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Controlling Four Agent Formations

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Abstract: This paper considers formation shape control of a team of four point agents, for the most part in the plane. Control laws based on specified interagent distances are used. For a complete graph, specification of all interagent distances determines the formation shape uniquely. Krick, Broucke and Francis showed that for a now almost standard control law, there may exist equilibrium formation shapes with incorrect interagent distances. This paper studies such equilibria, shows that in some cases they are necessarily unstable.

1. INTRODUCTION

Autonomous vehicle formations, often perhaps functioning as mobile sensor networks for surveillance purposes or high resolution Earth and deep-space imaging, present significant technical challenges. Frequently, a fixed shape for a formation is desired, to ensure optimal sensing, and a basic task for autonomous vehicle formations is formation shape control. This paper analyses stability properties of a control law, now almost standard, for a simple formation, namely one defined by four agents in which all interagent distances are to be preserved, i.e. the associated graph is K_4, the complete graph on 4 vertices. The restriction to a simple formation is made because we do not know how to handle more complicated formations, and in the belief that treatment of a simple formulation may help provide the tools for addressing general formations.

In this paper, we envisage that the shape of a formation shape is controlled by actively controlling a certain set of interagent distances using relative position measurements. The objective is to design motion control laws for each agent such that the agents cooperatively and autonomously achieve a specified desired formation shape, with each agent working to change its associated interagent distances to the correct values. Early work within this framework includes Eren et al. [2002], Olfati-Saber and Murray [2002], Baillieul and Suri [2003]. Eren et al. [2002] propose the use of graph rigidity theory (see e.g. Graver et al. [1993]) for modeling information architectures. Olfati-Saber and Murray [2002] also utilize graph rigidity theory and propose gradient control laws based on structural potential functions which are generated from the graph. Baillieul and Suri [2003] also utilize graph rigidity theory and discuss application to formations of non-holonomic robots.

It has recently been observed that there is a fundamental distinction between formations where distances are maintained by both agents of a pair and formations are maintained by one agent of a pair Hendrickx et al. [2007], Yu et al. [2007, 2009]. The results in this paper are for formations where bidirectional control is used.

A significant recent work was that of Krick et al. [2009] which provides a complete analysis showing that the desired formation shape is *locally* asymptotically stable under a gradient control law, provided that the information architecture is rigid–the term rigid has a technical meaning in this context, consistent with common usage, see the paper in question. [They emphasize that the desired formation shape is a three-dimensional equilibrium manifold, and therefore since the resulting linearized system is non-hyperbolic, a non-trivial application of center manifold theory is required to establish stability.] However, the global stability properties of the desired formation shape remain a challenging open problem. Their contribution also considers by way of example a four-agent formation with a complete graph information architecture (i.e. all interagent distances are actively controlled). A simulation shows that the formation appears to converge to an incorrect shape (i.e. to a formation with interagent distances not all the same as those in the desired shape), and they conclude that the desired shape is not globally asymptotically stable.

In this paper, we elaborate on this four-agent example from Krick et al. [2009], and expand and correct earlier recent work of our own on this problem Summers et al. [2009]. This earlier work failed to include a correct proof of a claim on the instability of equilibria of incorrect shape when those shapes included certain acute angles. Our specific contributions are as follows.

(1) We provide properties relevant to a test for the instability of an incorrect equilibrium for any four agent formation (the test itself was stated in Summers et al. [2009]); in particular, we indicate the Hessian of a cost function whose nonzero eigenvalues at an equilibrium may indicate the instability of the equilibrium

(2) We show there is necessarily a topological distinction between the shapes associated with an incorrect and a correct equilibrium
(3) We show that any incorrect equilibrium which is rectangular must correspond to a correct equilibrium which is rectangular (the case considered in Krick et al. [2009]) and the incorrect equilibrium is necessarily unstable

The paper is organized as follows. Section 2, largely review, records the equations of motion for the four-agent formation shape control problem and examines the example from Krick et al. [2009]. We provide an easily checkable condition for local instability of an equilibrium shape and consider the general class of rectangular desired formations. In Section 3, we analyze the stability condition in some detail, and conclude the existence of a topological difference between the shapes associated with an incorrect and a correct equilibrium. Section 4 shows that the rectangular formation of Krick et al. [2009] cannot have a stable incorrect equilibrium and Section 5 offers concluding remarks and future research considerations.

2. EQUATIONS OF MOTION AND EXAMPLES

In this section, we present equations of motion for the four-agent formation shape control problem. We then examine the example from Krick et al. [2009] that illustrates existence of an incorrect equilibrium formation shape and show that this shape is an unstable saddle.

2.1 Equations of Motion

Let $p = [p_1, p_2, p_3, p_4]^T \in \Re^8$ be a vector of the four agent positions in the plane. Following Krick et al. [2009], we use a single integrator agent model to describe the motion of each agent $\dot{p}_i = u_i$ where u_i is the control input to be specified. Let $\bar{d} = [\bar{d}_{12}, \bar{d}_{13}, \bar{d}_{14}, \bar{d}_{23}, \bar{d}_{24}, \bar{d}_{34}]^T$ be a vector of desired interagent distances that define the formation shape. Let $d = [d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}]^T$, sometimes written as $d(p)$, denote instantaneous interagent distances, which are to be actively controlled to obtain $\bar{d}$. We assume that the entries of $\bar{d}$ correspond to a realizable shape.

Evidently,

$$d^2(p) = [||p_1 - p_2||^2, ||p_1 - p_3||^2, ||p_1 - p_4||^2, ||p_2 - p_3||^2, ||p_2 - p_4||^2, ||p_3 - p_4||^2]^T \quad (1)$$

We define also the error function

$$e(p) = d^2(p) - \bar{d}^2 = [e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}]^T \quad (2)$$

The desired formation shape is a three-dimensional manifold in $\Re^8$ given by

$$P_d = \{p \in \Re^8 | d^2(p) = \bar{d}^2\} \quad (3)$$

which is non-empty for a realizable $\bar{d}$. Note the following symmetry for any planar formation: two distinct formation orientations both correspond to a correct formation shape: one with given orientation and one that is the reflection in the plane, as illustrated in Figure 1.

Now consider the potential function

$$V(p) = \frac{1}{2}||e(p)||^2 = \frac{1}{2}(e_{12}^2 + e_{13}^2 + e_{14}^2 + e_{23}^2 + e_{24}^2 + e_{34}^2) \quad (4)$$

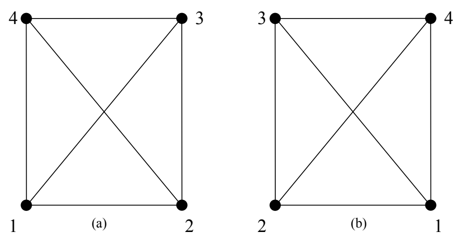

Fig. 1. A desired formation configuration and its reflection in the plane both satisfy the same prescribed interagent distance constraints.

The function quantifies the total interagent distance error between the current formation and the desired formation $\bar{d}$. Note that $V \geq 0$ and $V = 0$ if and only if $e(p) = 0$, that is if and only if the formation is in the desired shape. Thus, V is a suitable potential function from which to derive a gradient control law. Accordingly, let the control input be given by

$$u = -\nabla V(p)^T. \quad (5)$$

Then the closed-loop system is given by

$$\dot{p} = -\nabla V(p)^T = -[J_e(p)]^T e(p) \quad (6)$$

where $J_e(p)$ is the Jacobian of the error function $e(p)$ (also known as the *rigidity matrix*). This can be expressed in the following form

$$\dot{p} = -(E(p) \otimes I_2)p \quad (7)$$

where the matrix $E(p)$ is given by

$$E(p) = \quad (8)$$

$$\begin{bmatrix} e_{12}+e_{13}+e_{14} & -e_{12} & -e_{13} & -e_{14} \\ -e_{12} & e_{12}+e_{23}+e_{24} & -e_{23} & -e_{24} \\ -e_{13} & -e_{23} & e_{13}+e_{23}+e_{34} & -e_{34} \\ -e_{14} & -e_{24} & -e_{34} & e_{14}+e_{24}+e_{34} \end{bmatrix}$$

where $\otimes$ is the Kronecker product.

We remark that the analysis above remains identical if we assume that the formation is planar but exists in $\Re^3$, or if it is intrinsically three dimensional, i.e. not coplanar. Of course, the p_i become 3-vectors.

2.2 Equilibrium points

It is a reasonably standard calculation appealing to the LaSalle Principle to establish that from any initial condition, the system tends to an equilibrium in which $e_{ij}(z_i - z_j) = 0$ for all ij pairs. The equilibrium points of the closed-loop system (7) are the same as the critical points of the potential function V. The Jacobian of the right side of (7), which we denote as $J_f(p)$, is the same as the negative of the Hessian of V, which we denote as $H_V(p)$. These are given by

$$H_V(p) = 2J_e(p)^T J_e(p) + E(p) \otimes I_2 = -J_f(p). \quad (9)$$

Therefore, a study of the stability of the equilibrium points of (7) amounts to a study of the nature of the critical points of V: minima are locally stable and maxima and saddle points are locally unstable, and the signs of the eigenvalues of H_V at an equilibrium may indicate the nature of the equilibrium. Note that the stability of an equilibrium point is independent of rotation and translation of the formation shape. In particular, only relative positions matter.

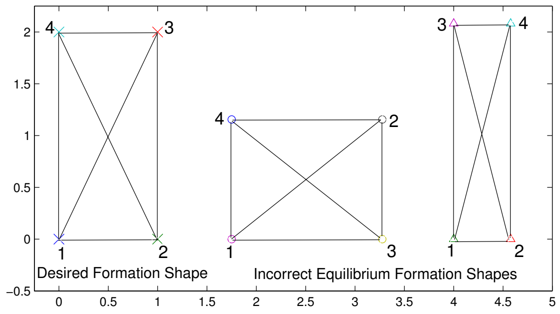

Fig. 2. The desired formation is rectangular and there are two possible "twisted rectangles" that are incorrect equilibrium shapes. Note that each formation has two different pairs of agents on the diagonals of the rectangle: (a) 13 and 24, (b) 12 and 34, and (c) 14 and 23.

2.3 Krick Example

Krick et al. [2009] prove a local asymptotic stability result for an n-agent formation under the gradient control law (5), given that the underlying information architecture is rigid. Since the desired formation shape given by (3) is a three-dimensional equilibrium manifold, the linearized system is non-hyperbolic and they utilize center manifold theory to obtain the result.

The same paper studies a four-agent example with K_4 information architecture where the formation appears to converge to an incorrect equilibrium formation shape from initial states remote from an equilibrium. The example is as follows. Suppose the desired formation is a 1×2 rectangle given by $\bar{d}^2 = [1, 5, 4, 4, 5, 1]^T$. It is easy to verify that when we use the desired distances specified by $\bar{d}$, any formation with distance set $d^{*2} = [\frac{11}{3}, \frac{7}{3}, \frac{4}{3}, \frac{4}{3}, \frac{7}{3}, \frac{11}{3}]^T$ is also an equilibrium with incorrect interagent distances. The incorrect equilibrium is a "twisted rectangle" with the term twisted referring to a change in agent ordering, as illustrated in the middle formation of Figure 2 (it turns out there is a further incorrect equilibrium shape, which we discuss later). Krick et al concluded that the desired shape is not globally attractive since the control law appears to cause convergence to an incorrect equilibrium shape. However, convergence is only apparent, occurring along the ridge of a saddle. The eigenvalues of the negative of the Hessian evaluated at the incorrect shape (i.e. $eig[J_f(p^*)] = eig[-H_V(p^*)]$ where p^* satisfies $d^2(p^*) = d^{*2}$), are $\{0, 0, 0, -22.78, -14.67, -6.56, 1.33, 5.33\}$. Since there are both positive and negative eigenvalues, one can see that the incorrect equilibrium shape is in fact a saddle and is therefore unstable. Much of this paper is concerned with examining this sort of phenomenon.

3. FURTHER PROPERTIES OF INCORRECT EQUILIBRIA

In this section, we shall first study the properties of the matrix $E(p)$ when evaluated at an incorrect equilibrium. We shall note some inequalities relating lengths in a correct and an incorrect equilibrium, and show that there is a topological distinction between a correct and an incorrect equilibrium.

3.1 Incorrect equilibria are to be expected

We noted already that there are two different possible orientations for a correct equilibrium; it is not in general possible to smoothly pass from one to the other while maintaining the equilibrium property. Hence there will be at least one boundary of the domains of attractivity of the two equilibria, Sastry [1999]. Such a boundary normally is an invariant set, on which one or more equilibria should lie. Such equilibria by definition can at best be saddle points, and certainly not attractive. This argument however does not settle the question of whether attractive but incorrect equilibria are to be expected or not.

3.2 The matrix $E(p^*)$ at an equilibrium p^*

Assume that there is an equilibrium point p^* other than the correct equilibrium. Here, $p^* = [p_1^{*T}\ p_2^{*T}\ p_3^{*T}\ p_4^{*T}]^T$ with $p_i^* = [x_i^*\ \ y_i^*]^T$. Suppose a correct equilibrium is defined by $\bar{p} = [\bar{p}_1^T\ \bar{p}_2^T\ \bar{p}_3^T\ \bar{p}_4^T]^T$. The six errors e_{ij} are defined by

$$e_{ij}(p^*) = ||p_i^* - p_j^*||^2 - ||\bar{p}_i - \bar{p}_j||^2 \quad (10)$$

and a 4×4 matrix $E(p^*)$ is defined as in (8); evidently $E(p^*)$ is symmetric and rows and columns sum to zero.

Consider (7) at the equilibrium point. Then clearly

$$[E(p^*) \otimes I_2]p^* = 0 \quad (11)$$

which means (together with the zero row sum property) that

$$E(p^*) \begin{bmatrix} 1 & x_1^* & y_1^* \\ 1 & x_2^* & y_2^* \\ 1 & x_3^* & y_3^* \\ 1 & x_4^* & y_4^* \end{bmatrix} = 0 \quad (12)$$

Hence generically $E(p^*)$ has rank 0 or 1. However, since p^* is an incorrect equilibrium point, at least one e_{ij} is nonzero, and thus $E(p^*)$ is not zero, and so must have rank 1.

We now have

Proposition 1. For the problem set-up as defined, suppose p^* is an incorrect equilibrium for the system (7). Then the associated matrix $E(p^*)$ as defined by (8) is negative semidefinite and of rank 1.

The proof is omitted due to space limitations. An extended version of the paper containing the proof can be obtained from the first author.

3.3 A calculation and further interpretation for the matrix $E(p^*)$

We will now give an interpretation for the vector whose outer product with itself is, with scaling, equal to $E(p^*)$.

This interpretation rests on two easy lemmas.

Lemma 2. Consider a triangle defined by vertices p_1, p_2, p_3 which occur in counterclockwise order. Then the area of this triangle given by

$$\Delta_{123} = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \quad (13)$$

Note that if we are prepared to tolerate negative area triangles, then the formula is valid without restriction on the ordering of the vertices, and a negative area triangle corresponds to p_1, p_2, p_3 occurring in clockwise order.

Lemma 3. With x_i^*, y_i^* fixed and a_i unknown for $i = 1, 2, 3, 4$, consider the equation

$$[a_1\ a_2\ a_3\ a_4]\begin{bmatrix} 1 & x_1^* & y_1^* \\ 1 & x_2^* & y_2^* \\ 1 & x_3^* & y_3^* \\ 1 & x_4^* & y_4^* \end{bmatrix} = 0 \tag{14}$$

Then this equation is satisfied by:

$$a_1 = \begin{vmatrix} 1 & x_2^* & y_2^* \\ 1 & x_3^* & y_3^* \\ 1 & x_4^* & y_4^* \end{vmatrix}, a_2 = -\begin{vmatrix} 1 & x_1^* & y_1^* \\ 1 & x_3^* & y_3^* \\ 1 & x_4^* & y_4^* \end{vmatrix}, \tag{15}$$
$$a_3 = \begin{vmatrix} 1 & x_1^* & y_1^* \\ 1 & x_2^* & y_2^* \\ 1 & x_4^* & y_4^* \end{vmatrix}, a_4 = -\begin{vmatrix} 1 & x_1^* & y_1^* \\ 1 & x_2^* & y_2^* \\ 1 & x_3^* & y_3^* \end{vmatrix}$$

Proof: Use Cramer's rule.

Assume that the vertices with coordinates $p_1^*, p_2^*, p_3^*, p_4^*$ occur as corners of a quadrilateral in counterclockwise order, and no one of the p_i^* is in the convex hull of the others. Then the 3×4 matrix in the definition of the a_i has columns spanning the nullspace of $E(p^*)$ and accordingly, $E(p^*)$ will be of the form $-\mu aa^T$, where $a = [a_1\ a_2\ a_3\ a_4]$ and μ is a positive scaling constant. We have just shown that the a_i can be related to certain triangles formed within the quadrilateral:

$$[a_1\ a_2\ a_3\ a_4] = [\Delta_{234}\ (-\Delta_{134})\ \Delta_{124}\ (-\Delta_{123})] \tag{16}$$

Other arrangements for the p_i^* will doubtless lead to other sign patterns for the a_i. The two positive terms are the areas of two triangles formed from the quadrilateral using one diagonal. The two negative terms are the areas of the other two triangles formed using the second diagonal. The sum of the a_i is therefore zero, as required by the defining equation.

The observation of this lemma leads to an interesting and simply obtained corollary, flowing from the equality of triangle areas when the equilibrium is a parallelogram:

Corollary 4. Suppose that an incorrect equilibrium is a parallelogram. Then the associated matrix $E(p^*)$ is of the form

$$E = e_{12}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \tag{17}$$

3.4 Length inequalities

In this subsection, we recall some inequalities between lengths in a correct and an incorrect equilibrium, which are derivable by considering the equilibrium conditions pertaining at each vertex. While obtained slightly tediously in Summers et al. [2009], the same inequalities however are also an immediate consequence of expressing $E(p^*)$ as $-\mu aa^T$ with the entries of a identified with the signed areas of certain triangles.

Lemma 5. Suppose there is an incorrect equilibrium with quadrilateral convex hull, and with agent pairs 13, 24 diagonally opposite. Let $\bar{d}_{ij}$ and d_{ij}^* denote the distance sets of the correct and incorrect equilibrium respectively. Then there holds

$$d_{12}^* < \bar{d}_{12}, d_{34}^* < \bar{d}_{34}, \tag{18}$$
$$d_{13}^* > \bar{d}_{13}, d_{14}^* > \bar{d}_{14}, d_{23}^* > \bar{d}_{23}, d_{24}^* > \bar{d}_{24}.$$

A corresponding lemma applies for formations with triangular convex hull, and the same derivation technique can be used:

Lemma 6. Suppose there is an incorrect equilibrium shape with agent 2 lying in the convex hull of agents 1,3 and 4. Let $\bar{d}_{ij}$ and d_{ij}^* denote the distance sets of the correct and incorrect equilibrium respectively. Then there holds

$$d_{12}^* > \bar{d}_{12}, d_{23}^* > \bar{d}_{23}, d_{24}^* > \bar{d}_{24}, \tag{19}$$
$$d_{13}^* < \bar{d}_{13}, d_{14}^* < \bar{d}_{14}, d_{34}^* < \bar{d}_{34}.$$

3.5 A twisting property for incorrect equilibria

In the second section where we described the rectangular formation example of Krick et al. [2009], we noted that the incorrect and correct equilibria were twisted. We now consider this notion more formally, and without restriction to rectangular formations.

Consider an equilibrium where the convex hull of the vertices is a quadrilateral (as opposed to a triangle; lines are excluded as being nongeneric). Suppose that the vertices in clockwise or counterclockwise order are 1,2,3,4 or 4,3,2,1 or a cyclic permutation of one of these possibilities. There are eight permutations covered by these possibilities. We say that a second equilibrium with quadrilateral convex hull is twisted (in relation to the first) if the ordering of its vertices does not correspond to one of these eight possibilities. To smoothly change between these two equilibria requires at some stage three of the agents to become collinear.

Observe that there are precisely two distinct twists of a quadrilateral, defined by 1,3,2,4 with cyclic permutation and reversal, and 1,2,4,3, again with cyclic permutation and reversal. Each corresponds to an equivalence class of eight different possible permutations.

We shall prove the following:

Theorem 7. Consider a four agent formation with a correct and an incorrect equilibrium which are both convex quadrilaterals. Then the two equilibria must be twisted.

To prove the theorem, we will appeal to a result by Connelly [2009]. We will first explain this result. Consider a framework in which some bars are replaced by either a *cable* or a *strut*. A cable (strut) enforces the constraint that the two agents, call them i and j, at each end can be no further apart (closer) than a nominated distance $\bar{d}_{ij}$, but may be closer (further apart). Such a framework is called a *tensegrity*. A tensegrity is rigid if any continuous motion of the framework satisfying the distance constraints preserves the shape, i.e. leads to a framework congruent with the original. A tensegrity framework is termed globally rigid if any other configuration satisfying the same constraints is congruent to the first. Connelly develops results establishing that certain tensegrity frameworks are globally rigid. To understand these results, we need to understand the concept of a *stress vector*, a *proper stress vector* and a *stress matrix*.

The concept of a stress vector applies to both an ordinary framework or a tensegrity framework. Let J_e be the rigidity matrix of a framework; any nonzero vector in the left kernel is a stress vector. Thus for a framework which is K_4, there is, up to multiplication by a nonzero scalar, a single stress vector. The concept of a proper stress vector applies to a tensegrity framework; note first that the ordering of the entries in a stress vector is associated with an ordering of the edges in the graph, viz. that used in defining the rigidity matrix. Thus each entry of the stress vector is associated with an edge of the graph. A proper stress vector is one where each graph edge which is a cable has a nonnegative entry and each graph edge which is a strut has a nonpositive entry. Let ω_{ij} denote the entry of a proper stress vector associated with edge ij. The stress matrix is defined by

$$\begin{aligned}\Omega_{ij} &= -\omega_{ij}, i \neq j\\ \Omega_{ii} &= \sum_{j\neq i}\omega_{ij}\end{aligned}$$

The key result is:

Theorem 8. Let $G(\bar{p})$ be a tensegrity, with the affine span of the vertex coordinates being R^2. Suppose that the rigidity matrix has a nontrivial proper stress vector ω and there is an associated stress matrix Ω. Suppose further that

(1) The matrix Ω is nonnegative definite
(2) The rank of Ω is $m-3$, there being m agents
(3) If the cables and struts in the tensegrity structure are replaced by bars, the resulting structure is rigid

Then $G(\bar{p})$ is globally rigid.

A particular example of a tensegrity structure is provided by a K_4 formation with convex hull a quadrilateral. Assume the ordering of the vertices is 1,2,3,4. The edges 12,23,34,41 are all cables, while the edges 13 and 24 are struts. It is nontrivial that the requirement of the theorem that a *proper* stress vector exist can be fulfilled. It is also nontrivial, but provable in the work of Connelly, that the first condition is satisfied. It is obvious that the second and third conditions of the theorem are satisfied.

In particular, it follows from the theorem that if there exists a second framework with edge lengths δ_{ij} and satisfying

$$\begin{aligned}&\delta_{12} \le \bar{d}_{12}, \delta_{23} \le \bar{d}_{23}, \delta_{34} \le \bar{d}_{34},\\ &\delta_{41} \le \bar{d}_{41}, \delta_{24} \ge \bar{d}_{24}, \delta_{13} \ge \bar{d}_{13},\end{aligned} \tag{20}$$

then necessarily, equality holds in every equation.

Proof of Theorem 7. Consider an incorrect equilibrium of a K_4 formation, with the four agents having a quadrilateral convex hull, with ordering 1,2,3,4. The interagent distances for the correct and incorrect equilibrium are $\bar{d}_{ij}$ and d^*_{ij}. Then (18) holds.

Now suppose that the correct equilibrium is a quadrilateral with the same vertex ordering as the incorrect equilibrium. Then by the Connolly result, identifying the d^*_{ij} of the incorrect equilibrium with the δ_{ij} in the Connolly result, we see that (18) imply the equations (20). But because K_4 is a tensegrity structure, this means $d^*_{ij} = d_{ij}$ for all ij, i.e. the incorrect equilibrium is actually a correct equilibrium. In other words, given a correct equilibrium with quadrilateral convex hull, there can be no incorrect equilibrium with quadrilateral convex hull with the same vertex ordering, and the theorem is proved.

A similar argument handles the case where the formation has a triangular convex hull.

4. RECTANGULAR FORMATIONS

In this section, we consider rectangular formations. In particular, we show that if an incorrect equilibrium formation is rectangular, the associated correct equilibrium formation must also be rectangular; that a rectangular correct equilibrium formation has two different associated incorrect rectangular equilibria (analytic formulas being presented); and that the incorrect equilibria are necessarily unstable (saddle points to be more precise).

Suppose that the incorrect equilibrium is defined by a rectangle with vertices 1,2,3,4 in counterclockwise order. In particular then,

$$d^*_{12} = d^*_{34}, d^*_{23} = d^*_{14}, d^*_{13} = d^*_{24} \tag{21}$$

Recall also the form of the matrix $E(p^*)$ in (17). The off diagonal entries are the quantities $-e_{ij}$. Evidently, $e_{12} = e_{34}$ and so $\bar{d}_{12} = \bar{d}_{34}$. Likewise, $e_{23} = e_{14}$ and $e_{13} = e_{24}$, which yield $\bar{d}_{23} = \bar{d}_{14}$ and $\bar{d}_{13} = \bar{d}_{24}$ respectively. It is not hard to check that, noting that for the correct equilibrium, the vertex ordering cannot be the same as that for the incorrect equilibrium, these equality conditions for the correct formation imply again that the associated equilibrium is a rectangle.

Now suppose that in the incorrect equilibrium, $d^*_{12} = a^*, d^*_{23} = b^*$. Consider a correct equilibrium with counterclockwise vertex ordering 1,2,4,3, and with $\bar{d}_{12} = \bar{a}, \bar{d}_{24} = \bar{b}$. Then the identity $e_{12} = e_{14}$ implies $a^{*2} - b^{*2} = -\bar{b}^2$ and the identity $e_{12} = -e_{13}$ implies $2a^{*2} + b^{*2} = \bar{a}^2 + \bar{b}^2$. Immediately, there results

$$a^{*2} = \frac{\bar{a}^2}{3},\ b^{*2} = \frac{\bar{a}^2}{3} + \bar{b}^2 \tag{22}$$

By symmetry, it is evident that the other possible twistedness relationship yields

$$a^{*2} = \bar{a}^2 + \frac{\bar{b}^2}{3},\ b^{*2} = \frac{\bar{b}^2}{3} \tag{23}$$

The example quoted in Section 2 corresponds to the first of these possibilities ($\bar{a}^2 = 4, \bar{b}^2 = 1, a^2 = 4/3, b^{*2} = 7/3$).

To examine the stability of the incorrect equilibrium, suppose that the rectangle has a side of length a on the x-axis. Reorder the rows and columns of the Hessian of $V(p)$ given in (9) so that the first four rows and columns are associated with x-coordinates and the second four rows and column s are associated with y-coordinates. Make a corresponding re-ordering of the columns of the rigidity matrix, so that the first four columns correspond to the x-coordinates of p_1, p_2, p_3, p_4. Then in obvious notation, the four by four subblock is

$$H_x(p^*) = 2J_{ex}(p^*)^T J_{ex}(p^*) + E(p^*) \tag{24}$$

It is straightforward to verify that if $w = [1 \ -\beta \ \beta \ 0]^T$, then

$$w^T[2J_{ex}(p^*)^T J_{ex}(p^*) + E(p^*)]w = 1 - 4\beta \tag{25}$$

so that if $\beta > a/4$, a negative value results. This shows that the equilibrium is not stable. In fact, it is a saddle point.

Arguing along similar lines, one can show that if the incorrect equilibrium is a parallelogram, then the correct equilibrium is either a parallelogram or an isosceles trapezoid. We have not yet verified whether the incorrect equilibrium is or is not stable. Nor can we show that if the incorrect equilibrium has an *arbitrary* quadrilateral or triangle as its convex hull, then that equilibrium is not stable. Actually, if such a formation is considered as lying in R^3, a long argument will demonstrate a saddle point property.

Several more gaps in our understanding can be noted. In relation to a correct rectangular equilibrium, is it necessarily the case that any incorrect equilibrium is also rectangular? And in relation to any generic formation, how many incorrect equilibria are there, discounting the issue of their stability or otherwise?

5. CONCLUSIONS AND FUTURE WORK

Our ultimate goal is to show how to control an arbitrary formation to a shape that is uniquely specified, i.e. specified up to congruence, by a sufficiently large number of distance constraints (in effect, the associated graph must be what is known as *globally rigid*). A K_4 graph is the second simplest such graph, after the triangle, and one might reasonably expect that a general theory would need to properly encompass this special case. It is remarkable that the K_4 graph presents such a challenging task of global stability analysis. The most significant open problem is to show whether or not there can ever exist an incorrect attractive equilibrium. Other problems include counting the number of incorrect equilibria, and, perhaps less importantly, showing that correct equilibria with convex hull of a certain shape, e.g. rectangle, parallelogram, equilateral or isosceles triangle, imply some similar kind of restriction on the shape of the incorrect equilibrium.

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