MARKOV PARAMETER CHARACTERIZATION OF THE SYRICT POSITIVE PROBLEM

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Abstract

Suppose that a rational function $Z(s)$ is defined by a Laurent series, the coefficients of which are known. Several criteria are given in terms of these coefficients (the Markov parameters of $Z(s)$) to ensure that $R_e Z(j\omega) > 0$ for all real ω. The criteria are of two types - involving a Routh table with first two rows initialized using the coefficients, and Bezout matrices with entries which are the integral expressions in the coefficients.

I. INTRODUCTION

In this paper, we examine the question of how the positive real nature [1] of $Z(s)$ can be characterized in terms of the Markov parameters [2] of $Z(s)$. These are the constants γ_i in the expansion

$$Z(s) = \gamma_{-1}s + \gamma_0 + \gamma_1 s^{-1} + \gamma_2 s^{-2} + \cdots \qquad (1)$$

(without loss of generality, we can assume $\gamma_{-1} = 0$. For if not, $Z(s)$ is positive real if and only if $Z(s) - \gamma_{-1}s$ is positive real and $\gamma_{-1} > 0$. [1]).

Our main interest is in rational $Z(s)$. For such $Z(s)$, we recall that one can express the positive real property in several ways.
The various characterizations and tests have a variety of uses - e.g. for developing further theory, or as a basis for checking the positive real property.
Of course, knowing the γ_i and the degree of the denominator of a rational $Z(s)$, one could compute the numerator and denominator of $Z(s)$, [2], and then check positive realness via a standard procedure [1]. We aim here for a more direct procedure (The precise meaning of "direct" will be defined by the tests themselves).
We shall also be rather more concerned with examining strict positive realness, where all poles of $Z(s)$ must be in $R_e[s] < 0$ and $R_e Z(j\omega) > 0$ for all real ω. Strict positive realness tends to be more important in applications, [4-7]
In [8], a test based on the γ_i is provided for checking that the denominator zeros of a rational $Z(s)$ all lie in $R_e[s] < 0$.
In view of the existence of the test for the pole restriction on $Z(s)$ and our concentration on strict positive realness, we shall confine our attention for the most part to characterizing the properties of the γ_i which ensure that $R_e Z(j\omega) > 0$ for all real ω.

II. Reformulation of the positivity criteria

Suppose it is known that

$$Z(s) = \frac{q(s)}{p(s)} \qquad (2)$$

for some real polynomials q, p with also $Z(\infty) < \infty$. Then

$$R(\omega) \triangleq R_e Z(j\omega) = \frac{q(j\omega)p(-j\omega) + q(-j\omega)p(j\omega)}{p(j\omega)p(-j\omega)} = \frac{f(\omega)}{d(\omega)} \qquad (3)$$

The main result on which the positivity test depends is as follows.

LEMMA 2.1 With $Z(s)$ as in (2) and with $Z(\infty) < \infty$, $R_e Z(j\omega) > 0$ for all real ω except where $j\omega$ is a pole of $Z(s)$ if and only if

$$\mathop{I}_{-\infty}^{+\infty} \frac{R'(\omega)}{R(\omega)} = 0, \; R(a) > 0 \qquad (4)$$

for some arbitrary real a. (The symbol $\mathop{I}_{-\infty}^{+\infty} x(\omega)$ for rational x denotes the Cauchy index of $x(.)$ over $(-\infty, +\infty)$ see [2])

Proof First, suppose that $Z(s)$ has no purely imaginary poles. Then in (3), $d(\omega) > 0$ for all ω. Then $R_e Z(j\omega) > 0$ for all real ω if and only if [2]

$$\mathop{I}_{-\infty}^{+\infty} \frac{f'(\omega)}{f(\omega)} = 0 \qquad f(a) > 0 \qquad (5)$$

for some arbitrary a. Now

$$\mathop{I}_{-\infty}^{+\infty} \frac{R'(\omega)}{R(\omega)} = \mathop{I}_{-\infty}^{+\infty} \frac{[f'(\omega)d(\omega) - d'(\omega)f(\omega)]/d^2(\omega)}{f(\omega)/d(\omega)}$$

$$= \mathop{I}_{-\infty}^{+\infty} \frac{f'(\omega)}{f(\omega)} - \mathop{I}_{-\infty}^{+\infty} \frac{d'(\omega)}{d(\omega)} \qquad (6)$$

Since $d(\omega) \neq 0$ for real ω, $\mathop{I}_{-\infty}^{+\infty} [d'(\omega)/d(\omega)] = 0$.
Now suppose $Z(s)$ has k pure imaginary poles. The number of distinct real zeros of $d(\omega) = p(-j\omega)p(j\omega)$ is equal to the number of distinct pure imaginary zeros of $p(j\omega)$. Thus [2],

$$\mathop{I}_{-\infty}^{+\infty} \frac{d'(\omega)}{d(\omega)} = k \qquad (7)$$

Also, $f(\omega) = q(j\omega)p(-j\omega) + q(-j\omega)p(j\omega)$ cleary has a zero at any pure imaginary zeros of $p(j\omega)$
If $f(\omega)$ has a zero at any other value of ω, then $R_e Z(j\omega)$ will be zero, and conversely. So to have $R_e Z(j\omega) > 0$ for all ω other than those for which $j\omega$ is a pole of $Z(s)$, it is necessary and sufficient that $f(\omega)$ have precisely k distinct real zeros, i.e.

$$\mathop{I}_{-\infty}^{+\infty} \frac{f'(\omega)}{f(\omega)} = k \qquad (8)$$

Use of (6), which holds irrespective of where the zeros of $p(s)$ are located then yields the main result.

Remarks 1: Using the Markov coefficients, it is easy to verify whether or not $R(a) > 0$ for any a. For if $Z(s)$ is as in (1), we have

$$R(\omega) = \gamma_0 - \gamma_2\omega^{-2} + \gamma_6\omega^{-4} - \gamma_6\omega^{-6} + \cdots\cdots \tag{9}$$

and the following alternative conditions guarantee $R(\Omega) > 0$ for sufficiently large Ω.

$$\gamma_0 > 0 \text{ or } \gamma_0 = 0,\ \gamma_2 < 0 \text{ or } \gamma_0 = 0,\ \gamma_2 = 0,\ \gamma_4 > 0,\ \cdots\cdots \tag{10}$$

III. STURM SEQUENCE METHOD

Now suppose that we are given two Laurent series

$$\begin{aligned} R_1(\omega) &= \alpha_0 + \alpha_1\omega^{-1} + \alpha_2\omega^{-2} + \cdots, \\ R_2(\omega) &= \beta_0 + \beta_1\omega^{-1} + \beta_2\omega^{-2} + \cdots \end{aligned} \tag{11}$$

with R_1, R_2 both known to be rational, suppose that R_2/R_1 is proper, i.e. finite at $\omega = \infty$. We shall be interested in the evaluation of the Cauchy index

$$\mathop{I}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)} = \mathop{I}_{-\infty}^{+\infty} \frac{\beta_0 + \beta_1\omega^{-1} + \cdots}{\alpha_0 + \alpha_0\omega^{-1} + \cdots} \tag{12}$$

Without loss of generality, we may assume $\alpha_0 \neq 0$ (Else replace $R_1(\omega)$, $R_2(\omega)$ by $\omega^k R_1(\omega)$ and $\omega^k R_2(\omega)$ for an appropriate value of k)

A computational approach to the index evaluation will now be derived. Suppose that

$$R_i(\omega) = \frac{h_i(\omega)}{\gamma(\omega)} \qquad i = 1, 2 \tag{13}$$

for some polynomials h_1, h_2 and γ (Note: one or both of the pairs h_1, γ and h_2, γ may not be coprime). Via a Euclidean algorithm, define h_3, h_4, ... through

$$h_{i+1}(\omega) = q_i(\omega)h_i(\omega) - h_{i-1}(\omega) \tag{14}$$

until $h_{N+2}(\omega) = 0$. Then [2]

$$\mathop{I}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)} = \mathop{I}_{-\infty}^{+\infty} \frac{h_2(\omega)}{h_1(\omega)} = V_h(a) - V_h(b) \cdots \tag{15}$$

when $V_h(a)$ denotes the number of variations in sign of $\{h_1(\omega), h_2(\omega), \cdots, h_{N+1}(\omega)\}$ evaluated at a.

Now introduce rational functions

$$R_i(\omega) = \frac{h_i(\omega)}{\gamma(\omega)} \qquad i = 3, 4, \cdots\cdots \tag{16}$$

and assume that $\gamma(a)\gamma(b) \neq 0$. Then clearly,

$$R_{i+1}(\omega) = q_i(\omega)R_i(\omega) - R_{i-1}(\omega) \cdots\cdots \tag{17}$$

and

$$\mathop{I}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)} = V_R(a) - V_R(b) \tag{18}$$

Now suppose that $\gamma(\omega)$, $h_i(\omega)$ have degree n and n_i. Then each $R_i(\omega)$ has the form

$$R_i(\omega) = \alpha_{i,0}\omega^{-(n-n_i)} + \alpha_{i,1}\omega^{-(n-n_i)-1} + \cdots\cdots \tag{19}$$

if (11) and (17) are used to compute the successive R_i

Now observe that the computation of the $q_i(\omega)$ does not need the $h_i(\omega)$: it is enough to use the Laurent series (11) and (19). The key to selecting $q_i(\omega)$ knowing $R_i(\omega)$ and $R_{i-1}(\omega)$ is to ensure that $q_i(\omega)R_i(\omega) - R_{i-1}(\omega)$ has a Laurent series the first term in which is of higher power in ω^{-1} than $R_i(\omega)$. Also, the computations end after a finite number of iterations. In fact, if one knows that R_2/R_1 has denominator degree not greater than N', then we have $N' >> N$.

Our interest is in $\mathop{I}_{-\infty}^{+\infty} R_2(\omega)/R_1(\omega)$. Together, (18) and (19) imply

$$\begin{aligned} \mathop{I}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)} &= V\{(-1)^{n_1}\alpha_{1,0},\ (-1)^{n_2}\alpha_{2,0},\ \cdots\ (-1)^{n_{N+1}}\alpha_{N+1,0}\} \\ &\quad - V\{\alpha_{1,0},\ \alpha_{2,0},\ \cdots,\ \alpha_{N+1,0}\} \end{aligned} \tag{20}$$

These observations now allow us to establish a Routh table-like method for evaluating the Cauchy index (6). In case $\gamma_0 \neq 0$ in (6), observe that $R'(\omega)/R(\omega)$ has no singularity at $\omega = 0$, so that

$$\mathop{I}_{-\infty}^{+\infty} \frac{R'(\omega)}{R(\omega)} = \mathop{I}_{-\infty}^{+\infty} \frac{\omega^2 R'(\omega)}{R(\omega)} = \mathop{I}_{-\infty}^{+\infty} \frac{\gamma_2\omega^{-1} - 2\gamma_4\omega^{-3} + \cdots}{\gamma_0 - \gamma_2\omega^{-2} + \gamma_4\omega^{-4} - \cdots} \tag{21}$$

Set $a_i = \gamma_{2i}$, $b_i = i\gamma_{2i}$ and

$$R_1(\omega) = R(\omega) = a_1 - a_2\omega^{-2} + a_3\omega^{-4} - \cdots\cdots \tag{22a}$$

$$R_2(\omega) = \omega^2 R'(\omega) = b_1\omega^{-1} - b_2\omega^{-3} + b_3\omega^{-5} - \cdots \tag{22b}$$

In case $\gamma_0 = 0$, there exists an even power of ω, ω^{2k} say, such that $\omega^{2k}R(\omega)$ has a Laurent series with leading coefficient nonzero. Then we set

$$R_1(\omega) = \omega^{2k}R(\omega) = a_1 - a_2\omega^{-2} + a_3\omega^{-4} - \cdots\cdots \tag{23a}$$

$$R_2(\omega) = \omega^{2k}R'(\omega) = b_1\omega^{-1} - b_2\omega^{-3} + b_3\omega^{-5} - \cdots\cdots \tag{23b}$$

We now form the following table

row					
1	a_1	a_2	a_3	a_4	······
2	b_1	b_2	b_3	b_4	······
3	c_1	c_2	c_3	c_4	······
⋮	⋮	⋮	⋮	⋮	
N+1	p_1	p_2	p_3		
N+2	0	0	0		

$$c_i = \frac{-1}{b_1}\begin{vmatrix} a_1 & a_{i+1} \\ b_1 & b_{i+1} \end{vmatrix} \quad d_i = \frac{-1}{c_1}\begin{vmatrix} b_1 & b_{i+1} \\ c_1 & c_{i+1} \end{vmatrix} \text{ and so on.}$$

Then

$$\mathop{I}_{-\infty}^{+\infty} \frac{R'(\omega)}{R(\omega)} = \mathop{I}_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)} = N - 2V\,(a_1, b_1, c_1, \cdots\cdots, p_1) \tag{24}$$

IV. BEZOUTIAN APPROACH

We shall base our development on the following result.

LEMMA 4.1[9] Let $h_1(x)$, $h_2(x)$ be real polynomials with $\deg h_1 = n$, $\deg h_2 = n$. Define an $n \times n$ matrix $C = (c_{ij})$ by

$$\frac{h_2(y)h_1(x) - h_1(y)h_2(x)}{x - y} = \sum_{ij=1}^{n} c_{ij} x^{n-i} y^{n-j} \tag{25}$$

Then

$$I_{-\infty}^{+\infty} \frac{h_2(x)}{h_1(x)} = \text{signature } C \tag{26a}$$

$$N \triangleq n - \deg\,[\text{g.c.d of } h_1(x), h_2(x)] = \text{rank } C \tag{26b}$$

Now let $R_1(\omega)$, $R_2(\omega)$ be two real proper rational functions with $R_2(\omega)/R_1(\omega)$ proper and

$$\begin{aligned} R_1(\omega) &= \alpha_0 + \alpha_1 \omega^{-1} + \cdots\cdots \\ R_2(\omega) &= \beta_0 + \beta_1 \omega^{-1} + \cdots\cdots \end{aligned} \tag{27}$$

Since we are interested in the Cauchy index of R_2/R_1, we can, as earlier, assume $\alpha_0 \neq 0$ without loss of generality. Define the Bezoutian matrix of $R_1(\omega)$, $R_2(\omega)$ to be $D = (d_{ij})$ where

$$\frac{R_1(x)R_2(y) - R_1(y)R_2(x)}{x - y} = \sum_{ij=1}^{\infty} d_{ij} x^{-i} y^{-j} \tag{28}$$

We shall now relate D to C. Suppose that

$$R_i(\omega) = \frac{h_i(\omega)}{\gamma(\omega)} \qquad i = 1, 2, \tag{29}$$

for polynominals $h_i(\omega)$ and $\gamma(\omega)$. We have $n = \deg h_1 = \deg \gamma \geqq \deg h_2$.

Then, (25), (28) and (29) yield

$$\begin{aligned} \sum_{ij=1}^{\infty} d_{ij} x^{-i} y^{-j} &= \frac{1}{\gamma(x)\gamma(y)} \sum_{ij=1}^{n} c_{ij} x^{n-i} y^{n-j} \\ &= \frac{\pi'(y)}{\gamma(y)} C \frac{\pi(x)}{\gamma(x)} \end{aligned} \tag{30}$$

where $\pi(x) = [x^{n-1}, x^{n-2}, \cdots\cdots, x, 1]'$

Now define $\hat{\gamma}_i$ by

$$\frac{1}{\gamma(x)} = \hat{\gamma}_0 x^{-n} + \hat{\gamma}_1 x^{(n+1)} + \hat{\gamma}_2 x^{-(n+2)} + \cdots\cdots$$

where $\hat{\gamma}_0 \neq 0$. Then

$$\frac{1}{\gamma(x)}\pi(x) = \begin{bmatrix} \hat{\gamma}_0 & \hat{\gamma}_1 & \hat{\gamma}_2 & \cdot & \cdot & \cdot \\ 0 & \hat{\gamma}_0 & \hat{\gamma}_1 & & & \\ 0 & 0 & \hat{\gamma}_0 & & & \\ \cdot & \cdot & \cdot & & & \\ 0 & 0 & 0 & & & \end{bmatrix} \begin{bmatrix} x^{-1} \\ x^{-2} \\ x^{-3} \\ \cdot \\ \cdot \end{bmatrix}$$

$$\triangleq [\Gamma_0 \quad \Gamma_1] x_\infty$$

Combining this with (30) shows that

$$D = \begin{bmatrix} \Gamma_0' \\ \Gamma_1' \end{bmatrix} C\,[\Gamma_0\ \Gamma_1] \tag{31}$$

and so

$$\text{rank } D = \text{rank } C, \quad \text{signature } D = \text{signature } C \tag{32}$$

With this observation and Lemma 4.1, we have therefore established the following result:

THEOREM 4.2 Let $R_1(\omega)$, $R_2(\omega)$ be two real rational proper functions with $R_2(\omega)/R_1(\omega)$ proper and $R_1(\infty) = 0$. Then with $D = (d_{ij})$ defined by (28),

rank D = degree of denominator of R_2/R_1 after cancellations (33a)

$$\text{signature } D = I_{-\infty}^{+\infty} \frac{R_2(\omega)}{R_1(\omega)} \tag{33b}$$

The application to the positivity problem of Section 2 is clear.

Remark Let $\hat{D}$ denote the $n \times n$ top left corner of D. Then (31) and the nonsingularity of Γ_0 imply

$$\text{rank } D = \text{rank } \hat{D} \qquad \text{signature } D = \text{signature } \hat{D} \tag{34}$$

The key to the importance of these equations is that the entries of D are readily computable from the Markov parameters α_i, β_i associated with the $R_i(\omega)$ in (27).

For the $R_i(\omega)$ in (22), we have

$$\begin{aligned} d_{i+1,j} - d_{i,j+1} &= \overline{(-1)^{(i/2)} a_{(i/2)+1} (-1)^{(j-1)/2}} \\ &\quad\ b_{(j+1)/2} \ (i\text{:even}, j\text{:odd}) \\ &= -(-1)^{j/2} a_{(j/2)+1} (-1)^{(i-1)/2} \\ &\quad\ b_{(i+1)/2} \ (i\text{:odd}, j\text{:even}) \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{35}$$

V. CONCLUSION

We have stated two separate tests for strict positivity of $R_e z(j\omega)$ for all ω, all ω other than these for which $j\omega$ is a pole of $Z(s)$, in terms of the Markov coefficients associated with $Z(s)$.

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