Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

Solving Discrete Algebraic Riccati Equations: A New Recursive Method

Yantao Feng, Brian D. O. Anderson and Weitian Chen

***Abstract*— In this paper, an iterative algorithm is proposed to solve discrete time algebraic Riccati equations (DARE) with a sign indefinite quadratic term, which arise from linear discrete time H_∞ control. By constructing two positive semidefinite matrix sequences, we obtain the stabilizing solution of the given DARE. The algorithm has a global convergence property.**

I. Introduction

We consider the following DARE

$$0 = -\Pi + C^T C + A^T \Pi \left[I + (B_2 B_2^T - B_1 B_1^T)\Pi \right]^{-1} A, \quad (1)$$

where A, B_1, B_2, C are given real matrices with compatible dimensions, and Π is the stabilizing and nonnegative definite solution we seek. Equation (1) has many applications such as in a two players zero-sum difference game [1], and linear discrete time H_∞ control [2]–[6]. Typically, analyzing and solving the DARE (1) is more difficult than the corresponding H_2 DARE since there is an indefinite nonlinear term in (1) and we cannot generally assert the existence of a solution of (1).

The motivation of this paper comes from our previous work in [8], [9]. In the algorithm of [8], [9], we replace the problem of solving an H_∞ continuous time algebraic Riccati equation (CARE) by the problem of solving a sequence of H_2 CAREs; then the stabilizing solution of the original CARE can be obtained by using the solutions of these H_2 CAREs. Compared with some other existing algorithms to solve H_∞ CAREs, the algorithm in [8], [9] has some advantages such as local quadratic rate of convergence, a simple initialization, and a natural game theoretic interpretation [8], [9]. The question arising here is whether we can extend the algorithm in [8], [9] to solve H_∞ DARE (1). The purpose of this paper is to provide an answer to this question. To this end, we will present an iterative algorithm to solve (1), which can be regarded as a discrete time version of the algorithm in [8], [9]. Not surprisingly, the new algorithm will have some similar properties as those of the algorithm in [8], [9] which will be shown later.

Before going further, we first review some existing algorithms to solve (1), in particular the recursive algorithm in [2] and the non-recursive algorithms in [7]. In the algorithm of [2], under some appropriate assumptions, a monotonic increasing matrix sequence is constructed to approximate the stabilizing solution of (1). It is claimed in [2] that the algorithm has a simple initialization and an exponential convergence rate; however, it has been observed in Example 1 of [7] that this algorithm has some difficulties in numerical convergence. In [7], based on a transformation between H_∞ CAREs and H_∞ DAREs, a non-recursive algorithm is developed. It is proved in [7] that the stabilizing solution of an H_∞ DARE can be obtained by solving a corresponding H_∞ CARE.

Recently, a model-free Q-learning method has been developed to solve H_∞ DAREs (see [13]). Using this method, DAREs can be solved by recursively approximating a so called "Q-function", which is a function of the inputs and state of the system. In so doing, DAREs can be solved without knowing the system dynamical matrices. It is clear that an advantage of the method in [13] is its "model-free" property; however, except got this advantage, its numerical properties are still not clear since no comparison between this method and other methods has been done.

The algorithm we will present is based on the mappings between CAREs and DAREs and such mappings have been discussed extensively in literature (see [7], [14]–[16], [18]). In this paper, we are interested in developing a direct recursive algorithm to solve (1), which is a discrete time version of the algorithm in [8], [9]. To do this, we will first set up a mapping between the DARE (1) and its corresponding CARE, then we show that this CARE is a typical H_∞ CARE (i.e. it has a sign indefinite quadratic term and a positive semidefinite constant term); after that, we will construct a monotone non-decreasing matrix sequence by using the mapping between the CARE and DARE, which converges to the stabilizing solution of (1).

The structure of the paper is as follows. Section 2 will establish some preliminary results which will be used in the main theorem. Section 3 will give the main results with a proof of global convergence and Section 4 will give the algorithm. Section 5 will present the theorem for the local quadratic rate of convergence. Section 6 gives a numerical example. Section 7 provides some concluding remarks.

II. Preliminary results

We firstly introduce some notation: Let $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices; $\mathbb{Z}$ denotes the set of integers with $\mathbb{Z}_{\geq a}$ denoting the set of integers greater or equal to $a \in \mathbb{R}$; $\rho[\cdot]$ denotes the spectral radius of a square matrix; $\overline{\sigma}(\cdot)$ denotes the maximum singular value of a matrix; I denotes the identity matrix. Let $Y \in \mathbb{R}^{n\times n}$ be a real matrix, and let $n_+(Y)$ denote the number of eigenvalues of Y with positive real parts, $n_0(Y)$ denote the number of eigenvalues of Y with zero real parts, and $n_-(Y)$ denote the number of eigenvalues of Y with negative real parts.

We first make the following assumption:

Assumption 1: The matrix A in (1) has no eigenvalues at -1.

The Assumption 1 is without loss of generality since in the discrete-time H_∞ control problem associated with (1), one can always apply a pre-feedback law to relocate the eigenvalues of A that are at -1, provided that $(A,\ B)$ is D-stabilizable (see Definition 8).

We now consider the system Σ_d, which has a close relationship with (1).

$$\Sigma_d : \begin{cases} x_{k+1} = Ax_k + B_1 w_k + B_2 u_k \\ y_k = \begin{pmatrix} I \\ 0 \end{pmatrix} x_k + \begin{pmatrix} 0 \\ I \end{pmatrix} w_k \\ z_k = Cx_k + D_1 u_k + D_2 w_k \end{cases} \quad (2)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is the disturbance input, $z \in \mathbb{R}^p$ is the controlled output and $y \in \mathbb{R}^{n+q}$ is the measurement, $A,\ B_1,\ B_2,\ C$ are matrices appearing in (1) and $D_1,\ D_2$ are given matrices with suitable dimensions and subject to certain constraints which we now detail. It is well known that the system Σ_d represents the full information case in discrete time H_∞ control. Suppose there exists a stabilizing solution $\Pi > 0$ of (1)(i.e. all eigenvalues of $[I+(B_2B_2^T - B_1B_1^T)\Pi]^{-1}A$ are inside the unit circle), we make some standard assumptions:

Assumption 2: $D_2 = 0$ and $D_1^T[C \quad D_1] = [0 \quad I]$

Assumption 3: $(A,\ B_2)$ is D-stabilizable (see Definition 8).

Assumption 4: $(A,\ B,\ C,\ D_1)$ is left invertible with no invariant zeros on the unit circle.

Assumption 5: $I - B_1^T(I + \Pi B_2 B_2^T)^{-1}\Pi B_1 > 0$

Assumption 2 is without loss of generality since if they are not fulfilled, some algebraic manipulations can be executed to produce an equivalent problem in which they are fulfilled (see [17]) we make them here. Assumptions 3-5 are necessary conditions for the existence of the stabilizing solution to the $H_\infty-$DARE (9) for the full information problem (see [7]).

The following lemma constructs a mapping between (1) and an H_∞ CARE.

Lemma 6: [7] Consider the system Σ_d and (1). Suppose Assumption 1-5 hold and there exists a stabilizing solution Π of (1). Let $M := A + I$ and $N := A - I$. Define

$$\tilde{A} := M^{-1}N, \quad (3)$$
$$\tilde{B}_1 := 2M^{-2}B_1, \quad (4)$$
$$\tilde{B}_2 := 2M^{-2}B_2, \quad (5)$$
$$\tilde{C} := C, \quad (6)$$
$$\tilde{D}_1 := D_1 - CM^{-1}B_2, \quad (7)$$
$$\tilde{D}_2 := D_2 - CM^{-1}B_1 \quad (8)$$

Then

(1). There exists a stabilizing solution for the following CARE

$$0 = \tilde{\Pi}\tilde{A} + \tilde{A}^T\tilde{\Pi} + \tilde{C}^T\tilde{C} - \begin{pmatrix} \tilde{B}_2^T\tilde{\Pi} + \tilde{D}_1^T\tilde{C} \\ \tilde{B}_1^T\tilde{\Pi} + \tilde{D}_2^T\tilde{C} \end{pmatrix}^T \tilde{G}^{-1} \begin{pmatrix} \tilde{B}_2^T\tilde{\Pi} + \tilde{D}_1^T\tilde{C} \\ \tilde{B}_1^T\tilde{\Pi} + \tilde{D}_2^T\tilde{C} \end{pmatrix} \quad (9)$$

where

$$\tilde{G} = \begin{pmatrix} \tilde{D}_1^T\tilde{D}_1 & \tilde{D}_1^T\tilde{D}_2 \\ \tilde{D}_2^T\tilde{D}_1 & \tilde{D}_2^T\tilde{D}_2 - I \end{pmatrix}$$

(2). Solutions Π of (1) and solutions $\tilde{\Pi}$ of (9) are related by

$$\Pi = 2M^{-T}\tilde{\Pi}M^{-1} \quad (10)$$

(3). $\tilde{D}_1$ is injective, $(\tilde{A}, \tilde{B},\ \tilde{C},\ \tilde{D}_1)$ has no invariant zeros on the $j\omega-$axis and

$$\tilde{D}_2^T(I - \tilde{D}_1(\tilde{D}_1^T\tilde{D}_1)^{-1}\tilde{D}_1^T)\tilde{D}_2 < I. \quad (11)$$

Proof: See [7]. ■

Remark: Suppose $\tilde{D}_1$ has m_2 columns and $\tilde{D}_2$ has m_1 columns, then since $\tilde{D}_1$ is injective and (11) holds, after some straightforward computations, we can obtain that $\tilde{G}$ has m_2 eigenvalues with positive real part and m_1 eigenvalues with negative real parts. Furthermore, it is also worth pointing out that equation (9) is associated with the continuous-time H_∞ full information feedback control problem of the following system,

$$\Sigma_c : \begin{cases} \dot{x} = \tilde{A}x + \tilde{B}_1 w + \tilde{B}_2 u \\ y = \begin{pmatrix} I \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} w \\ z = \tilde{C}x + \tilde{D}_1 u + \tilde{D}_2 w \end{cases} \quad (12)$$

Definition 7: [2] A solution Π of (1) is called ***stabilizing*** if it is such that

$$A_{cl} = \left[I + (B_2B_2^T - B_1B_1^T)\Pi\right]^{-1} A \quad (13)$$

has all its eigenvalues inside the unit circle.

Definition 8: A matrix $A \in \mathbb{R}^{n\times n}$ is said to be ***C-stable*** if all its eigenvalues have negative real part; a matrix $A \in \mathbb{R}^{n\times n}$ is said to be ***D-stable*** if all its eigenvalues are inside the unit circle. A matrix pair (A, B) is called ***C-stabilizable*** if there exists a matrix D such that $(A + BD)$ is C-stable; a matrix pair (A, B) is called ***D-stabilizable*** if there exists a matrix E such that $(A + BE)$ is D-stable.

The following lemma gives a uniqueness result of the stabilizing solution of (1).

Lemma 9: [2] Suppose there exists a stabilizing solution to (1); then this solution must be unique (i.e. no other stabilizing solution to (1) exists.)

Proof: See [2]. ■

The next lemma shows that the constant term

$$S := \tilde{C}^T(I - [\tilde{D}_1 \quad \tilde{D}_2]\tilde{G}^{-1}[\tilde{D}_1 \quad \tilde{D}_2]^T)\tilde{C} \quad (14)$$

in (9) is positive semidefinite.

Lemma 10: Consider (9). Suppose $(\tilde{A}, \tilde{B}_2)$ is C-stabilizable, $\tilde{D}_1$ is injective and $(\tilde{A}, \tilde{B}_2, \tilde{C}, \tilde{D}_1)$ has no invariant zeros on the $j\omega-$axis. Suppose (11) holds. Then S defined by (14) in(i.e. the term obtained by setting $\tilde{\Pi}$ equal to zero in (9)) is positive semidefinite.

Proof: After some straightforward computations, we obtain the constant term of (9) as

$$\begin{aligned} S : &= \tilde{C}^T(I - [\tilde{D}_1 \quad \tilde{D}_2]\tilde{G}^{-1}[\tilde{D}_1 \quad \tilde{D}_2]^T)\tilde{C} \\ &= \tilde{C}^T X \tilde{C} \end{aligned} \quad (15)$$

with $X := (I - [\tilde{D}_1 \quad \tilde{D}_2]\tilde{G}^{-1}[\tilde{D}_1 \quad \tilde{D}_2]^T)$. To show S is positive semidefinite, we only need to show X is positive semidefinite. Now we consider the matrix

$$Q := \begin{pmatrix} I & \tilde{D} \\ \tilde{D}^T & \tilde{G} \end{pmatrix} \tag{16}$$

where $\tilde{D} = [\tilde{D}_1 \quad \tilde{D}_2]$. Suppose the identity matrix I in (16) has p columns, $\tilde{D}_1$ has m_2 columns and $\tilde{D}_2$ has m_1 columns. Next we will show that the matrix Q has p positive eigenvalues, m_2 zero eigenvalues and m_1 negative eigenvalues. To do this, we note that condition (11) holds and it is straightforward to verify $\tilde{G}$ is invertible. We consider the matrix

$$W := \begin{pmatrix} I & 0 \\ -\tilde{D}^T & I \end{pmatrix} \tag{17}$$

Then we have

$$WQW^T = \begin{pmatrix} I & 0 \\ 0 & \tilde{G} - \tilde{D}^T\tilde{D} \end{pmatrix} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_{m_1} \end{pmatrix} \tag{18}$$

So the matrix Q has p positive eigenvalues, m_2 zero eigenvalues and m_1 negative eigenvalues. Now we consider the matrix

$$S = \begin{pmatrix} X & 0 \\ 0 & \tilde{G} \end{pmatrix} \tag{19}$$

It is clear that we have $S = VQV^T$ with $V = \begin{pmatrix} I & -\tilde{D}\tilde{G}^{-1} \\ 0 & I \end{pmatrix}$. Note that $\tilde{G}$ is invertible, $\tilde{D}_1$ is injective and (11) holds, it can be asserted that $\tilde{G}$ has m_2 eigenvalues with positive real parts and m_1 eigenvalues with negative real parts. It is clear that Q and S must have the same eigenvalue sign patterns, and we have

$$\begin{aligned} n_+(X) + n_+(\tilde{G}) &= n_+(Q) \Longrightarrow n_+(X) = p - m_2 \\ n_0(X) + n_0(\tilde{G}) &= n_0(Q) \Longrightarrow n_0(X) = m_2 \\ n_-(X) + n_-(\tilde{G}) &= n_-(Q) \Longrightarrow n_-(X) = 0 \end{aligned}$$

So we conclude that X has no negative real part eigenvalues and thus is positive semidefinite. ■

The following lemma provides a concise expression for (9), which will be extensively used in deriving our main results.

Lemma 11: Consider the equation (9). Suppose $(\tilde{A}, \tilde{B}_2)$ is C-stabilizable, $(\tilde{A}, \tilde{B}_2, \tilde{C}, \tilde{D}_1)$ has no invariant zeros on the $j\omega$ axis and (11) holds. Suppose $\tilde{\Pi}$ is the stabilizing solution of (9). Then (9) can be rewritten as

$$0 = \tilde{\Pi}\bar{A} + \bar{A}^T\tilde{\Pi} - \tilde{\Pi}(\bar{B}_2\bar{B}_2^T - \bar{B}_1\bar{B}_1^T)\tilde{\Pi} + S \tag{20}$$

where

$$\bar{A} = \tilde{A} - [\tilde{B}_2 \quad \tilde{B}_1]\tilde{G}^{-1}[\tilde{D}_1 \quad \tilde{D}_2]^T\tilde{C} \tag{21}$$

$$\bar{B}_2\bar{B}_2^T - \bar{B}_1\bar{B}_1^T = [\tilde{B}_2 \quad \tilde{B}_1]\tilde{G}^{-1}[\tilde{B}_2 \quad \tilde{B}_1]^T \tag{22}$$

$$S = \tilde{C}^T(I - [\tilde{D}_1 \quad \tilde{D}_2]\tilde{G}^{-1}[\tilde{D}_1 \quad \tilde{D}_2]^T)\tilde{C} \tag{23}$$

with S positive semidefinite. Furthermore, $(\bar{A}, \bar{B}_2)$ is C-stabilizable and $(S^{\frac{1}{2}}, \bar{A})$ has no unobservable modes on the $j\omega-$axis.

Proof: (21)-(23) can be obtained by a straightforward computation. Note that it is shown in Lemma 10 that

$$X := I - \begin{pmatrix} \tilde{D}_1 & \tilde{D}_2 \end{pmatrix} \tilde{G}^{-1} \begin{pmatrix} \tilde{D}_1^T \\ \tilde{D}_2^T \end{pmatrix} \tag{24}$$

is positive semidefinite, so $S = \tilde{C}^T X \tilde{C}$ in (20) is also positive semidefinite. Since $\tilde{\Pi}$ is the stabilizing solution of (20), we can conclude that $(\bar{A}, \bar{B}_2)$ is stabilizable and $(S^{\frac{1}{2}}, \bar{A})$ has no unobservable modes on the $j\omega-$axis by using an argument in standard H_∞ control theory (see [17] for example). ■

Motivated by the right hand side of (20), we define a function Θ, which will be extensively used in our later results.

Definition 12: Let $\bar{A},\ \bar{B}_1,\ \bar{B}_2,\ \tilde{C},\ X$ be the matrices appearing in (20) and $\tilde{P} \in \mathbb{R}^{n\times n}$. Define $\Theta : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ as

$$\begin{aligned} \Theta(\tilde{P}) = &\tilde{P}\bar{A} + \bar{A}^T\tilde{P} \\ &- \tilde{P}(\bar{B}_2\bar{B}_2^T - \bar{B}_1\bar{B}_1^T)\tilde{P} + \tilde{C}^T X \tilde{C} \end{aligned} \tag{25}$$

We now recall Lemma 1 and Lemma 2 in [8], [9].

Lemma 13: [8], [9] Let Θ be defined as (25). Suppose that $\bar{A},\ \bar{B}_1,\ \bar{B}_2,\ \tilde{C},\ X$ are the given matrices appearing in (20). Let $\tilde{P} = \tilde{P}^T,\ \tilde{Z} = \tilde{Z}^T \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned} \Theta(\tilde{P} + \tilde{Z}) = &\Theta(\tilde{P}) + \tilde{Z}\hat{A} + \hat{A}\tilde{Z} \\ &- \tilde{Z}\left(\bar{B}_2\bar{B}_2^T - \bar{B}_1\bar{B}_1^T\right)\tilde{Z}, \end{aligned} \tag{26}$$

where $\hat{A} = \bar{A} + \bar{B}_1\bar{B}_1^T\tilde{P} - \bar{B}_2\bar{B}_2^T\tilde{P}$.

Furthermore, if $\tilde{P} = \tilde{P}^T,\ \tilde{Z} = \tilde{Z}^T \in \mathbb{R}^{n\times n}$ satisfy

$$0 = \Theta(\tilde{P}) + \tilde{Z}\hat{A} + \hat{A}\tilde{Z} - \tilde{Z}\bar{B}_2\bar{B}_2^T\tilde{Z}, \tag{27}$$

then

$$\Theta(\tilde{P} + \tilde{Z}) = \tilde{Z}\bar{B}_1\bar{B}_1^T\tilde{Z} \tag{28}$$

and

$$\rho[\Theta(\tilde{P} + \tilde{Z})] = \overline{\sigma}(\bar{B}_1^T\tilde{Z})^2 \tag{29}$$

Proof: (27) can be obtained by direct algebraic manipulations. (28) and (29) are then trivial. ■

The following lemma sets up some basic relationships between the stabilizing solution $\tilde{\Pi}$ to equation (1) and the matrices $\tilde{P}$, $\tilde{Z}$ satisfying equation (27).

Lemma 14: [8], [9]Suppose that $\bar{A}, \bar{B}_1, \bar{B}_2, \tilde{C},\ X$ are the matrices appearing in (20), $P = P^T \in \mathbb{R}^{n\times n}$ and $Z = Z^T \in \mathbb{R}^{n\times n}$ satisfying equation (27), and a stabilizing $\tilde{\Pi} = \tilde{\Pi}^T \in \mathbb{R}^{n\times n}$ satisfying equation (20). Let

$$\check{A} = \bar{A} + \bar{B}_1\bar{B}_1^T(\tilde{P} + \tilde{Z}) - \bar{B}_2\bar{B}_2^T\tilde{\Pi}$$

and

$$\acute{A} = \bar{A} + \bar{B}_1\bar{B}_1^T\tilde{P} - \bar{B}_2\bar{B}_2^T\tilde{\Pi}.$$

Then

(i) $\tilde{\Pi} \geq (\tilde{P} + \tilde{Z})$ if $\acute{A}$ is C-stable,

(ii) $\check{A}$ is C-stable if $\tilde{\Pi} \geq (\tilde{P} + \tilde{Z})$.

Proof: See [8], [9]. ■

III. Main results

In this section, we set up the main theorem by constructing two positive semidefinite matrix sequences P_k and Z_k ($k = 0, 1, \cdots$), and we also prove that the sequences P_k is monotonically non-decreasing and converges to the unique stabilizing solution Π (which is also positive semidefinite) of DARE (1) if such a solution exists.

Theorem 15: Let $A, B_1, B_2, C,\ D_1$ be the real matrices appearing in the system Σ_d. Suppose assumptions 1-5 hold and there exists a stabilizing solution Π, which is also positive semidefinite, of DARE (1). Define $\Theta : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ as in (25). Let $\bar{A},\ \bar{B}_1,\ \bar{B}_2$ be the matrices appearing in (20). Suppose that by virtue of the mapping defined in Lemma 6, solutions Π of (1) and solutions $\tilde{\Pi}$ are related by (10).
Then

(I) two square matrix sequences Z_k and P_k can be defined for all $k \in \mathbb{Z}_{\geq 0}$ recursively as follows:

$$P_0 = \tilde{P}_0 = 0, \tag{30}$$
$$\tilde{A}_k = \bar{A} + \bar{B}_1\bar{B}_1^T\tilde{P}_k - \bar{B}_2\bar{B}_2^T\tilde{P}_k, \tag{31}$$

$\tilde{Z}_k \geq 0$ is the unique stabilizing solution of

$$0 = \Theta(\tilde{P}_k) + \tilde{Z}_k\tilde{A}_k + \tilde{A}_k\tilde{Z}_k - \tilde{Z}_k\bar{B}_2\bar{B}_2^T\tilde{Z}_k, \tag{32}$$
$$Z_k = 2M^{-T}\tilde{Z}_kM^{-1}, \tag{33}$$

and then

$$P_{k+1} = P_k + Z_k; \tag{34}$$

(II) the two sequences P_k and Z_k in part (I) have the following properties:
1) $(A + B_1B_1^TP_k, B_2)$ is D-stabilizable $\forall k \in \mathbb{Z}_{\geq 0}$,
2) $A+B_1B_1^TP_k-B_2B_2^TP_{k+1}$ is D-stable $\forall k \in \mathbb{Z}_{\geq 0}$,
3) $\Pi \geq P_{k+1} \geq P_k \geq 0\ \forall k \in \mathbb{Z}_{\geq 0}$;

(III) the limit

$$P_\infty := \lim_{k\to\infty} P_k$$

exists with $P_\infty \geq 0$. Furthermore, $P_\infty = \Pi$ is the unique stabilizing solution of DARE (1), which is also positive semidefinite.

Proof: The proof can be obtained by using Lemma 9-Lemma 14 and the argument used in the proofs of Theorem 3 in [8], [9]. It is omitted here for brevity. ■

The following corollary gives one condition under which there does *not* exist a stabilizing solution $\Pi \geq 0$ to (1).

Corollary 16: Let A, B_1, B_2, C, D_1 be the real matrices appearing in the system Σ_d. Suppose assumptions 1-5 hold. Define $\Theta : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ as in (25). If $\exists k \in \mathbb{Z}_{\geq 0}$ such that $(A+B_1B_1^TP_k, B_2)$ is not D-stabilizable, where P_k arises from iterating (31)-(34) and as defined in Theorem 15 Part (II), then there does not exist a stabilizing solution $\Pi \geq 0$ to (1).

Proof: Restatement of Theorem 15, implication $(II1)$. ■

IV. Algorithm

Let A, B_1, B_2, C, D_1 be the real matrices appearing in the system Σ_d. Suppose assumptions 1-5 hold; define $\Theta : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ as in (25). Then an iterative algorithm for finding the positive semidefinite stabilizing solution of equation (1), when it exists, is given as follows:

1) Let $P_0 = \tilde{P}_0 = 0$ and $k = 0$.
2) Compute $\bar{A},\ \bar{B}_1,\ \bar{B}_2,\ X$ and set
$$\tilde{A}_k = \bar{A} + \bar{B}_1\bar{B}_1^T\tilde{P}_k - \bar{B}_2\bar{B}_2^T\tilde{P}_k \tag{35}$$
3) Use some existing algorithms (for example the Schur method in [21]) to compute the unique real symmetric stabilizing solution $\tilde{Z}_k \geq 0$ which satisfies
$$0 = \Theta(\tilde{P}_k) + \tilde{Z}_k\tilde{A}_k + \tilde{A}_k\tilde{Z}_k - \tilde{Z}_k\bar{B}_2\bar{B}_2^T\tilde{Z}_k \tag{36}$$
4) $Z_k = 2M^{-T}\tilde{Z}_kM^{-1}$ and set $P_{k+1} = P_k + Z_k$.
5) If $\overline{\sigma}(Z_k)^2 < \Delta$, then set $\Pi = P_{k+1}$ and exit. Otherwise, go to step 6.
6) If $(A + B_1B_1^TP_{k+1}, B_2)$ is D-stabilizable, then increment k by 1 and go back to step 3. Otherwise, exit as there does not exist a real symmetric stabilizing solution $\Pi \geq 0$ satisfying (1).

From Corollary 16 we see that if the stabilizability condition in step 6 fails at some $k \in \mathbb{Z}_{\geq 0}$, then there does not exist a stabilizing solution $\Pi \geq 0$ to (1) and the algorithm should terminate (as required by step 6). But when this stabilizability condition is satisfied $\forall k \in \mathbb{Z}_{\geq 0}$, construction of the series P_k and Z_k is always possible (with conditions $(II2)$ and $(II3)$ holding at every step) and either P_k converges to Π (which is captured by step 5) or P_k just diverges to infinity, which again means that there does not exist a stabilizing solution $\Pi \geq 0$ to (1).

V. Rate of Convergence

The following theorem states that the local rate of convergence of the algorithm given in Section IV is quadratic.

Theorem 17: Given the suppositions of Theorem 15, and two sequences P_k, Z_k as defined in Theorem 15 Part I. Then there exists a $\theta > 0$ such that the rate of convergence of the series P_k is quadratic in the region $\|P_k - \Pi\| < \theta$.

Proof: The proof can be obtained by using the argument in the proof of Theorem 5 in [9] and it is omitted here for brevity. ■

VI. A Numerical Example

In this section, we will give a numerical example to show the efficiency of our algorithm. Choose

$$A = \begin{pmatrix} -1.2705 & -0.5412 & -0.0113 & -0.2640 \\ -1.6636 & -1.3335 & -0.0008 & -1.6640 \\ -0.7036 & 1.0727 & -0.2494 & -1.0290 \\ 0.2809 & -0.7121 & 0.3966 & 0.2431 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0.1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -0.1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; D_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; D_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is simple to verify that (A, B_2, C, D_1) satisfy assumptions given in this paper. For our proposed algorithm, after 5 iterations, we have

$$\Pi_1 = \begin{pmatrix} 860.1271 & 777.6440 & -86.2002 & 516.6765 \\ 777.6440 & 704.5985 & -78.4084 & 467.4395 \\ -86.2002 & -78.4084 & 8.8516 & -51.7702 \\ 516.6765 & 467.4395 & -51.7702 & 311.7143 \end{pmatrix}.$$

When the transient discrete time Riccati difference equation (see [2]) is used to solve (1), after 25 iterations, we obtain

$$\Pi_2 = \begin{pmatrix} 860.1256 & 777.6424 & -86.2000 & 516.6754 \\ 777.6424 & 704.5969 & -78.4082 & 467.4384 \\ -86.2000 & -78.4082 & 8.8515 & -51.7701 \\ 516.6754 & 467.4384 & -51.7701 & 311.7136 \end{pmatrix}.$$

If we use MATLAB command DARE (which is a standard direct method) to solve (1) with $A,\ B_1,\ B_2,\ C$ given above, we can obtain a solution Π which is almost identical to Π. The normalized error between Π and Π_1 is

$$\epsilon_1 = \frac{\|\Pi_1 - \Pi\|}{\|\Pi\|} = 1.0739 \times 10^{-11}.$$

The normalized error between Π and Π_2 is

$$\epsilon_2 = \frac{\|\Pi_2 - \Pi\|}{\|\Pi\|} = 2.0583 \times 10^{-6}.$$

From this example, we can see that our algorithm uses only 5 iterations to solve a DARE with a satisfactory accuracy, but the algorithm in [2] will need 25 iterations to obtain a solution with less accuracy. However, one potential drawback of our proposed algorithm might be its high computational burden per iteration since many matrix transformations are involved in iterations. For this example, although our algorithm only needs 5 iterations to obtain an accurate solution, the computational cost at each iteration is higher than that of the transient discrete Riccati difference equation (see [2]). To have a clearer picture, we compare the computational complexity of the algorithm in [2] with that of our algorithm. The total flop count for one iteration of the algorithm in [2] is

$$\frac{28}{3}n^3 + 2n^2 r + O(n^2) \tag{37}$$

And the total flop count for an iteration of our algorithm is

$$\frac{241}{3}n^3 + 4n^2(m_1 + m_2) + 2(n^2 r + 2nr^2) + O(n^2) \tag{38}$$

Comparing (38) with (37), we find that for each iteration our algorithm typically needs a greater flop count than the method in [2]. However, simulation results show that when the method in [2] is used to solve an H_∞ DARE, typically far more iterations will be needed when compared with our algorithm. The price to be paid for our algorithm is that more computational burden is required for each iteration. For this particular example, our algorithm has higher accuracy than the algorithm in [2] at the price of computational burden.

VII. Conclusions

In this paper, we proposed an iterative algorithm to solve DAREs arising from standard H_∞ control problems and two player zero-sum difference games. We have proved its convergence for a simple initialization $P_0 = 0$. Moreover, we have also shown that the convergence is also locally quadratic. Our algorithm also has a game theoretic interpretation, which is not provided here due to lack of space.

References

[1] T. Başar and P. Bernhard. *H_∞ Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach,* Boston: Birkhäuser, 1991.

[2] A. A. Stoorvogel and A. J. T. M. Weeren. The discrete-time Riccati equations related to H_∞ control problems. *IEEE Trans. Auto. Contr.*, Vol.39, No.3, March 1994.

[3] T. Basar. A dynamic games approach to controller design: Disturbance rejection in discrete time. Proc. CDC, Tampa, FL, 1989, pp.407-414.

[4] P. A. Iglesias and K. Glover. State space approach to discrete time H_∞ control. Int. J. Contr., vol. 54, pp. 1031-1073, 1991.

[5] D. J. N. Limebeer, M. Green, and D. Walker. Discrete time H_∞ control. Proc. CDC, Tampa, FL, 1989, pp. 392-396.

[6] A. A. Stoorvogel. The discrete time H_∞ control problem with measurement feedback. *SIAM J. Contr. Opt.,* vol. 30, pp. 182-202, 1992.

[7] B. M. Chen, A. Saberi and Y. Shamash. A Non-recursive Method for Solving the General Discrete-time Riccati Equations Related to the H_∞ Control Problem. *International Journal of Robust and Nonlinear Control,* Vol.4, 503-519, 1994.

[8] A. Lanzon, Y. Feng and B. D. O. Anderson. An iterative algorithm to solve Algebraic Riccati equations with an indefinite quadratic term. *Proceedings of European Control Conference 2007*, Greece.

[9] A. Lanzon, Y. Feng, B. D. O. Anderson and M. Rotkowitz. Computing the Positive Stabilizing Solution to Algebraic Riccati Equations with an Indefinite Quadratic Term via a Recursive Method. *IEEE Trans. Automat. Contr.* Vol. 53, No. 10, pp. 2280-2291, Novermber 2008.

[10] G. A. Hewer. An Iterative Technique for the Computation of the Steady State Gains for the Discrete Optimal Regulator. *IEEE Trans. on Auto. Contr.*,August, 1971.

[11] P. Lancaster, A. C. M. Ran and L. Rodman. Hermitian solutions of the discrete algebraic Riccati equation. *Int. J. Contr.*, vol.44, pp. 777-802, 1986.

[12] L. Silverman. Discrete Riccati equations: Alternative algorithms, asymptotic properties, and sytem theory interpretations. *Control Dynamical Systems,*vol. 12, pp.313-386, 1976.

[13] A. Al-Tamimi, F. L. Lewis and M. Abu-Khalaf. Model-free Q-Learning designs for linear discrete-time zero-sum games with application to H-infinity control. *Automatica* 43 (2007) 473-481.

[14] B. D. O. Anderson and J. B. Moore. *Linear Optimal Control*. Prentice-Hall, Englewood Cliffs, N.J., 1971.

[15] B. D. O. Anderson and S. Vongpanitlerd.*Network Analysis and Synthesis: A Modern Systems Approach.* Prentice-Hall, Englewood Cliffs, N.J., 1973.

[16] B. D. O. Anderson, K. L. Hitz, and N. D. Diem. Recursive algorithms for spectral factorization. *IEEE Trans. and Circuits Systems*, CASC21 (6): 742-750, November 1974.

[17] K. Zhou, J. C. Doyle and K. Glover, *Robust and optimal control*, Prentice Hall, Upper Saddle River, NJ; 1996.

[18] K. L. Hitz and B. D. O. Anderson. An iterative method of computing the limiting solution of the matrix Riccati differential equation. Proc. IEE, 119 (9): 1402-1406, September 1972.

[19] G. H. Golub and C. F. Van Loan. Matrix Computations. The Johns Hopkins University Press. 1989.

[20] Grigori, Laura, Nuentsa, Desire W., and Xiang Hua. Saving Flops in LU Based Shift-and-Invert Strategy (2008).

[21] A. J. Laub. A Schur Method for Solving Algebraic Riccati Equations. *IEEE Trans. Auto. Contr.*, Vol. AC-24, No.6, December 1979.