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Network Localizability with Link or Node Losses

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Abstract— **The ability to localize a sensor network is important for its deployment. A theoretical result exists defining necessary and sufficient conditions for network unique localizability (for inter-sensor range-based localization); it has its roots in Graph Rigidity Theory where sensors and links/measurements are modelled as vertices and edges of a graph, respectively. However, critical missions do require a level of robustness for localizability, ensuring that localizability is retained in the event of link (edge) losses and/or sensor (vertex) losses. This work characterizes this robustness through a novel notion of redundant localizability, which is backed by redundant rigidity. Analogously to two well-known types of result for rigidity characterization, similar results are developed for edge redundant rigidity; they are supplemented by rather fewer results dealing with vertex redundant rigidity. These preliminary results may shed a light for any further study of redundant localizability.**

Index Terms— **Rigidity Theory, Network Localizability, Sensor Network Localization**

I. Introduction

Obtaining location information is a fundamental task in a sensor network, since otherwise the sensed data will become much less valuable. The location of every sensor node, if not already known from the deployment of the network or directly from GPS, can only be determined from a process based on measurements, the network structure and partially known location information. This process is referred to as localization, and the network property that governs the feasibility of localizing the entire network given the measurement is called localizability.

Most localization algorithms are based on range-measurements, or measurements that can be converted to inter-node distance measurements.

Aspens et al [2] formally prove that for a 2D localization problem, a necessary and sufficient condition for localizability given inter-node distance measurements is that a network graph must be globally rigid (the concept is reviewed below) and at least three noncollinear sensors must have known location information (hence they are anchors).

Having just global rigidity may be unrealistic in practical scenarios, as not only can the localization algorithms demand very high computational complexity, but also because the global rigidity property (hence localizability) can easily be lost if some node or measurement becomes unavailable, manifesting some type of node/link failure in the network.

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The first problem has attracted great interest among researchers. Anderson et al [1] discussed various graphical properties of easily localizable sensor networks. A main result is that by doubling or tripling the sensing radius, the new network graph acquires special properties that provide guarantees on the computational complexity, sometimes linear in the number of sensors. In [14], limitations of classical trilateration algorithms are analyzed, proving their insufficiency in even recognizing the localizability of a graph. A novel localization method generalizing trilateration is proposed based on aggregated knowledge of the subnetwork formed by all 1-hop neighbors of each node and the node itself; roughly speaking, the induced subgraph of the neighbors and the node is actually a wheel graph that is globally rigid.

The second problem, viz. ensuring tolerance of node or link failure, has been rarely addressed. Recent study of redundant rigidity [13] has started to shed light on a new direction, using a graph theoretical approach. It raises the question, pursued in depth in this paper: *what are the conditions for preserving localizability in the event of loss of up to p link length measurements and q nodes (together with all their associated distance measurements)?*

This work studies the level of redundancy that can be built into localizability, such that, the network is guaranteed to be localizable when the loss of nodes and/or links is allowed, up to a certain maximum number in each case. As the result is more of a fundamental characterization than a practical algorithm, the typical problems in practical localization, such as dealing with noisy measurements, characterization of RMS error and bias in position estimates, error propagation, and computational complexity, are not considered and remain for the future.

The rest of this paper is organized as follows. The preliminaries, especially relevant results of graph rigidity theory, are introduced in Section II. The connections between redundant rigidity, redundant connectivity, global rigidity and redundant localizability are established in Section III. Based on these, Section IV discusses the relationship between vertex and edge redundancy. Two characterizations of $(1,p)$-rigidity, a concept associated with the loss of links, are then given in Section V and Section VI, respectively. Section VII sets out the relationship between redundant connectivity and $(1,p)$-rigidity on the one hand and redundant global rigidity and localizability on the other. Finally, the conclusions and proposals for future work are presented. Due to space limitation, all proofs are omitted and can be found in an extended version by a request to the first coauthor.

II. PRELIMINARIES

We will model a network as an undirected graph $G = (V, E)$ where the nodes are elements of the vertex set V, and an edge $e_{ij} \in E$ if the distance between nodes i and j is available. In the sequel for $X \subset V$, $i_G(X)$ will be the number of edges in the graph induced by the vertex set X in G. Further, with $V_i \subset V$, the edge set in the graph induced by the vertex set V_i in G will be denoted by $E_G(V_i)$. Similarly, with $E_i \subset E$, the vertex set in the graph induced by the edge set E_i in G will be denoted by $V_G(E_i)$.

As noted in the introduction a result due to [2] proves that a two-dimensional network $G = (V, E)$ is localizable iff it is globally rigid and has at least three noncollinear anchor nodes. Thus in section II-A we provide some preliminary information about rigidity and global rigidity.

In subsection II-B we provide two alternative though, equivalent characterizations of rigidity. The first, a celebrated result due to Laman, [10] involves edge counts on subgraphs induced by components of vertex partitions. The second, primarily due to Lovasz and Yemeni [11] involves vertex counts on subgraphs induced by components of edge partitions.

Subsequently, in Section III we provide pertinent redundancy definitions and an important linkage between redundant rigidity, redundant connectivity and global rigidity, that sets up the remainder of the paper.

A. Rigidity and Global Rigidity

In this subsection, we recall briefly the notions and some properties of (minimal) rigidity and global rigidity. Our description will be largely based on graphs that model the network under consideration. Formal definitions of graph rigidity and global rigidity, which involve the theory of graph representations, can be found in, e.g., [4], [5], [12].

A graph is *rigid* if the only edge-length preserving smooth motions in a generic network[1] modelled by the graph are translation and rotation, thus only result in congruent graphs. A graph is *minimally rigid* if it is rigid and no single edge can be removed without losing rigidity.

Close inspection of the rigidity definition reveals that the consideration is made for smooth motions. Under discontinuous transformation of a network corresponding to a rigid graph, it is possible to have a non-congruent network that preserves the edge-lengths due to flip and/or flex ambiguity.

To eliminate these ambiguities and to uniquely determine the relative positions of any vertex given the set of distance measurements (corresponding to the edge-length set), a stronger definition is desirable. A graph is *globally rigid* if any two realizations of a network (with prescribed edge lengths and modelled by the graph) of the one edge-length set are congruent, i.e., differ at most by translation, rotation or reflection. We refer the reader to [4], [8] for source material on global rigidity.

[1]The graph theory literature generally uses a different term than network in discussing rigidity, the word framework being commonly employed. Because of this paper's connection with sensor networks, we retain the word network below.

B. Rigidity Characterization

We provide now two different sets of necessary and sufficient conditions for a graph to be rigid. The first due to Laman, [10] involves edge counts on graphs induced by vertex subsets.

Theorem 2.1: [Laman's Theorem] A graph $G = (V, E)$ is minimally rigid iff $|E| = 2|V| - 3$ and for all $X \subset V$, $i_G(X) \leq 2|X| - 3$. The graph is rigid iff there is an $E' \subset E$, such that $G' = (V, E')$ is minimally rigid.

In the sequel, we will also need the following result that presents Henneberg Operations for growing a rigid graph [7].

Theorem 2.2: [Vertex Addition] Consider a graph $G = (V, E)$, a node $k \notin V$ and two edges e_{ik} and e_{jk}, with i and j elements of V. Then G is (minimally) rigid iff $(V \bigcup k, E \bigcup \{e_{ik}, e_{jk}\}$ is (minimally) rigid.

Theorem 2.3: [Edge Splitting] Consider a graph $G = (V, E)$, a node $t \notin V$ and three edges e_{ti} e_{tj} and e_{tk}, with i j and k elements of V and e_{ij} element of E. Then G is (minimally) rigid if $(V \bigcup t, E \backslash e_{ij} \bigcup \{e_{ti}, e_{tj}, e_{tk}\}$ is (minimally) rigid.

There are efficient algorithms for checking the edge count condition underlying Laman's theorem [9].

The second necessary and sufficient condition due to Yemeni and Lovasz [11], involves vertex counts on graphs induced by edge subsets. Though less succinctly phrased, it may be easier to check than Laman's edge count result, and references to algorithms for checking the condition are provided in [11]. Jackson and Jordan [8] provide an elegant interpretation of this result using the theory of matroids.

Definition 2.1: Consider a graph $G = (V, E)$ and $E_i \subset E$. A set of subgraphs $P = \{G_i = (V_G(E_i), E_i)\}$ is termed an m-Admissible Decomposition (AD) of G if the following hold.

(i) $|E_i| > 0$.
(ii) $\bigcup_i E_i = E$.
(iii) P has at least m elements.

If $m = 1$, then P is simply called an AD of G.

Next we define the index of an m-AD that reflects an associated vertex count.

Definition 2.2: Consider a graph $G = (V, E)$ and $E_i \subset E$. We call $r(P)$ the index of an m-AD $P = \{G_i = (V_G(E_i), E_i)\}_{i=1}^{k}$ of G where $k \geq m$ is defined as

$$r(P) = \sum_{i=1}^{k} (2|V_G(E_i)| - 3)\,. \qquad \text{(II.1)}$$

To state the result we need one more definition.

Definition 2.3: Consider a graph $G = (V, E)$ and $E_i \subset E$. For a given m, an m-AD $P = \{G_i = (V_G(E_i), E_i)\}_{i=1}^{k}$ of G, $k \geq m$ is termed m-minimizing if $r(P)$ is the smallest among the indices of all possible m-AD's of G. Such an index will be denoted as $r_m(G)$.

Now we provide the promised second characterization of rigidity that follows from results in [8], [11] and [6].

Theorem 2.4: A graph G is rigid iff $r_1(G) = 2|V| - 3$. Further, the edge sets E_i underlying any minimizing admissible decomposition are disjoint.

Observe one cannot say that $r_1(G) = 2|V| - 3$ implies *minimal* rigidity. Indeed consider the graph $G = K_4$. Then with $P = \{G\}$, $r(G) = 2|V| - 3$, even though G is not minimally rigid.

We conclude this section with an associated definition from [8] and two results, one from [8], that will assist us in later development.

Definition 2.4: Consider a graph $G = (V, E)$. Suppose S is a non-empty subset of E, and H is the subgraph induced by the edge set S. Then S is an independent subset of E if $i_H(X) \leq 2|X| - 3$ for all $X \subset V_G(S)$, as long as $|X| \geq 2$. The null set is also an independent subset of E.

The following result is a direct consequence of Laman's theorem.

Lemma 2.1: Suppose a graph $G = (V, E)$ is minimally rigid. Then for any $S \subset E$, S is a maximally independent subset of S in the subgraph of G, induced by S.

The final result in this section is a translation of Lemma 2.4 from [8].

Lemma 2.2: Consider a graph $G = (V, E)$ with $|E| \geq 1$. Suppose $S \subset E$ is a maximally independent subset of E. Then $r_1(G) = |S|$.

III. Redundant Global Rigidity

Generally, there are two types of redundancy: those involving loss of edges, and those involving the loss of vertices. As a generalization we work here with *mixed redundancy*. We begin with definitions of mixed redundant rigidity and mixed redundant connectivity.

Definition 3.1: A graph $G = (V, E)$ is (q, p)-rigid if for all $0 \leq l \leq q - 1$ and $0 \leq k \leq p - 1$, the induced subgraph obtained by removing any l vertices and k edges is rigid.

In particular, a $(q, 1)$-rigid graph is what has been defined in [13] as *q-vertex rigid*, or simply q-rigid. Similarly a $(1, p)$-rigid graph is known as *p-edge rigid*. There is also a comparable definition for redundant connectivity.

Definition 3.2: A graph $G = (V, E)$ is (q, p)-connected if for all $0 \leq l \leq q - 1$ and $0 \leq k \leq p - 1$, the induced subgraph obtained by removing any l vertices and k edges is connected.

These definitions bring us to our first new result, aimed at tightening them. Specifically, we observe below that these definitions are stronger than is needed. Indeed it is clear from Laman's theorem that if a graph is not rigid then the removal of a further edge cannot make it rigid. The same fact applies to connectivity. Thus it is possible to alter the definition of $(1, p)$-rigidity/connectivity to simply require that the graph remain rigid/connected after the removal of any $p - 1$ edges, as opposed to *up to p-1 edges* as required by the foregoing definitions.

On the other hand, it is entirely possible for a nonrigid or a disconnected graph to gain rigidity or connectivity after losing vertices. Thus consider a rigid graph (V, E). Suppose $v \notin V$ and let e be an edge connecting v to a vertex in V. Then it is easy to see that the graph $(V \bigcup \{v\}, E \bigcup \{e\})$ is not rigid.

Similarly if the graph (V, E) is connected, the graph $(V \bigcup \{v\}, E)$ is not connected as the vertex v is isolated in this augmented graph.

We now assert that in fact, barring small graphs, even with q-vertex rigidity it suffices to check if rigidity is retained after deleting any $q - 1$-vertices.

Theorem 3.1: A graph $G = (V, E)$, with $|V| > q + 1$ is (q, p)-rigid iff it is rigid after removing any $q - 1$ vertices and $p - 1$ edges.

A similar result obtains for redundant connectivity.

Theorem 3.2: A graph $G = (V, E)$, with $|V| > q$ is (q, p)-connected iff it is connected after removing any $q-1$ vertices and $p - 1$ edges.

We now recount a result from [8] that ties connectivity and rigidity to global rigidity.

Theorem 3.3: A graph $G = (V, E)$ is globally rigid iff it is $(1, 2)$-rigid and $(3, 1)$-connected.

We now define *redundant global rigidity*, which as noted in section II is equivalent to redundant localizability given three or more noncollinear anchors.

Definition 3.3: A graph $G = (V, E)$ is (q, p)-globally rigid if it is globally rigid and for all $0 \leq l \leq q - 1$ and $0 \leq k \leq p - 1$, the induced subgraph obtained by removing any l vertices and k edges is globally rigid.

In view of theorems 3.1-3.3 one has the following result.

Theorem 3.4: A graph $G = (V, E)$, with $|V| > q + 1$ is (q, p)-globally rigid iff it is rigid after removing any $q - 1$ vertices and p edges and is connected after removing any $q + 1$ vertices and $p - 1$ edges.

Thus redundant global rigidity has two components: Redundant rigidity and redundant connectivity.

Finally, we come to redundant localizability.

Definition 3.4: A sensor network with a given set of inter-node distance measurements is (q, p) redundantly localizable if it remains localizable after removing any $q - 1$ nodes and any $p - 1$ edges.

Since a network is localizable if and only if it is globally rigid and has three or more noncollinear anchors, the previous theorem immediately yields:

Theorem 3.5: A two-dimensional sensor network with a given set of anchors and inter-node distance measurements and at least $q + 2$ nodes is (q, p) redundantly localizable if and only if its associated graph is rigid after removing any $q-1$ vertices and p edges, connected after removing any $q+2$ vertices and $p - 1$ edges, and the network retains three or more noncollinear anchors after removing any $q-1$ vertices.

The rest of the paper is largely concerned with the characterization of the redundant rigidity.

IV. Connections between vertex and edge redundancy

In this section we summarize relationships between vertex and edge redundancy. The uniform message is that vertex redundancy conditions are stronger than their edge counterparts.

First we present a result concerning redundant rigidity.

Theorem 4.1: A graph $G = (V, E)$, with $|V| \geq p + q + 3$ that is $(q + s, p)$-rigid for some $s > 0$, is $(q, p + s)$-rigid.

Next we address connectivity.

Theorem 4.2: A (q, p)-connected graph is $(q - s, p + s)$-connected for all $0 \leq s < q$.

We conclude this section by the following theorem that directly follows from Theorems 4.1 and 4.2, together with the characterization of redundnant global rigidity in terms of redundant rigidity and connectivity of Theorem 3.4.

Theorem 4.3: A graph $G = (V, E)$, with $|V| \geq p + q + 3$ that is $(q + s, p)$-globally rigid for some $s > 0$, is $(q, p + s)$-globally rigid.

V. Redundant Edge Rigidity: The Laman Approach

This section focuses on seeking a *Laman type* necessary and sufficient condition for $(1, p)$-rigidity. First we provide a natural definition of *minimal (1,p)-rigidity* generalized from the standard notion of (nonredundant) minimal rigidity, i.e. when $p = 1$.

Definition 5.1: A graph $G = (V, E)$ is minimally $(1, p)$-rigid if it is $(1, p)$-rigid and loses rigidity after the removal of any set of p edges.

It is not immediately clear that minimally $(1, p)$-rigid graphs actually exist for $p > 1$. It might be that a $(1, p)$-rigid graph necessarily has some sets of p edges whose removal renders the graph nonrigid, and some sets of p edges whose removal leaves the graph rigid. This is indeed the case for $p > 2$, see Lemma 5.1 below (though we do not prove this here); however for $p = 2$, the definition is meaningful, [8].

We first provide a necessary and sufficient condition for minimal $(1, 2)$-rigidity [8].

Theorem 5.1: A graph $G = (V, E)$ is minimally $(1, 2)$-rigid if and only if both the conditions below hold:

(a) $|E| = 2|V| - 2$.
(b) For any $X \subset V$, $1 < |X| < |V|$, $i_G(X) \leq 2|X| - 3$.

This result is in fact quite powerful, in that beyond a precise edge count, all it requires is that Laman's condition be satisfied by the graph induced by any proper subset of V.

Rigidity, as opposed to minimal rigidity, is characterized through Laman's theorem using the property that for a rigid graph $G = (V, E)$ there must exist an $E' \subset E$, such that $G = (V, E')$ is minimally rigid. The example depicted in Fig. 1(a) shows however, that *this is not true in general* for $(1, p)$-rigidity when $p > 1$.

To analyze the graph $G = (V, E)$, depicted in Fig. 1(a) observe that every vertex in the first and last rows of this nine vertex graph has an edge to every vertex in the second row. Thus in all there are 18 edges in this graph. Obviously, as

$$18 > 2 \times 9 - 4 + 2,$$

this graph cannot be minimally $(1, 2)$-rigid. It is also clear that there is no $E' \subset E$ such that (V, E') is minimally $(1, 2)$-rigid. This is so as were such an E' to exist it, $E \backslash E'$ must have precisely two elements in it. It is clear that the removal of any edge from G leaves at least one vertex with degree equal to two. Then no matter what further edge is removed, the resulting subgraph cannot be $(1, 2)$-rigid.

We now assert that G is in fact $(1, 2)$-rigid. To see this first observe that the graph induced by the vertices $\{1, \cdots, 6\}$ and that induced by $\{4, \cdots, 9\}$ are each minimally rigid. Indeed the graph in Fig. 1(b) is minimally rigid. The graph induced by $\{1, \cdots, 6\}$ can be built from this by two successive edge-splitting operations. The minimal rigidity of the graph induced by $\{4, \cdots, 9\}$ similarly follows.

Also note that there is a complete symmetry between the edges, in that if we show that the graph obtained by removing the edge e_{14} is rigid, this implies that a graph obtained by removing any single edge is rigid. It is easy to see that if edge e_{14} is removed, the graph remains rigid. Thus indeed the graph in Fig. 1 is $(1, 2)$-rigid.

The above observation shows that a $(1, p)$-rigid graph may not contain a minimally $(1, p)$-rigid graph. However in addition, the definition of minimally $(1, p)$-rigid graph is restricted to $p \leq 2$ only. Seeking an alternative definition remains open problem.

Lemma 5.1: For every integer $p > 2$, there is no minimally $(1, p)$-rigid graph satisfying Definition 5.1.

VI. Redundant Edge Rigidity: The Lovasz-Yemeni Approach

In this section, we characterize $(1, p)$-rigidity using the Lovasz-Yemeni approach. As in section V we will state the result attending first to minimally $(1, 2)$-rigid graphs.

Lemma 6.1: Let $G = (V, E)$ be a minimally $(1, 2)$-rigid graph. Then $r_2(G) = 2|V| - 2$, where $r_2(G)$ is defined in Definition 2.3.

We next state a result that shows that $r_p(G) \geq 2|V| - 4 + p$ is a sufficient condition for $(1, p)$-rigidity.

Theorem 6.1: Let $G = (V, E)$ be a graph with $r_p(G) \geq 2|V| - 4 + p$. Then G is $(1, p)$-rigid.

Combining these two results we immediately obtain the following necessary and sufficient condition for minimal $(1, 2)$-rigidity.

Theorem 6.2: A graph $G = (V, E)$ is minimally $(1, 2)$-rigid iff $r_2(G) \geq |E| = 2|V| - 2$.

The question remains whether $r_p(G) = 2|V| + p - 4$ is a necessary condition for *nonminimal* $(1, p)$-rigidity (assuming that $|E| \neq 2|V| + p - 4$). The graph $G = (V, E)$ in Fig. 2 serves as a counter example.

Consider the three edge partitions: $E_1 = E_G(\{1, 2, 4, 5\})$, $E_2 = E_G(\{3, 7, 4, 8\})$ and $E_3 = E_G(\{8, 9, 6, 5\})$. $P = \{(V_G(E_i), E_i)\}_{1=1}^{3}$ is a 2-AD for G. In this case:

$$r(P) = 3 \times (8 - 3) = 15 < 2 \times 9 - 2 = 16.$$

The graph is clearly rigid after removing any single edge other than from the set e_{45}, e_{48}, e_{58}. Now without sacrificing generality remove the edge e_{45}. Observe it can be reconstructed by an edge-splitting operation on $(V \setminus \{2\}, E_G(V \setminus \{2\}))$. Thus this graph is $(1, 2)$-rigid.

To this point we have shown that both Lovasz-Yemeni and Laman type necessary and sufficient conditions exist for minimal $(1, p)$-rigidity ($p = 2$), the Lovasz-Yemeni

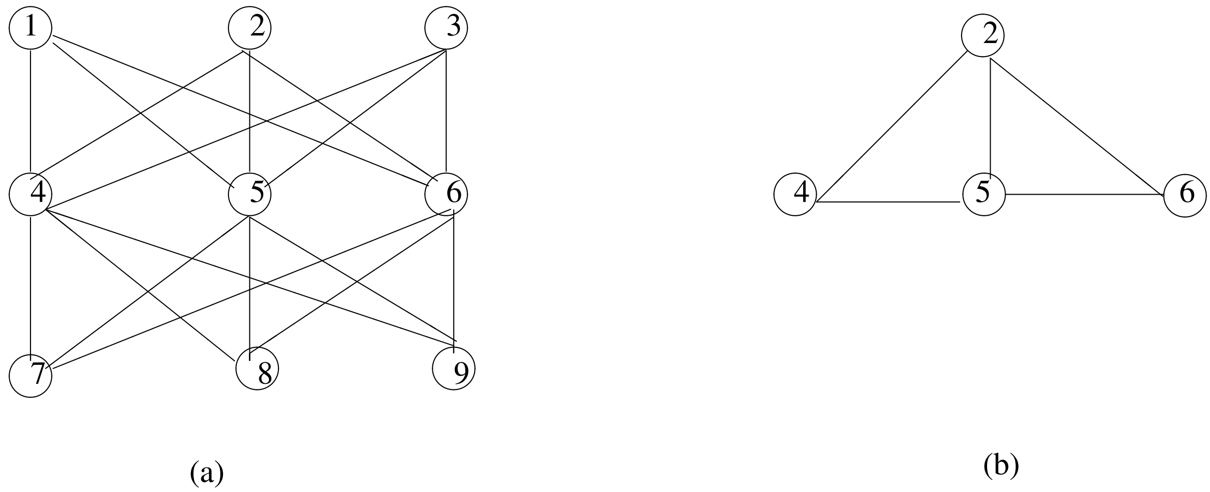

Fig. 1. (a) A $(1,2)$-rigid graph that does not contain a minimally $(1,2)$-rigid subgraph; (b) A graph used in proving claim for graph of (a).

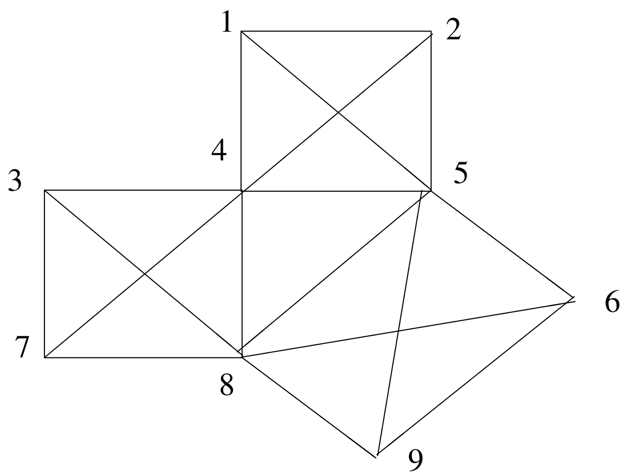

Fig. 2. A $(1,2)$-rigid graph with $r_2(P) < 2|V| - 2$.

type sufficient condition also characterize nonminimal $(1,p)$-rigidity for all positive integer p.

Recall however, that the direct subject of investigation in this paper is redundant global rigidity, which beyond redundant rigidity also requires redundant connectivity. It is instructive to note that the example of Fig. 1 is *(3,1)-connected, though that of Fig. 2 is not*. It is also possible to show that the graph in Fig. 1 does satisfy the corresponding Lovasz-Yemeni sufficient condition for $(1,2)$-rigidity. Indeed the next section demonstrates that a $(1,p)$-rigid graph that has an additional redundant connectivity property must have $r_p(G) \geq 2|V| - 4 + p$.

Clearly, having an efficient algorithm for computing r_p is important. While we fail to include such an algorithm in this paper, we would like to conjecture that it exists: Lovasz and Yemeni [11] claimed that $r_1(G)$ is easy to compute, and by definition finding $r_p(G)$, $p > 1$ involves computation of sums of vertex cardinalities involving a set of subgraphs that is also a subset of those needed to compute $r_1(G)$.

VII. Lovasz-Yemeni type condition for redundant rigidity with redundant connectivity

This section considers levels of connectivity needed for a $(1,p)$-rigid graph to satisfy $r_p(G) \geq 2|V| - 4 + p$. At the end of this section we will tie the results here to redundant global rigidity.

We have the following result connecting Yemeni-Lovasz type conditions to $(1,p)$-global rigidity for $p \leq 4$.

Theorem 7.1: Suppose, for integer p with $2 \leq p \leq 4$, $G = (V,E)$ is $(3,p-1)$-connected and $(1,p)$-rigid. Then $r_p(G) \geq 2|V| - 4 + p$.

There remains the question whether the result of Theorem 7.1 will hold for $p > 4$. We now present a counterexample for $p = 5$.

Example 7.1: The graph in question, $G = (V,E)$ has $|V| = 33$, and has a vertex cover $V_1, \cdots, V_6$, such that each $|V_i| = 7$, each subgraph $(V_i, E_G(V_i))$ is a K_7 graph, and the sets $E_i = E_G(V_i)$ consitute a partition of E. Further, defining

$$X_i = V_i \cap (\cup_{j \neq i} V_j)\,, \qquad \text{(VII.2)}$$

one has $X_1 = \{1,2,3\}$, $X_2 = \{1,4,5\}$, $X_3 = \{2,6,7\}$, $X_4 = \{3,8,9\}$, $X_5 = \{4,6,8\}$ and $X_6 = \{5,7,9\}$. In other words V_i comprises X_i together with four other vertices that are not in any other V_i. Thus there are 24 vertices each belonging to just one V_i, and 9 which are in exactly two.

To see that this choice of X_i is consistent with the fact that the E_i partition E, observe for all $i \neq j$,

$$|X_i \cap X_j| \leq 1. \qquad \text{(VII.3)}$$

Further vertices in each V_i connect to the others through the elements of X_i. Thus, indeed this defintion is consistent with the requirement that the E_i are disjoint.

In effect then G comprises six K_7 subgraphs, connected through the vertices in the X_i. Each V_i has three vertices through which it connects to other V_j, there being one vertex in common with each of three different V_j.

One can show that G is both $(3,4)$-connected and $(1,5)$-rigid. Since each subgraph $(V_i, E_G(V_i))$ is K_7, it also follows that $V_i = V_G(E_i)$. Thus as the E_i partition E, we have that

$$P = \{(V_i, E_G(V_i))\}_{i=1}^{6}$$

is a 6-AD of G. Hence,

$$r_5(G) \leq r(P) = 6(14-3) = 66 = 2|V| < 2|V| - 4 + 5,$$

and Theorem 7.1 is violated for $p = 5$.

Nonetheless we now show that for $p > 4$, a stronger redundant connectivity condition suffices for $r_p(G) \geq 2|V| - 4 + p$ to hold given $(1,p)$-rigidity.

Theorem 7.2: Suppose, for integer $p \geq 3$, $G = (V,E)$ is $(4, p-2)$-connected and $(1,p)$-rigid. Then $r_p(G) \geq 2|V| - 4 + p$.

As global rigidity is equivalent to the combined properties of $(3,1)$-connectivity and $(1,2)$-rigidity, in view of the two theorems in this section we have the following key result.

Theorem 7.3: A graph $G = (V,E)$ is $(1,p)$-globally rigid if it is $(3,p)$-connected and $r_{p+1}(G) \geq 2|V| - 3 + p$. For $1 \leq p \leq 3$ it is $(1,p)$-globally rigid only if $r_{p+1}(G) \geq 2|V| - 3 + p$. For $p \geq 4$ a $(4, p-2)$-connected $G = (V,E)$ is $(1,p)$-globally rigid only if $r_{p+1}(G) \geq 2|V| - 3 + p$.

Given the relation between global rigidity, anchor count and localizability, there is an obvious extension of this to a corresponding characterization of redundantly localizable two-dimensional sensor networks.

We comment that in a sensor network in which loss of nodes is contemplated, it may be that loss of anchor nodes can be ruled out on reliability or other grounds. What is needed for redundant localizability in addition to redundant global rigidity is that after loss of up to q-1 nodes, at least three noncollinear anchor nodes remain. The issue of which nodes fail does not enter the picture when the redundancy is all with respect to edge loss of course.

VIII. Conclusions and Future Work

This work presents preliminary results on edge-redundant global rigidity, which is essential to the problem of seeking a solution to guaranteing network localizability in the event of link losses. The particular notion that is studied in detail is $(1,p)$-rigidity, for which two different types of characterizations were obtained: Laman type and Lovasz-Yemeni type. For the latter, further connection was established between redundant edge rigidity and redundant connectivity. We also show that a seemingly obvious definition of minimal $(1,p)$ rigidity is meaningless for $p > 2$.

As discussed in the beginning of this paper, it is of great importance to also study redundant vertex rigidity, i.e. $(q,1)$-rigidity. Some discussion is provided in the text but its characterization remains largely open. Also, one will need computationally efficient algorithms to check for satisfaction of these conditions (e.g. computing $r_p(G)$) for redundant localizability of a network. Moreover, operations are required for one to construct, augment, merge and split such networks, while ensuring the level of redundant localizability remain unchanged. Results along the lines of [1] providing simple sufficient conditions for redundant global rigidity, largely in terms of local properties of graphs, would also be welcome. At some point too, random geometric graphs should be investigated.

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