Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

Multi-realization of nonlinear systems

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Abstract— **The system multi-realization problem is to find a state-variable realization for a set of systems, sharing as many parameters as possible. A multi-realization can be used to efficiently implement a multi-controller architecture for Multiple Model Adaptive Control (MMAC). We extend the linear multi-realization problem to nonlinear systems. The problem of minimal multi-realization of a set of MIMO systems is introduced and solved for feedback linearizable systems.**

I. INTRODUCTION

For the implementation of multiple model adaptive control (MMAC) [1] [2] [6] [15] with switching control actions [14] [13], Morse [14] [13] showed the multi-controller (for SISO case) can be efficiently implemented by using a parameter-dependent feedback structure. As argued in [13], because at any instant of time only one of the constituent controllers is to be applied to the plant, it is only necessary to generate one candidate control signal. That is, instead of implementing each of the controllers in the family as a separate dynamic system, one can often achieve the same results using a single controller with adjustable parameters.

This implementation strategy motivates the multi-realization problem [4] [18]. In standard linear system realization, we only need to find a state space realization to realize one transfer function. For a multi-realization, we need to find a parameter dependent state space description for a finite collection of systems, which may be those of the family of controllers.

Most literature on system realization deals with the implementation of a single linear time invariant (LTI) system [3] [5] [9] [12] [19] [20] based on one of a state space description approach or matrix fraction description approach. Morse [13] presented some results for the multi-realization of several linear SISO systems in the context of examining MMAC for scalar plants. Papers [4] [18] investigate the multi-realization of several linear multiple input multiple output (MIMO) systems. The results are applicable to MMAC problems for linear MIMO plants. In this paper, we give a modest extension to the nonlinear case. Specifically, we consider the multi-realization of a finite set of feedback linearizable systems [7] [8].

In the next section, we recall the result on multi-realization of a set of linear SISO systems by state sharing and feedback. In Section III, we introduce the problem of feedback multi-realization of a set of nonlinear systems, and present the result on minimal multi-realization of a set of feedback linearizable nonlinear systems. An illustrative example is given in Section IV.

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II. MULTI-REALIZATION FOR LINEAR SISO SYSTEMS

Suppose that it is desired to implement a finite number of SISO linear proper rational systems with transfer functions $\kappa_i(s) = \frac{n_i(s)}{d_i(s)} (i \in \mathscr{I})$, where $(n_i(s), d_i(s))$ are coprime polynomials. Assuming an upper bound n for the McMillan Degree of the $\kappa_i(s)$, it is shown in [13] that we can always find an n-dimensional controllable pair (A_0, b_0) with $Re\{\lambda_i(A_0)\} < 0$ such that $\{A_0 + b_0 k_{q_i}, b_0, c_{q_i}, d_{q_i}\}$ is a state space realization of each transfer function $\kappa_i(s)$, with corresponding adjustable parameters $k_{q_i} \in \mathscr{R}^n, c_{q_i} \in \mathscr{R}^{1\times n}, d_{q_i} \in \mathscr{R},\ i \in \{1, 2, \ldots, N\}$. There is a MIMO generalization of the problem, see [4] [18] and we will return to this later.

III. EXTENSION TO NONLINEAR SYSTEMS

Firstly, we introduce some basic notation and facts of differential geometry drawn from [8].

A. Some notations in differential geometry

A *smooth vector field* f, defined on an open set U of $\mathscr{R}^n$, can be intuitively interpreted as a smooth mapping assigning the n-dimensional vector $f(x)$ to each point x of U. Suppose now that d smooth vector fields $f_1, \cdots, f_d$ are given, all defined on the same open set U and note that, at any fixed point x in U, the vectors $f_1(x), \cdots, f_d(x)$ span a vector space (a subspace of the vector space in which all the $f_i(x)'s$ are defined, i.e. a subspace of $\mathscr{R}^n$). Let this vector space, which depends on x, be denoted by $\Delta(x)$, i.e. set

$$\Delta(x) = span\{f_1(x), \cdots, f_d(x)\}$$

and note that, in doing this, we have essentially assigned a vector space to each point x of the set U. The object thus characterized, namely the assignment of the subspace spanned by the values at x of some smooth vector fields defined on U, is called a *smooth distribution*.

Next, we recall some facts about Lie derivatives of functions (Page 155-156 in [7]).

Given a smooth vector-field X on $\mathscr{R}^n$ and a smooth function h on $\mathscr{R}^n$ the Lie derivative of h with respect to X is the function

$$L_X h(x) = X(h)(x) = \frac{\partial h}{\partial x}(x) \cdot X(x).$$

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Similarly the functions $L_X^k h$ are defined as follows. By convention we set $L_X^0 h(x) = h(x)$ and inductively for $k \geq 1$,

$$L_X^k h(x) = L_X(L_X^{k-1}h)(x).$$

Analogously for a smooth mapping $h : \mathscr{R}^n \mapsto \mathscr{R}^p$ we define $L_X^k h(x) = L_X^k h$ component-wise, i.e. $L_X^k h = (L_X^k h_1, \cdots, L_X^k h_p)^T$. For the nonlinear system

$$P: \left\{ \begin{array}{lcll} \dot{x} & = & f(x)+g(x)u & x \in \mathscr{R}^{n_x}, u \in \mathscr{R}^m \\ y & = & h(x) & y \in \mathscr{R}^p, \end{array} \right. \tag{1}$$

we introduce (in the local coordinates x) the mapping

$$W^k(x) = \left[\begin{array}{c} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{k-1} h(x) \end{array} \right]. \tag{2}$$

For any two vector-fields of f and g on $\mathscr{R}^n$, the Lie bracket $[f,g]$ is a vector field as given in [17]

$$[f,g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

We define the repeated Lie bracket $ad_f^k g$, $k = 0,1,2,\cdots$, inductively as $ad_f^k g = [f, ad_f^{k-1} g]$, $k \geq 1$, with $ad_f^0 g = g$.

A set of vector fields $\{X_1, \cdots, X_m\}$ is said to be involutive if there are scalar fields α_{ijk} such that

$$[X_i, X_j] = \sum_{k=1}^{m} \alpha_{ijk} X_k.$$

More detailed statements about differential geometry can be found in [8], [10] and [7].

B. Multi-realization problem for nonlinear systems

We first define the problem of finding a multi-realization of a set of nonlinear systems as an extension of the linear multi-realization problem.

Definition 1: Assume that there are given a number of m-input p-output nonlinear systems P_i ($i \in \{1,2,\ldots,N\}$) described by

$$P_i: \left\{ \begin{array}{lcll} \dot{x} & = & f_i(x)+g_i(x)u & x \in \mathscr{R}^{n_{ix}}, u \in \mathscr{R}^m \\ y & = & h_i(x) & y \in \mathscr{R}^p. \end{array} \right. \tag{3}$$

Provided that a state space description

$$\tilde{P}_i: \left\{ \begin{array}{lcll} \dot{\xi} & = & f_{s0}(\xi)+g_{s0}(\xi)v_{q_i} & \xi \in \mathscr{R}^{n_\xi}, v_{q_i} \in \mathscr{R}^m \\ y & = & h_{s0q_i}(\xi) & y \in \mathscr{R}^p. \end{array} \right. \tag{4}$$

with coordinate transformation (a smooth invertible transformation $x = \Phi_i(\xi)$ in a neighborhood of ξ_0 [8]) and state feedback ($v_{qi} = \alpha_{q_i}(\xi) + \beta_{q_i}(\xi)u$) can "realize" (locally or globally) each systems P_i (where the functions $\alpha_{q_i}(\xi)$, $\beta_{q_i}(\xi)$ and $h_{soq_i}(\xi)$ are adjustable), then we call the state space description (4) a multi-realization of the set of systems P_i ($i \in \{1,2,\ldots,N\}$). If the unforced system $\dot{\xi} = f_{s0}(\xi)$ is asymptotically stable, we say that the state space description (4) is a *stably-based* multi-realization of the set of systems P_i ($i \in \{1,2,\ldots,N\}$).

Furthermore, if the dimension of the system $\tilde{P}_i$ (n_ξ) is the smallest of all such *stably-based* multi-realizations, then we call the state space description (4) a *minimal stably-based* multi-realization of the set of nonlinear systems P_i ($i \in \{1,2,\ldots,N\}$).

Note 2: In the statement of Definition 1, the word "realize" means that system $\tilde{P}_i$ can be transformed into P_i by means of feedback and change of coordinates in the state space.

The problem of multi-realization of nonlinear systems is obviously more complicated and difficult than in the linear case, especially the nonlinear minimal *stably-based* multi-realization problem. In the next section, we will only consider a special nonlinear multi-realization problem: the multi-realization of state equations of feedback linearizable nonlinear systems.

C. The multi-realization of state equations of feedback linearizable nonlinear systems

Problem 3: Assume that there are given a number of state equations (without output equations) of nonlinear systems P_i ($i \in \{1,2,\ldots,N\}$) described by

$$P_i: \quad \dot{x} \quad = \quad f_i(x)+g_i(x)u \quad x \in \mathscr{R}^{n_{ix}}, u \in \mathscr{R}^m. \tag{5}$$

Find (if possible) a state equation (6)

$$\tilde{P}_i: \quad \dot{\xi} \quad = \quad A_0\xi + B_0 v_i \quad \xi \in \mathscr{R}^{n_\xi}, v_i \in \mathscr{R}^m. \tag{6}$$

with coordinate transformation (a smooth invertible transformation $x = \Phi_i(\xi)$ in a neighborhood of ξ_0) and state feedback ($v_i = \alpha_i(\xi) + \beta_i(\xi)u$) to "realize" (locally or globally) each state equation (5), where the functions $\alpha_i(\xi)$ and $\beta_i(\xi)$ are adjustable.

Furthermore, find a stable A_0 (all eigenvalues of A_0 are in the left half plane) with the smallest possible dimension for equation (6) to realize (locally or globally) each state equation (5).

Note 4: 1)The state feedback

$$v_i = \alpha_i(\xi) + \beta_i(\xi)u$$

in which

$$\alpha_i(\xi) = \left[\begin{array}{c} \alpha_{i1}(\xi) \\ \alpha_{i2}(\xi) \\ \vdots \\ \alpha_{im}(\xi) \end{array} \right], \quad \beta_i(\xi) = \left[\begin{array}{ccc} \beta_{i_{11}}(\xi) & \cdots & \beta_{i_{1m}}(\xi) \\ \cdots & \cdots & \cdots \\ \beta_{i_{m1}}(\xi) & \cdots & \beta_{i_{mm}}(\xi) \end{array} \right]$$

are an $m \times 1$ vector and, respectively, an $m \times m$ matrix, and the entries of $\alpha_i(\xi)$ and $\beta_i(\xi)$ are smooth functions defined on an open subset of $\mathscr{R}^n$. In the following discussions, we also assume that the matrix $\beta_i(\xi)$ is nonsingular for all ξ. Accordingly, the feedback is called a "regular static state feedback" [8].

2)In the statement of Problem 3, the word "realize" means that the state equation (5) can be constructed by selecting **some or all** transformed (feedback and change of coordinates) states of the state equation (6). Hence, it is obvious that $n_\xi \geq n_{ix}, \forall i \in \{1,2,\cdots,N\}$.

In order to solve Problem 3, we present some results about feedback linearizable systems from [8].

Firstly, we introduce the so-called "State Space Exact Linearization Problem" [8].

Problem 5: Given a state equation (without output equation) of a nonlinear system P described by

$$P: \ \dot{x} \ = \ f(x)+g(x)u, \tag{7}$$

and an initial state x_0, find (if possible), a neighborhood U of x_0, a pair of feedback functions $\alpha(x)$ and $\beta(x)$ defined on U, a coordinate transformation $\xi=\Phi(x)$ also defined on U, a matrix $A \in \mathscr{R}^{n\times n}$ and a matrix $B \in \mathscr{R}^{n\times m}$, such that

$$[\frac{\partial\Phi}{\partial x}(f(x)+g(x)\alpha(x))]_{x=\Phi^{-1}(\xi)} \ = \ A\xi \tag{8}$$

$$[\frac{\partial\Phi}{\partial x}(g(x)\beta(x))]_{x=\Phi^{-1}(\xi)} \ = \ B \tag{9}$$

and

$$rank(BAB\cdots A^{n-1}B)=n.$$

The following results for the "State Space Exact Linearization Problem" are directly from [8].

Lemma 6: Suppose the matrix $g(x_0)$ has rank m. Then, the "State Space Exact Linearization Problem" is solvable if and only if there exist a neighborhood U of x_0 and m real-valued functions $h_1(x),\cdots,h_m(x)$ defined on U, such that the system

$$\left\{\begin{array}{lcl} \dot{x} & = & f(x)+g(x)u \\ y & = & h(x) \end{array}\right. \tag{10}$$

has some relative degree $\{r_1,r_2,\cdots,r_m\}$ at x_0 and $r_1+r_2+\cdots+r_m=n$.

Furthermore, the matrices A and B in equation (8) and equation (9) of Problem 5 could be in the form (Brunowsky canonical form):

$$\begin{array}{ll} A & =diag\{A_1,A_2,\cdots,A_m\}, \\ B & =diag\{b_1,b_2,\cdots,b_m\}, \end{array} \tag{11}$$

where A_i is the $r_i\times r_i$ matrix

$$A_i=\left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ & & & & \end{array}\right],$$

and b_i is the $r_i\times 1$ vector

$$b_i=[0,\cdots,0,1]^T.$$

Proof: See Page 246-248 in [8]. ■

Note 7: 1) From Lemma 6, we can see that the controllable pair $\{A,B\}$ of the feedback linearized system has controllability indices $\{r_1,r_2,\cdots,r_m\}$; this set is invariant under state feedback, input transformation, and coordinate transformation.

2) It is easily checked that the two transformations (state feedback and coordinate transformation) used in order to obtain the linear form can be interchanged (see Remark 2.1 in Page 158-159 of [8]).

The geometric conditions for the solution of "State Space Exact Linearization Problem" are presented in the following lemma [8].

Lemma 8: Suppose the matrix $g(x_0)$ has rank m. Then, there exists a neighborhood U of x_0 and m real-valued functions $\lambda_1(x),\lambda_2(x),\cdots,\lambda_m(x)$ defined on U, such that the system

$$\left\{\begin{array}{lcl} \dot{x} & = & f(x)+g(x)u \\ y & = & \lambda(x) \end{array}\right. \tag{12}$$

has some relative degree $\{r_1,r_2,\cdots,r_m\}$ at x_0, with

$$r_1+r_2+\cdots+r_m=n,$$

if and only if

i)for each $0\le i\le n-1$, the distribution G_i has constant dimension near x_0;

ii)the distribution G_{n-1} has dimension n;

iii)for each $0\le i\le n-2$, the distribution G_i is involutive.

Here,

$$\begin{array}{ll} G_0 & =span\{g_1,\cdots,g_m\} \\ G_1 & =span\{g_1,\cdots,g_m,ad_fg_1,...,ad_fg_m\} \\ \cdots & \\ G_i & =span\{ad_f^kg_j: 0\le k\le l, 1\le j\le m\} \end{array} \tag{13}$$

for $i=0,1,\cdots,n-1$.

Proof: See Page 249-256 in [8]. ■

Note 9: (1)This lemma presents geometric conditions for the solvability of the "State Space Exact Linearization Problem".

(2)The relative degrees $r_1,r_2,\cdots,r_m$ are directly identified in terms of the dimensions of the distributions $G_0,G_1,\cdots,G_{n-2}$ (see Remark 2.7 in Page 256 of [8]). Therefore, the relative degrees $r_1,r_2,\cdots,r_m$ are invariant under feedback and coordinate transformation.

Furthermore, the relative degrees are equal to the controllability indices of the controllable pair (A_0,B_0) for the linearized systems (see Note 7).

Now that the conditions for "State Space Exact Linearization Problem" are clear, the next step returns to the problem of multi-realization of a set of linear systems by using state feedback, input transformation and coordinate transformation.

Now, we give a definition for the minimal *stably-based* multi-realization (with input transformation) for a set of linear systems.

Definition 10: Assume that there are given a number of m-input p-output strictly proper real rational transfer function matrices P_i ($i\in\{1,2,\ldots,N\}$). Provided that there exist state-variable realizations $\{A_0+B_0K_{q_i},B_0G_{qi},C_{q_i}\}$ (with the pair (A_0,B_0) being controllable) that can realize all the systems P_i with adjustable parameters C_{q_i}, K_{q_i} and G_{qi}, then we call $\{A_0+B_0K_{q_i},B_0G_{qi},C_{q_i}\}$ a multi-realization (with input transformation) of the set of systems P_i ($i\in\{1,2,\ldots,N\}$). If all eigenvalues of A_0 are in the left half plane, we say that $\{A_0+B_0K_{q_i},B_0G_{qi},C_{q_i}\}$ is a stably-based multi-realization (with input transformation) of the set of systems P_i ($i \in$

$\{1,2,\ldots,N\}$). Furthermore, if the dimension of A_0 is the smallest of all such stably-based multi-realizations, then we call $\{A_0+B_0K_{q_i},B_0G_{qi},C_{q_i}\}$ a minimal stably-based multi-realization (with input transformation) of the set of systems P_i ($i\in\{1,2,\ldots,N\}$).

Lemma 11: Assume given a number of m-input p-output strictly proper linear systems P_i ($i\in\{1,2,\ldots,N\}$).

Then there exists a controllable pair (A_0,B_0) ($A_0\in\mathscr{R}^{n\times n}$, $B_0\in\mathscr{R}^{n\times m}$), and appropriately dimensioned real matrices C_{q_i}, K_{q_i} and G_{q_i} (for $i\in\{1,2,\ldots,N\}$) such that A_0 is stable, and $\{A_0+B_0K_{q_i},B_0G_{qi},C_{q_i}\}$ is a controllable realization of system P_i (for $i\in\{1,2,\ldots,N\}$), if and only if there exists a state space realization $\{A_i\in\mathscr{R}^{n\times n},B_i\in\mathscr{R}^{n\times m},C_i\in\mathscr{R}^{p\times n}\}$ (where the pair (A_i,B_i) is controllable, and B_i has full column rank.) for each system P_i such that all controllable pairs (A_i,B_i) (for each $i\in\{1,2,\ldots,N\}$) have identical controllability indices sets as $\{d_1,d_2,\ldots,d_m\}$ (without ordering requirement).

Proof: See Theorem 3 in Page 260-261 of [16]. The unordered controllability indices are invariant under feedback, input transformation and coordinate transformation. This shows the condition of the theorem is necessary.

Now, we prove the sufficiency. According to item (1)-(9) in Page 507-508 of [9] (and Theorem 3 in Page 260-261 of [16]), we can find a controller form realization $\{A'_0+B_0K'_{q_i},B_0G_{qi},C_{q_i}\}$ for each system P_i with $\{A'_0,B_0\}$ controllable by adjusting parameters K'_{q_i}, G_{qi} and C_{qi}. Furthermore, if A'_0 is not stable, we can select a stable A_0 and a new adjustable feedback gain matrix K_{q_i} such that $A'_0+B_0K'_{q_i}=A_0+B_0K_{q_i}$ because $\{A'_0,B_0\}$ is controllable. Thus, we can construct a state-variable description $\{A_0+B_0K_{q_i},B_0G_{qi},C_{q_i}\}$ which is a (multi-) realization of system P_i (for $i\in\{1,2,\ldots,N\}$). ■

Lemma 11 provides the necessary and sufficient condition for multi-realization of a set of linear systems by a state-variable realization $\{A_0+B_0K_{q_i},B_0G_{qi},C_{q_i}\}$ with the dimension of A_0 fixed to n. If the dimension of A_0 is not fixed to n, the necessary and sufficient condition in Lemma 11 can be fulfilled by dimension augmentation of each controllable pair (A_i,B_i) to ensure all (A_i,B_i) (for each $i\in\{1,2,\ldots,N\}$) have identical controllability indices sets as $\{d_1,d_2,\ldots,d_m\}$ (without ordering requirement). Certainly, the minimal dimension augmentation is desired, and this is not difficult to achieve. The key to doing this is the following lemma.

Lemma 12: Assume given N ordered sets S_i of cardinality m of positive integers with $S_i=\{r_{i1},r_{i2},r_{i3},\cdots,r_{im}\}$, and with $r_{i1}\le r_{i2}\le\cdots\le r_{im},\forall i\in\{1,2,\cdots,N\}$. Let $\{d_{i1},d_{i2},\cdots,d_{im}\}$ denoting an arbitrary reordering of S_i. Then

$$\sum_{j=1}^{m}\max_{1\le i\le N}(r_{ij})\le\sum_{j=1}^{m}\max_{1\le i\le N}(d_{ij}). \qquad (14)$$

Proof: This is a special case of Theorem **D.7.a** in Page 155 of [11]. ■

Theorem 13: Assume given N distinct m-input p-output systems P_i ($i\in\{1,2,\cdots,N\}$) described by $A_i\in\mathscr{R}^{n_i\times n_i}$, $B_i\in\mathscr{R}^{n_i\times m}$ and $C_i\in\mathscr{R}^{p\times n_i}$, and suppose that the pairs (A_i,B_i) and (C_i,A_i) are controllable and observable, and B_i has full column rank. Further, assume the controllability indices for the pair (A_i,B_i) are $d_{i1},d_{i2},\cdots,d_{im}$, and define the ordered set $\{r_{i1},r_{i2},r_{i3},\ldots\}$ to be the set of controllability indices reordered so that

$$r_{i1}\le r_{i2}\le\cdots\le r_{im},\forall i\in\{1,2,\cdots,N\}.$$

Then the dimension of the minimal stably-based multi-realization (with input transformation) of the set of systems P_i ($i\in\{1,2,\cdots,N\}$) is equal to

$$\bar{n}=\sum_{i=1}^{N}\max_{1\le j\le m}(r_{ij}).$$

Proof: According to Lemma 11, in order to obtain a stably-based multi-realization (with input transformation), we need to find another state space realization $\{\bar{A}_i\in\mathscr{R}^{\bar{n}\times\bar{n}},\bar{B}_i\in\mathscr{R}^{\bar{n}\times m},\bar{C}_i\in\mathscr{R}^{p\times\bar{n}}\}$ (with the pair $(\bar{A}_i,\bar{B}_i)$ controllable, and $\bar{B}_i$ of full column rank.) for each systems P_i such that all the controllable pairs $(\bar{A}_i,\bar{B}_i)$ ($i\in\{1,2,\ldots,N\}$) have an identical controllability indices set as $\{\bar{d}_1,\bar{d}_2,\ldots,\bar{d}_m\}$ (without order requirement). In order to find a **minimal** stably-based multi-realization (with input transformation), we need to find a minimal dimension ($\bar{n}$) for all $\{\bar{A}_i\in\mathscr{R}^{\bar{n}\times\bar{n}},\bar{B}_i\in\mathscr{R}^{\bar{n}\times m},\bar{C}_i\in\mathscr{R}^{p\times\bar{n}}\}$. The minimal dimension ($\bar{n}$) has been given in Lemma 12 which deals with an equivalent problem of finding the minimal dimension $\bar{n}$. ■

Lemma 6 and 8 solve the "State Space Exact Linearization Problem". They provide the conditions based on which a nonlinear system can be linearized by using "regular static state feedback"and coordinate transformation. Then, based on Lemma 11 and Theorem 13, we can give the answer for Problem 3, "the multi-realization of state equations of feedback linearizable nonlinear systems".

Theorem 14: (Main Result) Suppose the matrices $g_i(x_0),\forall i\in\{1,2,\cdots,N\}$ in equation (5) of Problem 3 have rank m. Then, Problem 3 is solvable if and only if the following equivalent conditions (a) or (b) hold:

(a) For each system P_i (described by equation (5)), there exist a neighborhood U of x_0 and m real-valued functions $h_{i1}(x),\cdots,h_{im}(x)$ defined on U, such that the system

$$\left\{\begin{array}{rcl}\dot{x}&=&f_i(x)+g_i(x)u\\ y&=&h_i(x)\end{array}\right. \qquad (15)$$

has some relative degree $\{r_{i1},r_{i2},\cdots,r_{im}\}$ at x_0 and $r_{i1}+r_{i2}+\cdot,+r_{im}=n_i$.

(b) i)for each $i\in\{1,2,\cdots,N\}$ and $l\in\{1,2,\cdots,n-1\}$, the distribution G_{il} has constant dimension near x_0;

ii)the distribution $G_{i,n-1}$ ($i\in\{1,2,\cdots,N\}$) has dimension n;

iii)for each $i\in\{1,2,\cdots,N\}$ and $l\in\{1,2,\cdots,n-1\}$, the distribution G_{il} is involutive.

Here,

$$\begin{array}{ll}G_{i0}&=span\{g_{i1},\cdots,g_{im}\}\\ G_{i1}&=span\{g_{i1},\cdots,g_{im},ad_{f_i}g_{i1},...,ad_{f_i}g_{im}\}\\ &\cdots\\ G_{il}&=span\{ad^k_{f_i}g_{ij}:0\le k\le l,1\le j\le m\}.\end{array} \qquad (16)$$

Furthermore, the smallest possible dimension for equation (6) to realize (locally or globally) each state equation (5) is equal to

$$\bar{n} = \sum_{i=1}^{N} \max_{1 \le j \le m} (r_{ij}).$$

Proof: From Lemma 6 and 8, we conclude that the set of nonlinear state space equations (see equation (5)) can be expressed as linear state space equations (with feedback and coordinate transformation) if and only if Conditions i), ii) and iii) are satisfied. Then, according to Lemma 11 and Theorem 13, we can find a minimal stably-based multi-realization for these linearized systems. Further, considering that the state feedback and coordinate transformation are interchangeable (see Note 7), we conclude that Problem 3 is solvable under these conditions.

Because Conditions i), ii) and iii) of b) are necessary for the set of nonlinear state space equations (see equation (5)) being linearized. Therefore, Conditions i), ii) and iii) of b) are necessary for the solvability of Problem 3 ■

IV. Example

We show the multirealization of the following two feedback linearizable systems (the symbols and notations are defined as in Definition 1):

$$P_1 : \begin{cases} \dot{x} = \begin{bmatrix} -x_1^3 + x_2 + u_1 \\ x_1 - x_2^3 + u_2 \end{bmatrix} \\ y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{cases} \quad P_2 : \begin{cases} \dot{x} = \begin{bmatrix} x_2^3 \\ x_1^2 + x_2^2 + u_1 \\ x_2 + x_3^2 + u_2 \end{bmatrix} \\ y = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}. \end{cases}$$

We construct $\tilde{P}_i$ as follows:

$$\begin{aligned} \dot{\xi} &= f_{s0}(\xi) + g_{s0}(\xi) v_{qi} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{qi1} \\ v_{qi2} \end{bmatrix}. \end{aligned}$$

To implement P_1, we can use the following settings:

$$\begin{cases} v_{q1} = \alpha_{q1}(\xi) + \beta_{q1}(\xi) u \\ \quad = \begin{bmatrix} -\xi_1^3 + 3\xi_2 + \xi_1 \\ \xi_1 - \xi_2^3 + 3\xi_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\ x = \phi_1(\xi) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \\ y = h_1(x) = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = h_{soq1}(\xi) = \begin{bmatrix} \xi_2 \\ \xi_3 \end{bmatrix}. \end{cases}$$

To implement P_2, we can use the following settings:

$$\begin{cases} v_{q2} = \alpha_{q2}(\xi) + \beta_{q2}(\xi) u \\ \quad = \begin{bmatrix} 3\xi_1^2 \xi_2^{\frac{2}{3}} + 3\xi_2^{\frac{4}{3}} + \xi_1 + 2\xi_2 \\ \xi_2^{\frac{1}{3}} - \xi_3^2 + 3\xi_3 \end{bmatrix} + \begin{bmatrix} 3\xi_2^{\frac{2}{3}} & 0 \\ 0 & 1 \end{bmatrix} u, \\ x = \phi_2(\xi) = \begin{bmatrix} \xi_1 \\ \xi_2^{\frac{1}{3}} \\ \xi_3 \end{bmatrix}, \\ y = h_2(x) = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = h_{soq2}(\xi) = \begin{bmatrix} \xi_1 \\ \xi_3 \end{bmatrix}. \end{cases}$$

V. Conclusion

In this paper, the problem of multi-realization of a set of linear SISO systems is reviewed. Then, the problem of multi-realization of a set of nonlinear systems is introduced. A minimal stably-based multi-realization of state equations of feedback linearizable nonlinear systems is achieved. These results will make the MMAC more efficient and practical.

VI. Acknowledgments

The first author would like to express his best thanks to Dr. Thomas Brinsmead (Division of Energy Technology, CSIRO, Australia) for the valuable discussions and suggestions. The work of Prof. Brian Anderson has been supported by ARC Discovery Project Grant DP0664427.

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