}. A sequence of allowable gossips pairs is *complete* if the gossip graph which the sequence generates is a connected spanning subgraph of the graph of all allowable gossip pairs. A complete gossip sequence is minimally complete if there is no other complete gossip sequence of shorter length. {§II-B}. An infinite sequence of gossips is *repetitively complete* with period T if each successive subsequence of gossips of length T in the sequence is complete. Repetitively complete gossip sequences converge exponentially fast {§II-C}. Repetitively complete gossip sequences which are also periodic with period T are treated in Section III. The convergence rate of any such sequence is determined by T and by the second largest eigenvalue {in magnitude} of the stochastic matrix which the

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Proofs of the assertions in this paper are available from the first author upon request and will appear in a full length version of this paper.

gossips over a period defined. In the case when the allowable gossip graph is a tree and the sequence of gossips over a period is minimally complete, it is shown that the value of this eigenvalue does not depend on the order in which gossips over a period take place. This surprising result, which was first established in [17], is proved in [18] using different arguments. Section IV offers some concluding remarks and future work.

II. Gossiping

The type of gossiping we want to consider involves a group of n agents labeled 1 to n. Each agent i has control over a real-valued scalar quantity x_i called a *gossip variable* which the agent is able to update from time to time. A *gossip* occurs at time $t \in \{0, 1, 2, \dots\}$ between agents i and j if the values of both agents' variables at time $t+1$ equal the average of their values at time t. In other words $x_i(t+1) = x_j(t+1) = \frac{1}{2}(x_i(t) + x_j(t))$. If agent i does not gossip at time t, its gossip variable does not change; thus in this case $x_i(t+1) = x_i(t)$. Generally not every pair of agents is allowed to gossip. The edges of a given simple undirected n-vertex graph $\mathbb{A}$, called an *allowable gossip graph*, specify which gossip pairs are allowable. In other words a gossip between agents i and j is *allowable* if (i, j) is an edge in $\mathbb{A}$. Agent j is a *neighbor* of agent i if (i, j) is an edge in $\mathbb{A}$; we use the symbol $\mathcal{N}_i$ to denote the labels of the neighbors of agent i. One rule which sharply distinguishes a gossiping process from a more general consensus process is that in the case of gossiping, each agent is allowed to gossip with at most one of its neighbors at one time. This rule does not preclude the possibility of two or more pairs of agents gossiping at the same time, provided each of the two pairs have no agent in common. More precisely, two gossip pairs (i, j) and (k, m) are *noninteracting* if neither i nor j equal either k or m. When multiple noninteracting pairs of allowable gossip occur simultaneously, the event is called a *multi-gossip*. The goal of gossiping is for the n agents to reach a consensus in the sense that all n gossip variables ultimately reach the same value in the limit as $t \to \infty$. For this to be possible, no matter what the initial values of the gossiping variables are, it is clearly necessary that $\mathbb{A}$ be a connected graph.

A gossiping process can be modeled by a discrete time linear system of the form $x(t+1) = M(t)x(t), \quad t = 0, 1, \dots$, where $x \in \mathbb{R}^n$ is a state vector of gossiping variable and $M(t)$ is a matrix characterizing how x changes as the result of the gossips which take place at time t. If a single pair of distinct agents i and j gossip at time $t \geq 0$, then $M(t) = P_{ij}$ where P_{ij} is the $n \times n$ matrix for which $p_{ii} = p_{ij} = p_{ji} = p_{jj} = \frac{1}{2}$, $p_{kk} = 1, k \notin \{i, j\}$ and all remaining entries equal zero. We call such P_{ij} *primitive gossip matrices*. It will be convenient to include in the set of primitive gossip matrices, the $n \times n$ identity; the identity can be thought of as the correct update matrix to model the effect of each agent gossiping only with itself. If a multi-gossip occurs at time t, then as a consequence of noninteraction, $M(t)$ is simply the product of the primitive gossip matrices corresponding to the individual gossips comprising the multi-gossip; moreover because of noninteraction, the primitive gossip matrices in the product commute with each other and so any given permutation of the primitive matrices in the product determines the same matrix $M(t)$.

A. Doubly Stochastic Matrices

Each primitive gossip matrix is a nonnegative matrix whose row sums and column sums all equal one. Matrices with these two properties are called *doubly stochastic*. Note that the type of doubly stochastic matrix which characterizes a single gossip {i.e., a *primitive gossip matrix*} has two additional properties - it is symmetric and its diagonal entries are all positive. The same is true for the type of doubly stochastic matrices which characterize multi-gossips. Doubly stochastic matrices are special types of "stochastic matrices" where by a *stochastic matrix* is meant a nonnegative $n \times n$ matrix whose row sums all equal one. It is easy to see that a nonnegative matrix S is stochastic if and only if $S\mathbf{1} = \mathbf{1}$ where $\mathbf{1} \in \mathbb{R}^n$ is a column vector whose entries are all ones. Similarly a nonnegative matrix S is doubly stochastic if and only if $S\mathbf{1} = \mathbf{1}$ and $S'\mathbf{1} = \mathbf{1}$. Using these characterizations it is easy to prove that the class of stochastic matrices in $\mathbb{R}^{n \times n}$ is compact and closed under multiplication as is the class of doubly stochastic matrices in $\mathbb{R}^{n \times n}$. It is also true that the class of nonnegative matrices in $\mathbb{R}^{n \times n}$ with positive diagonals is closed under multiplication.

Mathematically, reaching a consensus by means of an infinite sequence of gossips or multi-gossips modeled by a corresponding infinite sequence of gossip matrices $M(0), M(1), M(2), \dots$ means that the sequence of matrix products $M(0)$, $M(1)M(0)$, $M(2)M(1)M(0)$, $M(3)M(2)M(1)M(0), \dots$ converges to a matrix of the form $\mathbf{1}c$ [16]. It turns out that if convergence occurs, the limit matrix $\mathbf{1}c$ is also a doubly stochastic matrix; this means that $c = \frac{1}{n}\mathbf{1}'$ and consequently that all n gossip variables will have converged to the average of their initial values. This particular fact further distinguishes a gossiping process from a more general consensus process, since in a consensus process the value to which all consensus variables typically converge is not necessarily the average of their initial values.

B. Gossiping Sequences

By a *gossiping sequence* we mean a sequence of allowable gossips corresponding to some or all of the edges in a given graph of allowable gossips $\mathbb{A}$. Note that a multi-gossip in which m single gossips take place simultaneously can be regarded as a sequence of m single gossips which have occurred one after another, in any order. Thus a sequence of k consecutive multi-gossips, in which m_i single gossips have taken place simultaneously within the ith multi-gossip, can be viewed as a sequence of $m_1+m_2+\cdots m_k$ single gossips which have occurred over the same period. For simplicity we will henceforth focus exclusively on single gossip sequences except when otherwise noted such as in Section III-B.

Corresponding to any given sequence of gossips $(i_1,j_1),(i_2,j_2),\ldots$ is a sequence of primitive gossip matrices $P_{i_1j_1},P_{i_2j_2},\ldots$ whose product $\cdots P_{i_2j_2}P_{i_1j_1}$ defines the mapping which assigns to any given initial vector of gossip variables, the vector of gossip variables which results from the gossips in the sequence. We call any such matrix product a *gossip matrix*. It is thus clear that a given allowable gossip graph has associated with it a family of gossip matrices whose elements are all products of all combinations of primitive gossip matrices of all lengths. These are the gossip matrices determined by $\mathbb{A}$. Conversely, any given sequence of individual gossips {or corresponding product of primitive gossip matrices} *induces* a spanning subgraph of $\mathbb{A}$ whose edges correspond to the gossips in the sequence. We say that a gossip sequence or corresponding gossip matrix is *complete* if the graph the gossips in the sequence induces is a connected spanning subgraph within $\mathbb{A}$. A gossip sequence and corresponding gossip matrix is *minimally complete* if it is complete and if there is no other gossip sequence of shorter length. It is easy to see that a gossip sequence is minimally complete if and only if the subgraph of $\mathbb{A}$ that it induces is a spanning tree in $\mathbb{A}$. In the special but important case when $\mathbb{A}$ is itself a tree $\mathbb{T}$, more can be said. In this case, a minimally complete gossip sequence is one in which, for each edge in $\mathbb{T}$, there is exactly one corresponding individual gossip in the sequence.

C. Convergence

Roughly speaking, if over a period T a complete gossip sequence has occurred, then each agent in the group will have been "in touch" with each other agent at least indirectly. It is not surprising then that complete gossip subsequences over successive periods in an infinite gossiping sequence should be sufficient for all gossiping variables in a gossiping process to converge to a common value. Prompted by this, let us call an infinite sequence of gossips *repetitively complete* with period T if each successive subsequence of gossips of length T in the sequence is complete. The following theorem implies that repetitively complete gossip sequences converge exponentially fast.

Theorem 1: Let $M(0),M(1),M(2),\ldots$ denote the gossiping matrices corresponding to an infinite sequence of single gossips which is repetitively complete with period T. Suppose that the vector of gossip variables $x(t)$ evolves according to $x(t+1)=M(t)x(t),\ t\geq 0$. There exists a real nonnegative number $\lambda<1$ such that for each initial value of $x(0)$, all n gossip variables converge to the average value

$$\frac{1}{n}\sum_{i=1}^{n}x_i(0)$$

as fast as λ^t converges to zero.

There are several different ways to prove this theorem; see for example [14], [15], [18].

As stated, the theorem is only for sequences of single gossips. However the same theorem also applies, with essentially the same proof, to sequences of multi-gossips. This is because the set of possible multi-gossips for a given graph $\mathbb{A}$ is finite and thus compact.

D. Tree Graphs

In graph theory, tree graphs {i.e., graphs without cycles} often lead to significant simplifications. This is also the case with gossiping. Let us note that a tree graph has the property that removal of any one edge (i,j) results in a disconnected graph[1]. This means that if $\mathbb{A}$ is a tree, a necessary condition for a finite sequence of gossips to be complete is that over the period during which the gossips occur, each agent must gossip with each of its neighbors at least once. If is clear that the converse is also true; i.e., if each agent gossips with each of its neighbors at least once during a given period, then the sequence of gossips which took place over that period must be complete. It is easy to see that in the case when $\mathbb{A}$ is a tree, a gossip sequence is minimally complete if and only if a gossip between each agent and each of its neighbors occurs exactly once in the sequence. Equivalently, a gossip matrix G for a graph $\mathbb{A}$, which is a tree, is minimally complete if and only if G is a product of all of the primitive gossip matrices associated with $\mathbb{A}$.

III. Periodic Gossiping

About the easiest way to guarantee a repetitively complete gossip sequence is to use a protocol which generates an infinite gossip sequence which on the one hand is "periodic" and on the other is complete on each successive period. Prompted by this, let us agree to call an infinite sequence of gossips *periodic* with period T if each gossip in the sequence occurs once every T time units; such a sequence is *periodically complete* if each subsequence consisting of T consecutive gossips is complete. It is clear that any periodically complete gossip sequence is repetitively complete. The converse of course is not true.

A. Convergence Rate

Corresponding to any T-periodic sequence of gossips is an infinite sequence of primitive gossip matrices; such a matrix sequence is *periodic* with period T in that each matrix within the sequence repeats itself every T time units. Suppose that $M(0),M(1),M(2),\ldots$ is such a T-periodic sequence. If $x(t+1)=M(t)x(t)\ t\geq 0$, it is clear that $x((i+1)T)=$

[1]If this were not so, then there would have to be at least two distinct paths between i and j which contradicts the requirement that a tree be acyclic.

$Nx(iT),\ i \geq 0$ where $N = M(T-1)M(T-2)\cdots M(0)$. Thus $x(iT) = N^i x(0),\ i \geq 0$ which means that both the convergence and convergence rate of a periodic gossiping sequence are completely determined by properties of N. Note that N is a doubly stochastic matrix because each of the matrices in the product defining it is doubly stochastic and because the class of $n \times n$ doubly stochastic matrices is closed under multiplication. Now because N is stochastic, it has an eigenvalue at 1 and its spectral radius is 1 [19]. We are interested in the case when $\lim_{i \to \infty} N^i = \frac{1}{n}\mathbf{1}\mathbf{1}'$ which is clearly just when all eigenvalues other than the one eigenvalue with value 1 have magnitudes strictly less than one. This is precisely the property of a complete gossip matrix.

Theorem 2: A gossip matrix is complete if and only if the magnitudes of all of its eigenvalues, with the exception of a single eigenvalue of value 1, are strictly less than 1.

A proof of this theorem can be found in [18].

For any doubly stochastic matrix S, let $\rho(S)$ denote the magnitude of the second largest eigenvalue {in magnitude} of S. It is clear that the rate at which x converges to a rank one matrix is $\rho^{\frac{1}{T}}(N)$. The following is the main result of [17].

Theorem 3: Suppose that the graph of allowable gossips $\mathbb{A}$ is a tree. Let $\mathcal{G}(\mathbb{A})$ be the set of all minimal complete gossip matrices determined by $\mathbb{A}$. Then $\rho(G)$ is fixed and independent of $G \in \mathcal{G}(\mathbb{A})$.

The theorem implies, somewhat surprisingly, that in the case when $\mathbb{A}$ is a tree, the order in which the gossips occur in a minimally complete gossip sequence does not affect the rate at which the corresponding infinite periodically complete gossip sequence converges. A new proof of this theorem is given in [18].

B. Multi-gossiping

It is clear from the preceding that the rate at which a periodically complete gossiping converges depends not only on $\rho(N)$ but also on T. For example, suppose that $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \ldots, \gamma_T, \gamma_1, \gamma_2, \ldots$ is an infinite periodically complete gossip sequence with period T. Suppose in addition that $\gamma_1, \gamma_2, \gamma_3$ are non-interacting gossips. Then these three gossips might be executed simultaneously, rather than sequentially, at the beginning of each period without in any way affecting the complete gossip matrix N corresponding to the original subsequence $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \ldots, \gamma_T$. In other words, rather than executing the T-periodic sequence $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \ldots, \gamma_T, \gamma_1, \gamma_2, \ldots$, the group could execute the sequence periodic $\mu, \gamma_4, \ldots, \gamma_T, \mu, \gamma_4, \ldots$ without changing the value of $\rho(N)$. The key point here is that this sequence has period $T-2$ rather than T. Thus by using multi-gossiping, the convergence rate for this gossiping process would be reduced from $\rho^{\frac{1}{T}}(N)$ to $\rho^{\frac{1}{T-2}}(N)$. It is obvious that in general, to get faster convergence, one would want to implement multi-gossiping sequences using the smallest number of distinct multi-gossips possible. For the case when $\mathbb{A}$ is a tree and the original sub-sequence $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \ldots, \gamma_T$ is minimally complete, we know that the order of the gossips in the sequence can be changed without changing $\rho(N)$. In this case, the minimal number of multi-gossips needed to implement the original sequence would be the same as the minimal number of colors needed to color the edges of $\mathbb{A}$ subject to the constraint that no two edges incident on any vertex have the same color, for edges of the same color would then correspond to those gossips which could be implemented together as a single multi-gossip. Edge coloring is a basic problem in graph theory. The minimal number of colors required to color a graph subject to this constraint is called the *chromatic index*. Vizing's theorem states that the chromatic index of an allowable gossip graph $\mathbb{A}$ is either $\mathbf{d}$ or $\mathbf{d}+1$ where $\mathbf{d}$ is the maximum vertex degree of $\mathbb{A}$ [20]. Moreover if $\mathbb{A}$ is a tree, the chromatic index is $\mathbf{d}$ because of König's theorem [21]. In other words, if $\mathbb{A}$ is a tree with maximum vertex degree $\mathbf{d}$, it is possible construct a periodic sequence of multi-gossips with period $T = \mathbf{d}$ which converges as fast as the sequence $\rho^{\frac{1}{\mathbf{d}}}(N), \rho^{\frac{2}{\mathbf{d}}}(N), \rho^{\frac{3}{\mathbf{d}}}(N), \ldots$ converges to zero where N is any minimally complete gossip matrix for the graph.

IV. Concluding Remarks

Let $\mathbb{A}$ be a given allowable gossip graph. Recall that a complete gossip sequence *minimal* if there is no shorter sequence of allowable gossips which is complete. It is easy to see that a complete gossip sequence will be minimal if and only if the gossip graph it induces is a spanning tree in $\mathbb{A}$. For a given allowable gossip graph there can be many minimal spanning trees and consequently many minimally complete gossip sequences. Moreover, there can be differing second largest eigenvalues {in magnitude} for the different doubly stochastic matrices associated with different complete minimal sequences. A useful exercise then would be to determine those complete minimal sequences whose associated second largest eigenvalues {in magnitude} are as small as possible.

One of the problems with the idea of gossiping, which apparently is not widely appreciated, is that it is difficult to come up with provably correct protocols for gossiping without making restrictive assumptions. For example, we are unaware of any deterministic request-based protocol which leads to exponential convergence which does not depend on at least the distinct neighbor event time assumption, and which does not require each agent to be aware of the values of the gossip variables on *all* of its neighbors. Whether or not such protocols can be devised remains to be seen.

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