ON THE RATE OF ERROR PROPAGATION IN MULTIHOP RANGE-BASED LOCALIZATION

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ABSTRACT

Error propagation is a greatly complicated problem arising in multihop sensor localization. In this paper, we focus on how certain key factors in a sensor network affect error propagation for a restricted range-based localization scenario and obtain the significant conclusion that localization errors measured by the Mean Squared Error are propagated at the rate of the cube of the minimal hop count to anchors. A simulation analysis based on actual localization processes and the Cramér-Rao Lower Bound verifies this result.

Index Terms— Range-based localization, Nonlinear estimation, Error analysis, Cramér-Rao Lower Bound.

1. INTRODUCTION

Range-based sensor localization is the problem of identifying the locations of sensor nodes, or simply sensors, given estimates of the distances between them, known as range measurements. The basic range-based localization algorithms include trilateration which in 2-dimensional (2D) space employs at least three range measurements from non-collinear nodes at known locations, termed anchors, to localize a sensor. Due to the existence of noises in range measurements, only location estimates as opposed to exact positions can be derived. If not every sensor can measure its distances to sufficient anchors, already localized sensors must be used as pseudo-anchors to help their neighboring sensors become localized; this process is called multihop sensor localization. As a result, localization errors of pseudo-anchors propagate into localization results of later localized sensors, a phenomenon which is called error propagation.

In the literature, discussions in [1, 2] have raised concern about error propagation. In particular, error propagation in 1-dimensional sensor networks was examined based on the Cramér-Rao Lower Bound (CRLB) in [3, 4]. This paper deals with error propagation in multihop range-based localization for a simple 2D sensor network. Relying on certain approximating techniques, we conclude that localization errors measured by the Mean Squared Error (MSE) are propagated at the rate of the cubic minimal hop count to anchors, a fact which is verified by extensive simulations based on both actual localization processes and the CRLB. To the best of our knowledge, this is the first study addressing the challenge of developing this style of characterization of error propagation.

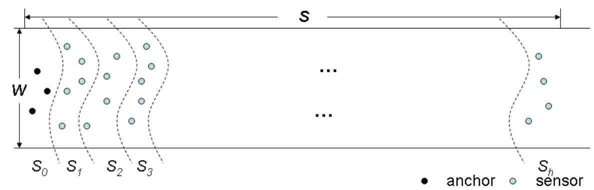

Fig. 1. A simple sensor network.

The next section establishes the problem model. Section 3 presents the result about the rate of error propagation. Section 4 provides simulations and Section 5 concludes the paper.

2. PROBLEM SETUP

Error propagation is determined by multiple factors, such as measurement noises, node densities and distributions, localization algorithms, etc. In order to disentangle these various factors, we abstract a simple localization scenario as shown in Fig. 1: sensors are deployed in a long rectangular ($w \ll s$) area and classified into h sets: $S_1, S_2, ..., S_h$, each of which, say S_i, comprises the sensors from which the minimal hop counts to anchors are i; anchors, denoted S_0, are confined to the leftmost area, which is essential to our analysis by enabling us to run an iterative localization process; the localization process involves h steps: in the i-th ($i \geq 1$) step, sensors in S_i are localized based on the range measurements from and positions of nodes in S_{i-1}, termed reference nodes. The analysis based on the strip-like scenario is indicative of error propagation in general sensor networks.

Aside from the geometric layout and the localization order, throughout the paper we assume that: *(1) the sensor distribution is generated by a homogeneous Poisson point process (HPPP) with intensity λ; (2) the ranging model is the unit disk model; (3) range errors are independent additive Gaussian with mean zero and σ as standard deviation.*

We shall use the following mathematical notation in this paper: $||\cdot||$, Euclidean norm; $|\cdot|$, dimension or cardinality; $Tr(\cdot)$, trace; $E(\cdot)$, expected value; $C(\cdot)$, covariance; $V(\cdot)$, sample variance; $M(\cdot)$, sample mean; $(\cdot)^T$, matrix transpose; I_m, $m \times m$ Identity matrix.

3. THE RATE OF ERROR PROPAGATION

In this section, we first formulate localization errors involved in one localization step and then present the propagation result through certain approximations. The calculations performed are not those performed in a localization algorithm, but analytical calculations for the purpose of obtaining conclusions on the performance of a localization algorithm.

Let θ_i ($i \geq 0$) be the variable vector representing positions of nodes in S_i and $\tilde{\theta}_i, \theta_i^0$ be their estimated and true positions. In the established localization scenario, the i-th ($i \geq 1$) step aims to localize sensors in S_i given the location vector $\tilde{\theta}_{i-1}$ of reference nodes in S_{i-1} and the associated noisy range measurement vector, denoted $\tilde{\gamma}_i$. Deriving $\tilde{\theta}_{i-1}$ only draws on $\tilde{\gamma}_1$, ..., $\tilde{\gamma}_{i-1}$ and hence, $\tilde{\theta}_{i-1}$ is independent of $\tilde{\gamma}_i$,..., $\tilde{\gamma}_h$.

Define the function vector $\gamma_i(\theta_i, \theta_{i-1})$ to calculate the Euclidean distances if corresponding range measurements exist in $\tilde{\gamma}_i$. The problem of simultaneously localizing all sensors in S_i can be formulated as a Least Squares (LS) problem

$$\min_{\theta_i} ||\tilde{\gamma}_i - \gamma_i(\theta_i, \tilde{\theta}_{i-1})||^2 \quad (1)$$

If $\tilde{\theta}_{i-1}$ and $\tilde{\gamma}_i$ are error free, the solution to (1), namely $\tilde{\theta}_i$, will be θ_i^0. If the errors are small, $\tilde{\theta}_i$ will be near θ_i^0. To understand how one error gives rise to another, we apply first-order Taylor Expansions on $\gamma_i(\theta_i, \theta_{i-1})$ around $\gamma_i^0 = \gamma_i(\theta_i^0, \theta_{i-1}^0)$:

$$\gamma_i(\theta_i, \theta_{i-1}) \approx \gamma_i^0 + J_i(\theta_i - \theta_i^0) + K_i(\theta_{i-1} - \theta_{i-1}^0) \quad (2)$$

where $J_i = \frac{\partial \gamma_i}{\partial \theta_i}(\theta_i^0, \theta_{i-1}^0)$ and $K_i = \frac{\partial \gamma_i}{\partial \theta_{i-1}}(\theta_i^0, \theta_{i-1}^0)$. Then, the LS problem (1) can be approximately formulated as

$$\min_{\triangle\theta_i} ||\tilde{\gamma}_i - \gamma_i^0 - J_i(\theta_i - \theta_i^0) - K_i(\tilde{\theta}_{i-1} - \theta_{i-1}^0)||^2 \quad (3)$$

and because $\tilde{\gamma}_i$ is independent of $\tilde{\theta}_{i-1}$, the solution to (3) is

$$\tilde{\theta}_i = \theta_i^0 + (J_i^T J_i)^{-1} J_i^T (\tilde{\gamma}_i - \gamma_i^0 - K_i(\tilde{\theta}_{i-1} - \theta_{i-1}^0)) \quad (4)$$

Define $\triangle\gamma_i = \tilde{\gamma}_i - \gamma_i^0$ and $\triangle\theta_i = \tilde{\theta}_i - \theta_i^0$. Then,

$$\triangle\theta_i = (J_i^T J_i)^{-1} J_i^T (\triangle\gamma_i - K_i \triangle\theta_{i-1}) \quad (5)$$

which reflects the relations of reference location errors $\triangle\theta_{i-1}$, range errors $\triangle\gamma_i$ and localization errors $\triangle\theta_i$. Due to $C(\triangle\gamma_i) = \sigma^2 I_{|\triangle\gamma_i|}$ and the independence between $\triangle\gamma_i$ and $\triangle\theta_{i-1}$, we can obtain

$$C(\triangle\theta_i) = \sigma^2 (J_i^T J_i)^{-1} + (J_i^T J_i)^{-1} J_i^T K_i C(\triangle\theta_{i-1}) K_i^T J_i (J_i^T J_i)^{-1} \quad (6)$$

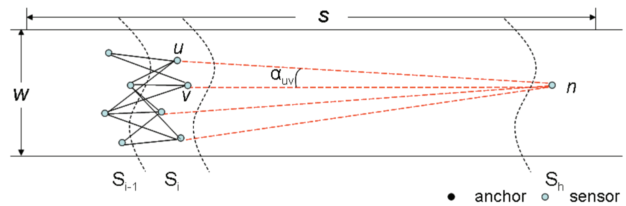

Fig. 2. A virtual localization relevant to V_i. Red dashed lines are precise distances and black lines are noisy distances.

If reference nodes are anchors, i.e. $\triangle\theta_{i-1} = \mathbf{0}$, we obtain

$$C(\triangle\theta_i) = \sigma^2 (J_i^T J_i)^{-1} \quad (7)$$

If range errors were zero, i.e. $\triangle\gamma_i = \mathbf{0}$, we would obtain

$$C(\triangle\theta_i) = (J_i^T J_i)^{-1} J_i^T K_i C(\triangle\theta_{i-1}) K_i^T J_i (J_i^T J_i)^{-1} \quad (8)$$

Based on these results, we can recursively calculate the covariance matrix for each sensor set, and in particular, Theorem 1 provides the covariance matrix for S_h.

Theorem 1 *In the localization scenario, the covariance matrix of localization errors of sensors in S_h, denoted Q_h, is*

$$Q_h = Q_h^1 + Q_h^2 + \cdots + Q_h^h \quad (9)$$

where $Q_h^i = L_i^T C(\triangle\gamma_i) L_i$ *(*$i = 1, 2, ..., h$*) and* $L_i = J_i (J_i^T J_i)^{-1} \prod_{j=i+1}^{h} (K_j^T J_j (J_j^T J_j)^{-1})$.

We make a preliminary observation concerning Q_h: suppose all range errors are 0 except $\triangle\gamma_i$; according to Theorem 1, all the terms on the right hand side of (9) vanish except Q_h^i, which means that Q_h^i is the covariance matrix for S_h produced by a *virtual localization problem* in the sense that all range measurements are precise except those relevant to $\triangle\gamma_i$.

In the virtual localization problem relevant to Q_h^i, sensors in $S_1, S_2, ..., S_{i-1}$ can be precisely localized, whereas locations of sensors in $S_i, S_{i+1}, ..., S_h$ have to be estimated. Similar to Q_h, accurately evaluating Q_h^i requires recursive calculations. Because range measurements are precise except those between sensors in S_{i-1} and S_i, it is possible to calculate the precise distances between sensors in S_i and sensors in S_h before the localization. Given these precise distances and location estimates of sensors in S_i, sensors in S_h can be directly localized and the resulting covariance matrix can be used to approximate Q_h^i. *Without loss of generality, we assume* $S_h = \{n\}$ *and* $S_i = \{s_1, s_2, ..., s_{q_i}\}$. Let $\triangle\theta'$ be localization errors of sensor n. Supposing the desired precise distances and the localization errors of sensors in S_i ($i < h$), denoted $\triangle\phi'$, are available, according to (5) we can obtain

$$\triangle\theta' = -(J'^T J')^{-1} J'^T K' \triangle\phi' \quad (10)$$

where J' and K' are Jacobians and $K'K'^T = I_{q_i}$. Due to precise reference locations, $C(\triangle\phi')$ is a block (2×2) diagonal matrix. For analytical convenience, we assume $C(\triangle\phi') = \delta_i^2 I_{2q_i}$ where $2\delta_i^2$ is the average MSE of localization errors of sensors in S_i and particularly, $Q_h^h = \delta_h^2 I_2$. Then we can obtain

$$C(\triangle\theta') = \delta_i^2 (J'^T J')^{-1} \tag{11}$$

Using $C(\triangle\theta')$ to approximate Q_h^i $(i < h)$, we can obtain

$$Tr(Q_h^i) \approx \frac{q_i \delta_i^2}{\sum\limits_{(u<v)\wedge(u,v\in S_i)} \sin^2 \alpha_{uv}} \tag{12}$$

where α_{uv} is the angle formed by u, v and n as shown in Fig. 2. If all the angles are close to 0 or π, the denominator is close to 0 so that $Tr(Q_h^i)$ is extremely large. If sensor n is far away from sensors in S_i, namely $h \gg 1$ and $i \ll h$, because w is small, all the angles are quite small; if sensor n is near sensors in S_i, the chance of having all the angles close to 0 or π is very low. Thus, if $h \gg 1$, $Tr(Q_h^i)$ with small i is far greater than that with large i. Since $Tr(Q_h) = \sum_{i=1}^{h} Tr(Q_h^i)$, the terms with small i dominate $Tr(Q_h)$.

If $i \ll h$, distances between sensor n and sensors in S_i are approximately equal, denoted l_i. We can derive

$$\sin \alpha_{uv} \approx \frac{|y_u - y_v|}{l_i} \tag{13}$$

Amorphous [5], a range-free localization algorithm, approximates the distance between two nodes by the minimal hop count between them multiplying the average hop size. Establishing a Cartesian coordinate system in a sensor network with randomly distributed sensors and one of them at the origin, let $\bar{x}$ be the maximal x-coordinate of 1-hop neighbors of this sensor. The average hop size is then $E(\bar{x})$, which can be calculated given a priori knowledge (i.e. sensor density λ). Let $l_i \approx E(\bar{x}) \times (h - i)$ and then

$$Tr(Q_h^i) \approx \frac{q_i \delta_i^2 (h-i)^2 [E(\bar{x})]^2}{\sum\limits_{(u<v)\wedge(u,v\in S_i)} (y_u - y_v)^2} \tag{14}$$

Observe that in (14) all the variables except h and i are random. Taking expectations on both sides of (14), we obtain

$$E(Tr(Q_h^i)) \approx E(\frac{q_i \delta_i^2 [E(\bar{x})]^2}{\sum\limits_{(u<v)\wedge(u,v\in S_i)} (y_u - y_v)^2})(h-i)^2 \tag{15}$$

In the localization scenario, $h \approx \frac{s}{E(\bar{x})}$ and the area covered by each sensor set can be approximated by dividing the total area by h, i.e. $wE(\bar{x})$. Hence, q_i, the number of sensors in S_i, can be regarded as a Poisson random variable (RV) with intensity $wE(\bar{x})\lambda$. Due to the distribution of nodes produced by a HPPP, the y-coordinates of sensors in S_i are equivalently independent uniform RVs and their probability density functions (pdf) depend on w. Moreover, δ_i^2 actually calculates the average MSE for 1-hop neighbors of anchors or sensors with precise locations, and its pdf has little relevance to i. In conclusion, the expectation term multiplying $(h-i)^2$ in (15) is a constant independent of i, call it c. Summing up (15) with $i = 1, 2, ..., h-1$ and $E(Tr(Q_h^h))$, we can obtain

$$E(Tr(Q_h)) \approx c \times \frac{(h-1)h(2h-1)}{6} + 2E(\delta_h^2) \tag{16}$$

When h is small, however, (15) is a poor approximation and so is (16); as h increases, the approximations of $Tr(Q_h^i)$ with small i (which dominate $Tr(Q_h)$) are close to their true values so that (16) becomes better; when h reaches a certain large value, because $\triangle\theta_i$ and $\triangle\theta_{i-1}$ in (2) can be so large that (2) collapses due to first-order Taylor Expansions, (16) becomes poor again. However, results concerning large localization errors or h are useless for practical applications.

Even though we do not explain here how c and $E(\delta_h^2)$ are precisely calculated, (16) describes the functional form of the relation between the localization error of a sensor (i.e. $E(Tr(Q_h))$) and its minimal hop count to anchors (i.e. h) and shows that errors are propagated at the rate of $\Theta(h^3)$.

4. SIMULATIONS

In this section, we verify the result by comparing localization errors from (16), the actual localization processes based on the Maximum Likelihood Estimator (MLE) and the CRLB. In the simulations: (1) each network instance is deployed in a 2×30 rectangle; (2) nodes are generated by a HPPP with intensity $\lambda = 6$ which is sufficiently high that the probability of having unlocalizable nodes is very small; the CRLB only exists if the whole network is localizable; (3) nodes in the leftmost 2×1 rectangle are anchors and the others are sensors; (4) $\sigma = 0.001, 0.01, 0.05$ are considered and for each σ, 1000 network instances are generated; (5) a node can measure its distance to any node as long as their distance is within 1.

Localizing one sensor via trilateration can be formulated as a non-linear LS problem using the MLE and solved by a Newton-Gauss iteration which requires an initial guess to start the iteration. Unlike the established localization scenario, the simulations use every possible range measurement for localizing each sensor. In the simulation of localizing a sensor, we randomly select 5 points as initial guesses from the square with the center at the true location of this sensor and the side length 0.2, and the solution minimizing the objective function of the problem formulation is chosen as the final solution.

We let $2E(\delta_h^2)$ equal the average MSE of 1-hop neighbors of anchors derived after the simulations. Given λ, w and s, we define a constant $\tilde{c}$ to approximate c:

$$\tilde{c} = \frac{E(\delta_i^2)[E(\bar{x})]^2}{E(\frac{1}{q_i} \sum\limits_{(u<v)\wedge(u,v\in S_i)} (y_u - y_v)^2)} \tag{17}$$

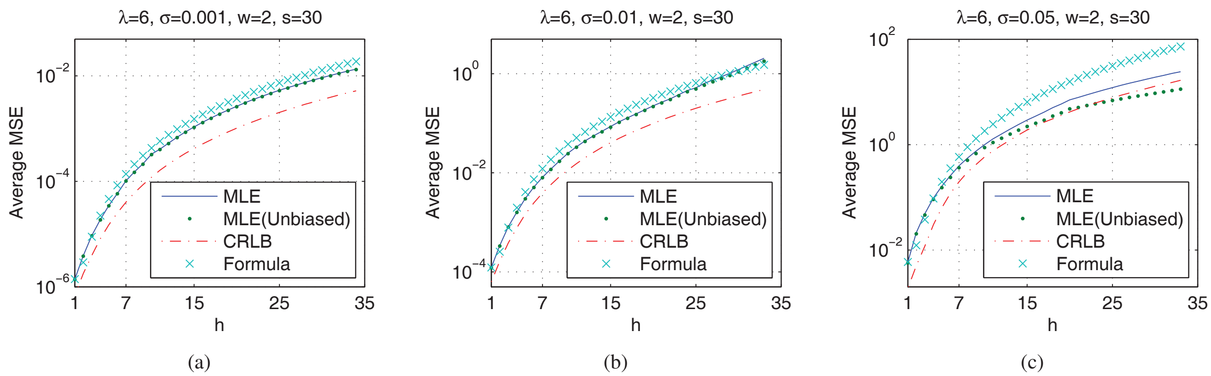

Fig. 3. Simulations with different noise levels: (a) $\sigma = 0.001$, (b) $\sigma = 0.01$ and (c) $\sigma = 0.05$.

Let $E(\delta_i^2) = E(\delta_h^2)$ and according to [5] $E(\bar{x})$ is around 0.8 when $\lambda = 6$. Because of the independence between the RVs representing the y-coordinates, we can obtain

$$E(\frac{1}{q_i}\sum_{(u<v)\wedge(u,v\in S_i)}(y_u - y_v)^2) = \frac{\lambda E(\bar{x})w - 1}{12}w^2 \quad (18)$$

The localization of each network instance is simulated 100 times with random range errors. Let e_x and e_y be vectors containing localization errors in x- and y-coordinates for one sensor. For this sensor, the unbiased MSE is $V(e_x) + V(e_y)$, the MSE is $V(e_x) + V(e_y) + (M(e_x))^2 + (M(e_y))^2$ and the MSE from the CRLB is the sum of the CRLBs on its x- and y-coordinates. Finally, for sensors with the same minimal hop counts to anchors in all network instances, the average MSE is calculated in terms of the MLE and the CRLB respectively.

Since $E(Tr(Q_h))$ in (16) represents the expected MSE for a sensor being h hops to nearest anchors, we compare it with the average unbiased MSE, the average MSE from the MLE and the average MSE from the CRLB in Fig. 3. Regardless of σ, the overall growth trends of curves of the formula, namely (16), and the CRLB are almost identical, and are quite similar to those of the MLE, which shows that errors are propagated at the same rate, namely $\Theta(h^3)$; when the average MSE from the MLE rises up to around 1, for example $\sigma = 0.01$ and $h > 25$ or $\sigma = 0.05$ and $h > 10$, the formula cannot exactly predict the trend of error propagation as well as when the MSE is less than 1, which verifies that the formula is not suitable for large localization errors; when h is small, say less than 7, the curves of the formula grow faster than those of the MLE and the CRLB, which is consistent with our statement; finally, the bias produced by the MLE can be observed (the gap between the two curves of the MLE). However, the bias does not affect our result.

5. CONCLUSIONS

In this paper, we obtained an analytic formula for error propagation in a simple localization scenario using a series of approximations and concluded that errors are propagated at the rate of the cube of the minimal hop count to anchors, a fact which was verified through extensive simulations. However, more effort is required to work out the details of the formula.

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