Localization Bias Correction in n-Dimensional Space

Yiming Ji, Changbin Yu, Brian D.O. Anderson
Research School of Information Sciences and Engineering
The Australian National University
Canberra ACT 2601, Australia
{yiming.ji, brad.yu, brian.anderson}@anu.edu.au

***Abstract*—In previous work we proposed a method to determine the bias in localization algorithms using 2 or 3 sensors, whose location have been already identified, for targets in 2-dimensional space by mixing Taylor series and Jacobian matrices. In this paper we extend the bias-correction method to n-dimensional space with N sensors. To illustrate this approach, we analyze the proposed method in three situations using localization algorithms. Monte Carlo simulation results demonstrate the proposed bias-correction method can correct the bias very well in most situations.**

***Keywords:* Localization; Bias correction; Sensor network.**

I. Introduction

Localization - determining the geographical localizations of targets - is a fundamental problem in many different application areas. Most existing localization algorithms cannot obtain the position of a target exactly; further, many can lead to biased estimates. In order to enhance the accuracy of localization algorithms, many techniques have been presented recently.

In almost all practical situations, measurement errors are inevitable, and these lead to errors in estimating the true target position. One common systematic contributor to localization errors is bias, and it is generally considered desirable to correct for it, if possible. Determination of bias has therefore attracted interest. In [1], Gavish et al. proposed an approach to analytically express the bias in localization algorithms based on bearing measurements. To obtain the analytical expression for the bias, they expand the first derivative of the maximum likelihood cost function by Taylor series. Three expansions of different orders were obtained separately. The final analytical expression for the bias involves the variance of the measurement noise and the derivatives of the cost function. In addition, though formulated for bearing algorithms, the analytical expression for the bias appears generic, which means it is independent of the localization algorithms or types of measurement.

In [2], S. P. Drake and K. Doğançay presents an introduction to tensor algebra with some application examples in estimation theory. One of the tensor algebra application proposed in the paper treats the bias in nonlinear systems with a noisy observable. They expand the non-linear function which maps measurements to target positions to second order in noisy case using a Taylor series. The expected value of the second order term is considered as the bias. However [2] mainly focus on the application of tensor algebra, rather than bias analysis.

In our previous work [3, 4], we proposed a novel method to correct the bias in localization algorithms. We first use a Taylor series to expand the localization mapping **g** which maps from the measurements to produce position estimates to second-order, and take the expected value of the second-order term as the bias. In this stage the expression of the bias involves the derivatives of the localization mapping which are generally hard to calculate. However the inverse mapping of **g** (call it **f**) which maps the target position to the noiseless measurements can be written down easily according to the geometric relationship between the target and sensors. Therefore we introduce the Jacobian matrix of **f** to calculate the derivatives of **g** in terms of the derivatives of **f**. Compared with the approach presented in [1], the Monte Carlo simulation results demonstrate a clearly better performance for our bias-correction method. However, the analysis of the proposed method is restricted in 2-dimensional space in [3, 4]. In this paper we will extend the bias-correction method to three (n=3) dimensional space with an arbitrary number of sensors.

The rest of the paper is organized as follows. In Section II review of the localization problem and the bias in localization are summarized. We analyze the proposed method in n-dimensional space with N sensors in Section III. The results of Monte Carlo simulations are provided in Section IV. Section V summarizes the paper and comments on future work.

II. Brief Review of Localization and Bias

In this section, a brief review of the localization problem and the bias in localization will be presented. All the analysis will be done in n-dimensional space ($n = 2$ or 3) with $N \geq n$ sensors whose location has been already identified.

A. Brief Review of the Localization

In the noiseless case, the localization problem can be formulated as follows. Suppose there is an emitter or target whose coordinate vector is $\mathbf{x} = (x_1, x_2, ..., x_i)^T$ where x_i denotes the ith-coordinate. Further a set of measurements $\Theta = (\theta_1, \theta_2, ..., \theta_N)^T$ can be obtained from N (generally $N \geq n$) sensors where $\theta_i (i = 1, 2, ..., N)$ denotes the measurement obtained from sensor i. Here the measurements can be of any form, such as distance, angle of arrival or bearing, etc. Now in the noiseless case we have

$$\Theta = \mathbf{f}(\mathbf{x}) \tag{1}$$

where $\mathbf{f} = (f_1, f_2, ..., f_N)^T$ denotes the mapping from the target position to the measurements. The function **f** is assumed (as is reasonable) to be obtained analytically according to the geometry of the target and sensors.

However, in practice measurement errors are inevitable. Therefore the mapping from the target position to the measurements can be described by a nonlinear equation as follows (where we use $\tilde{\Theta} = (\tilde{\theta}_1, \tilde{\theta}_2, ..., \tilde{\theta}_N)$ to denote the noisy measurements):

$$\tilde{\Theta} = \mathbf{f}(\mathbf{x}) + \delta\Theta \tag{2}$$

where $\delta\Theta = (\delta\theta_1, \delta\theta_2, ..., \delta\theta_N)^T$ denotes the measurement noise generally assumed to be zero-mean Gaussian with $N \times N$ covariance matrix $S = \text{diag}(\sigma^2_{\theta_1}, \sigma^2_{\theta_2}, ..., \sigma^2_{\theta_N})$. The covariance

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matrix will not be diagonal when the measurements are TDOA [5].

Generally, when $N \geq n+1$ equation (2) will have no solution in the noisy case. In order to obtain an approximate position estimate, various methods have been proposed such as maximum likelihood, least squares, etc [6, 7]. The main idea of these approaches is similar: convert the localization problem to an optimization problem as follows:

$$\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}} F_{\text{cost-function}}(\mathbf{x}, \tilde{\Theta}) \tag{3}$$

where $\tilde{\mathbf{x}}$ denotes the inaccurate target position estimate. By solving the above equation, which is often computationally difficult, we obtain the estimated position.

B. Brief Review of the Bias in Localization

Bias is a term in estimation theory which is defined as the difference between the expected value of a parameter estimate and the true value of the parameter [8]. In this subsection, a brief review of the bias in localization will be presented.

Assume $\mathbf{g} = (g_1, g_2, ..., g_n)$ denotes the localization mapping from the measurements to the target position estimates. In the noiseless case we have:

$$\mathbf{x} = \mathbf{g}(\Theta) \tag{4}$$

where $\mathbf{x}$ and Θ have the same meaning as above.

As mentioned in last subsection, in practice noise will always exist in the measurements. Therefore in the noisy case we have:

$$\tilde{\mathbf{x}} = \mathbf{g}(\Theta + \delta\Theta) = \mathbf{g}(\tilde{\Theta}) \tag{5}$$

where $\tilde{\mathbf{x}}$, $\tilde{\Theta}$ and $\delta\Theta$ have been defined above. In this paper we assume $\delta\Theta$ is subject to an independent Gaussian distribution with zero mean and known variance σ^2.

Suppose that in estimating a target position, in practice the measurement process is repeated M times. For each measurement set we obtain a estimated position of the target, giving M target position estimates. Generally we can average M target position estimates to obtain a single position which is then considered as the estimated position of the target. As $M \rightarrow \infty$, we would expect the estimate to go to :

$$E[\tilde{x}_i] = E[g_i(\tilde{\Theta})] \tag{6}$$

Now note that if g_i is nonlinear we have:

$$\begin{array}{ccccc} E[\tilde{x}_i] & = & E[g_i(\tilde{\Theta})] & \neq & g_i(E[\tilde{\Theta}]) \\ & = & g_i(\Theta) & = & x_i \end{array}$$

Therefore the bias appears in the estimation process.

$$Bias_{x_i} = E[\tilde{x}_i] - x_i \qquad i = 1, 2, ..., n \tag{7}$$

If computable, the bias can be used to systematically correct any single estimate from any single measurement set. From the above analysis, we can see that once (a) the localization mapping is nonlinear and (b) the measurements are noisy, bias is to be expected. In practice, these two factors are mostly present. The desirability of bias correction is the motivation, and the means to do so is the contribution of this paper.

III. Bias-Correction Method

In this section, we will extend the bias-correction method in localization algorithms proposed in [3, 4] to n-dimensional space with N sensors. The analysis will be done in three situations: $N = n$, $N = n+1$ and $N > n+1$. In principle, we could consider $N \geq n+1$ as one composite case; for simplicity, we consider in detail simply $N = n+1$.

A. N=n Situation

In this situation, the number of sensors equals to the number of obtained measurements. At that time, in the noisy case, we can obtain the position of the target by solving the following equation[1].

$$\tilde{\Theta} = \mathbf{f}(\tilde{\mathbf{x}}) \tag{8}$$

where $\tilde{\Theta} = \Theta + \delta\Theta$ and $\tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$. Here $\mathbf{f}$ can be easily written down analytically from the geometry.

Assume the localization mapping $\mathbf{g}$ is well-defined for each point and there are derivatives of any order of $\mathbf{g}$. Because $N = n$, $\mathbf{g}$ can be considered as the inverse mapping of $\mathbf{f}$. Thus

$$\tilde{\mathbf{x}} = \mathbf{g}(\tilde{\Theta}) \tag{9}$$

To determine the bias consider $x_i = g_i(\tilde{\theta})$. Because the localization mapping $\mathbf{g}$ is well defined, we can expand the function g_i by a Taylor series. Truncating at second order:

$$\begin{aligned} x_i + \delta x_i &= g_i(\tilde{\theta}_1, \tilde{\theta}_2, ..., \tilde{\theta}_N) \\ &= g_i(\theta_1 + \delta\theta_1, \theta_2 + \delta\theta_2, ..., \theta_N + \delta\theta_N) \\ &\approx g_i(\theta_1, \theta_2, ..., \theta_N) + \sum_{j=1}^{N} \frac{\partial g_i}{\partial d_j} \delta\theta_j \\ &\quad + \frac{1}{2!} \sum_{j=1}^{N} \sum_{l=1}^{N} \delta\theta_j \delta\theta_l \frac{\partial^2 g_i}{\partial\theta_j \partial\theta_l} \end{aligned}$$

The approximate bias expression is immediate:

$$E(\delta x_i) = \frac{1}{2!} \sum_{j=1}^{N} \sigma_j^2 \frac{\partial^2 g_i}{\partial\theta_j^2} \tag{10}$$

For range-measurement localization, it is not very difficult to compute the derivatives of $\mathbf{g}$. However, when considering e.g. a scenario involving TDOA in R^3, to obtain the analytical expression of $\mathbf{g}$ becomes very challenging. In contrast, $\mathbf{f}$ can be easily written down according to the geometric relationship between the target and sensors. Therefore we consider how to use $\mathbf{f}$ and its derivatives to calculate the derivatives of $\mathbf{g}$ resulting in an easy calculation of the bias. Because $\mathbf{f}$ and $\mathbf{g}$ are inverse mappings, the Jacobian identity holds:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_n} \\ . & ... & . \\ . & ... & . \\ . & ... & . \\ \frac{\partial f_N}{\partial x_1} & ... & \frac{\partial f_N}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & ... & \frac{\partial g_1}{\partial \theta_N} \\ . & ... & . \\ . & ... & . \\ . & ... & . \\ \frac{\partial g_n}{\partial \theta_1} & ... & \frac{\partial g_n}{\partial \theta_N} \end{bmatrix} = I_n \tag{11}$$

By solving the equation set (11) we can obtain the analytical expression for $\frac{\partial g_i}{\partial \theta_j}(i = 1, 2, ..., n; j = 1, 2, ..., N)$ in terms of $\frac{\partial f_i}{\partial x_j}(i = 1, 2, ..., N; j = 1, 2, ..., n)$. For ease of exposition we use $g^i_{\theta_j}$ to denote the expressions of $\frac{\partial g_i}{\partial \theta_j}$ as functions of $x_1, x_2, ..., x_n$. Here we take $\frac{\partial g_1}{\partial \theta_1}$ for example. We can obtain the following equation.

[1]When the measurement is the range between target and sensor, an ambiguity problem may be encountered: we may obtain two estimated target positions. At that time we assume further information such as a priori area restriction to resolve the ambiguity problem

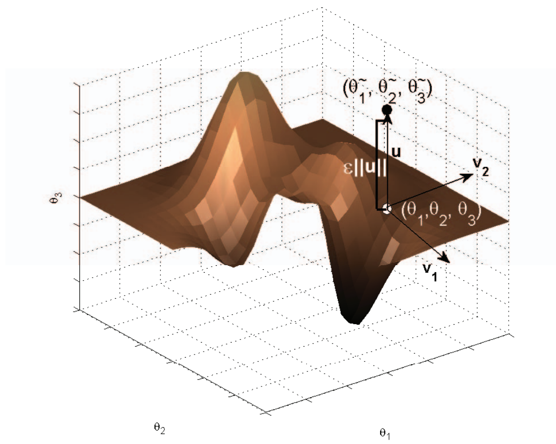

Fig. 1. Introduce One Extra Variable (Here N=3 and n=2)

$$\frac{\partial g_1}{\partial \theta_1} = g^1_{\theta_1} \tag{12}$$

Differentiating the equation (12) in respect to $x_1, x_2, ..., x_n$ respectively we can obtain an equation set as follows.

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_N}{\partial x_1} \\ . & ... & . \\ . & ... & . \\ . & ... & . \\ \frac{\partial f_1}{\partial x_n} & ... & \frac{\partial f_N}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 g_1}{\partial {\theta_1}^2} \\ . \\ . \\ . \\ \frac{\partial^2 g_1}{\partial \theta_1 \partial \theta_N} \end{bmatrix} = \begin{bmatrix} \frac{\partial g^1_{\theta_1}}{\partial x_1} \\ . \\ . \\ . \\ \frac{\partial g^2_{\theta_1}}{\partial x_n} \end{bmatrix} \tag{13}$$

Note that the quantities on the right side of this equation are all expressible analytically in terms of derivatives of the f_i, and so as functions of $x_1, x_2, ..., x_n$. Hence by solving the equation set (13), we can obtain a formula for $\frac{\partial^2 g_1}{\partial \theta_1^2}$ which only contains of the derivatives of f_i. The formulas for $\frac{\partial^2 g_i}{\partial \theta_j^2}$ can be obtained in the same way. Substituting the formulas into equation (10) we can finally obtain the easily-calculated equations for the bias.

B. N=n+1 Situation

One more sensor is introduced in this situation. In the noiseless case, a single well-defined position of the target can be obtained by solving the equation (1). However, in the noisy case generally there will be no solution for the equation (8). Further the equation (8) will become overdetermined which means there are more scalar measurements than there are unknowns. Because $N \neq n$, we cannot obtain the equation (11). In other words we cannot straightforwardly express the bias using the derivatives of **f**. At the same time, to calculate the localization mapping **g** directly becomes even harder, requiring analytical solution of (3). Therefore we adopt a method based on the least squares approach to introduce an extra variable into the equation (8).

Consider N-dimensional space, with axes corresponding to the N measurements. Assume a surface (shown in Fig. 1) consists of points which correspond to all sets of noiseless measurements $(\theta_1, \theta_2, ..., \theta_N)$. According to the least squares method, the cost function in equation (3) has the following form:

$$F_{\text{cost-function}}(\mathbf{x}, \tilde{\Theta}) = \sum_{i=1}^{N} (f_i - \tilde{\theta}_i)^2 = \sum_{i=1}^{N} \delta\theta_i^2 \tag{14}$$

In fact, the least squares method attempts to find a point $(\theta_1, \theta_2, ..., \theta_N)$ (the white point in Fig. 2) on the surface corresponding to an obtained set of noisy measurements $(\tilde{\theta}_1, \tilde{\theta}_2, ..., \tilde{\theta}_N)$ (the black point in Fig. 2 which is generically off the surface) to minimize the distance between the two points. Hence the white point must be the orthogonal projection of the black one on the surface, or the black point must be on the normal vector to the surface passing through the white one. Therefore the distance between the two points can be formulated as follows:

$$D_{\text{min}} = \sqrt{\sum_{i=1}^{N} \delta\theta_i^2} = \varepsilon||\mathbf{u}|| \tag{15}$$

where **u** denotes the normal vector at the white point and ε is a coefficient to set the distance. The normal vector **u** can be calculated as follows.

At the white point we can obtain n tangent vectors as follows:

$$\mathbf{v}_i = [\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, ..., \frac{\partial f_N}{\partial x_i}]^T \quad i = 1, 2, ..., n \tag{16}$$

By cross multiplying the n tangent vectors, we can obtain the normal vector **u** [9].

$$\mathbf{u} = [u_1, u_2, ..., u_N]^T = \mathbf{v}_1 \times \mathbf{v}_2 ... \times \mathbf{v}_n \tag{17}$$

Note that in the noiseless case $\Theta = \mathbf{f}$ where **f** can be written down easily according to the geometry of the sensors and target. Therefore for the black point we can obtain a new analytical mapping $\mathbf{F} = (F_1, F_2, ..., F_N)^T$ by moving from **f** along the normal vector for a minimal distance $\varepsilon||\mathbf{u}||$. The new mapping **F** is no longer overdetermined because an extra variable ε has been introduced into the mapping.

Now we have a new mapping **F**: $R^N \rightarrow R^N$ as follows.

$$\tilde{\Theta} = \mathbf{F}(\tilde{\mathbf{x}}, \varepsilon) = \mathbf{f}(\tilde{\mathbf{x}}) + \varepsilon\mathbf{u} \tag{18}$$

After introducing the extra variable ε, now $N = n$. Therefore we can consider the localization mapping (call it **G**) as the inverse mapping of **F**. We can then proceed along the same lines as previously.

C. N>n+1 Situation

With $N > n+1$, the situation is similar to $N = n+1$ case except that the extra variable ε is no longer a scalar. Instead it is a vector which can be defined as follows.

$$\varepsilon = [e_1, e_2, ..., e_{N-n}]^T \tag{19}$$

where e_i $(i = 1, 2, ..., N-n)$ denotes a coefficient to minimize the moved distance in each dimension of the normal.

The other processes are as same as the situation described in III. B. We omit the details here.

IV. Simulation Results

In this section, Monte Carlo simulation results will be provided. For ease of exposition we present two types of simulation results in 3-dimensional space. In the first situation, there are three sensors which means $N = n$. In the second situation, we add an extra sensor resulting in $N = n + 1$.

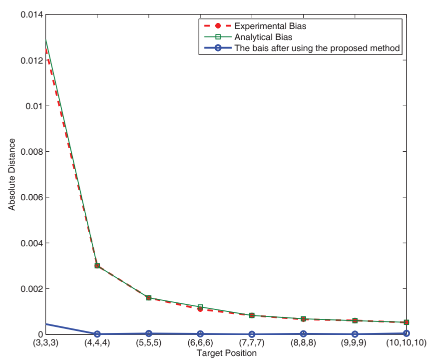

Fig. 2. Comparison of Experimental Bias and Analytical Bias

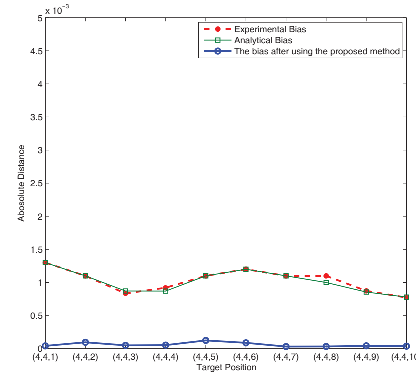

Fig. 3. Comparison of Experimental Bias and Analytical Bias

A. Simulation Assumptions

- Assume the measurements here are range measurements.
- The measurement error for each sensor are produced by independent Gaussian distributions with zero mean and variance σ^2=1.
- All the simulation results are obtained from 5000 Monte Carlo experiments.
- In the simulation the bias is considered as the absolute distance (average of 5000 experimental results) between the true target position and the estimated target position.
- Analytical bias denotes the bias obtained by using the analytical expression derived from our method.
- Experimental bias denotes the bias obtained by using simulations.

B. Three Sensors Situation

In this situation, we fix the three sensors at $(6, 0, 0)$, $(0, 6, 0)$ and $(0, 0, 6)$. The target position is changed from $(3, 3, 3)$ to $(10, 10, 10)$ in steps of $(1, 1, 1)$.

From Fig. 2 we see that by using the proposed method the bias can be corrected almost perfectly. The analytical bias obtained using the proposed method is consistent with the experimental bias. More simulation results demonstrate the proposed method can reduce the bias to a very low level except for adverse geometries, e.g. target remote from the plane containing the sensors. Though it is not very large as shown in the figure, the bias for x and y components is greater than 5% of their standard deviation [1] which means to correct the bias still makes sense.

C. Four Sensors Situation

In this situation, the four sensors are fixed at $(0, 0, 0)$, $(0, 8, 0)$, $(8, 4, 0)$ and $(4, 4, 8)$. Further we fix both the x and y value of the target at 4 while changing the z value from 1 to 10 with steps of 1.

Though an extra variable is introduced into the simulation, the results in Fig. 3 demonstrate the proposed method continues to yield good performance. By using the proposed method, the bias can be reduced to a very low level almost equal to zero.

V. Conclusion

In the previous work [3, 4], we proposed a method to correct the bias in localization algorithms in 2-dimensional space. The bias-correction method mixes Taylor series and Jacobian matrices to express the bias analytically in an easy way. In this paper we extend the bias-correction method to n-dimensional space with N sensors. We analyze the bias-correction method in three situations. In the first situation the dimension is equal to the number of sensors. Next we introduce an extra sensor resulting in an overdetermined problem. A method based on a least squares approach is adopted to solve the overdetermined problem. In the third situation, the number of sensors is greater than the dimension. Monte Carlo simulation results demonstrate the performance of the proposed bias-correction method. Our future work is aimed at improving the performance of the proposed method by using high-order term of the Taylor series; this may be important in high noise.

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