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# Analyzing Error Propagation in Range-based Multihop Sensor Localization

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*Abstract*— **Location information for sensors in wireless sensor networks (WSNs) is essential to many tasks. If sensors are mobile and are to be controlled to certain locations, localizability is indispensable. In a noisy environment, locations must be estimated, and the question of location error and its magnitude becomes important. Error is a function of the positioning and density of anchors in a WSN, and the error can propagate (i.e. increase) as sensors progressively more distant from anchors are localized. Understanding the semi-quantitative rules governing error propagation is quite helpful to designing WSNs and improving performances of localization systems. This paper deals with error propagation in 1-Dimensional WSNs with noisy range measurements in terms of the Cramér-Rao Lower Bound. Factors influencing error propagation are investigated carefully, and a divide and conquer method is presented to analyze error propagation in WSNs.**

## I. INTRODUCTION

In wireless sensor networks (WSNs), location information of sensors often plays a vital role. Knowledge of locations can be used to report the geographic origin of events, to assist in target tracking, to achieve geographic aware routing, to administer the WSN and to evaluate its coverage. When some of the sensors are mobile, especially when they are controllable to desired locations, localizability is essential. Therefore, considerable efforts have been invested in the development of localization protocols. In the literature, most studies assume the existence of a small fraction of anchor nodes, i.e. nodes whose locations are a priori known or which are equipped with additional sensing capability through which their locations are known. The remaining nodes, termed typically sensor nodes or simply sensors, besides being able to sense a variable of interest typically can sense a variable assisting in localization. Very commonly, this is range (or a surrogate, such as signal power level). The range in question is to anchors or sensors normally in the vicinity, with anchors acting as beacons which assist the normal sensors to estimate their locations [1]–[4].

The quality of the location estimates has received considerable attention in the literature. Since the Cramér-Rao Lower Bound (CRLB) gives a lower bound on the error covariance matrix for an unbiased estimate of certain parameters, it is widely used to evaluate the fundamental difficulty of an estimation problem and has becomes a powerful tool in the analysis of error behavior during sensor localization [5]–[8]. Both the Root Mean Squared Error for all sensors in a WSN and the error behaviors involving a single sensor and its neighbors are considered. What is somewhat underdeveloped is theory dealing with optimization, or approximate optimization of sensor locations, as well as rules of thumb involving the anchor density and the error statistics of range measurements giving guidance as to the WSN performance.

A preliminary network-wide issue relevant to these considerations is what is known as error propagation. If not every sensor can refer to sufficient anchors to localize itself, already localized sensors must be used as pseudo-anchors to help their unlocalized neighboring sensors become localized; this process is called multihop sensor localization. However, error propagation arises because the errors in pseudo-anchor locations are propagated into location estimates of later localized sensors. The error propagation problem was addressed and certain strategies were proposed to mitigate error propagation in some localization systems [9]–[11]. These papers were only concerned with local areas (i.e. neighborhood of sensors) in an algorithm-dependent manner. However, the overall properties of error propagation (for example rate of growth in terms of anchor density, hop counts to anchors, and connectivity) are still unknown.

In this paper, we attempt to study fundamentals of error propagation in 1-Dimensional (1-D) WSNs in an algorithm-independent manner (using the CRLB). Due to the fact that the research is just beginning, we currently restrict our work to 1-D scenarios. Specifically, we formulate the CRLB to measure the errors in location estimates and discuss how different factors influence the CRLB, namely error propagation. Moreover, a divide and conquer method is presented to analyze error propagation in WSNs and further directions are also proposed. Note that the physical embodiment of a 1-D WSN does not necessarily involve an ideal straight line. It might involve an irregular boundary of a region, or a coastline for example, as long as the curvature is small.

The remainder of this paper is organized as follows. Section II provides the formulation of the CRLB in 1-D WSNs with noisy range measurements. Section III discusses factors influencing the CRLB and introduces some basic properties. Section IV presents a divide and conquer method to analyze error propagation in WSNs. Section V concludes the paper and sheds light on future work.

## II. FORMULATION OF THE CRLB

We indicate in this section how to formulate the CRLB in 1-D WSNs.

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### A. The Problem Model of 1-D WSNs

Define a 1-D WSN $\mathcal{N}$ to be a triple $(A_m, S_n, M_l)$, where $A_m$ denotes a set with $m$ anchors, $S_n$ with $n+1$ sensors and $M_l$ with $l$ range measurements. Assume that

- all nodes, including both anchors and sensors, are deployed along a straight line;
- the ranging model is the unit disk model and range measurements are symmetric ("symmetric" means if node $i$ has the possibly noisy measurement between itself and node $j$, node $j$ is assumed to have the same measurement);
- anchor locations are known precisely by sensors within the sensing radius, and all sensor locations are unknown;
- noises in range measurements are independent additive Gaussian with mean zero and standard deviation $\sigma$;

Construct a graph $G = (V, E)$ for $\mathcal{N}$, where $V = A_m \bigcup S_n$ and $E = \{e_{ij} \mid i < j$ *and the range measurement between two nodes corresponding to vertices* $i$ *and* $j$ *exists in* $M_l\}$. Because a disconnected $G$ can be regarded as a group of connected graphs and can be studied separately in terms of connected graphs, we assume that *in this paper* $G$ *is connected*, and say $\mathcal{N}$ is connected for simplicity.

### B. CRLB

Suppose a 1-D WSN $\mathcal{N} = (A_m, S_n, M_l)$ conforms to the problem model. Given sufficient range measurements, all sensor locations can be estimated. In order to derive the CRLB in $\mathcal{N}$, it is convenient to calculate its Fisher Information Matrix (FIM) first. Define the following notation:

- $n+1$ sensors are labeled as $0, 1, 2, ..., n$ in order and starting with the leftmost one, and $S_n = \{0, 1, ..., n\}$;
- $m$ anchors are labeled as $n+1, n+2, ..., n+m$ and $A_m = \{n+1, n+2, ..., n+m\}$;
- the true location of $i$ $(0 \le i \le n+m)$ is $x_i$; $X = \{x_0, x_1, ..., x_n\}$;
- the true and noisy range measurements between $i$ and $j$ $(i < j)$ are $d_{ij}$ and $\overline{d_{ij}}$ respectively; $M_l = \{\overline{d_{ij}} | i < j (i, j \in A_m \bigcup S_n)$ *and* $\overline{d_{ij}}$ *is the range measurement between* $i$ *and* $j\}$;
- $p_{ij}$ is the probability density function of $\overline{d_{ij}}$;
- the FIM is $J_n$, an $(n+1) \times (n+1)$ matrix;

In terms of estimation terminologies, suppose $X$ is the set of parameters to be estimated and $M_l$ is the set of observations. Because of the independent Gaussian noises, the *logarithmic* likelihood function is

$$\ln f(X|M_l) = \sum_{\overline{d_{ij}} \in M_l} \ln p_{ij} \tag{1}$$

$$p_{ij} = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{(\overline{d_{ij}} - |x_i - x_j|)^2}{2\sigma^2}\} \tag{2}$$

where $i, j = 0, 1, ..., n+m$. Given a square matrix $B_n$ with order $n+1$, let $B_{ijn}$ be the entry in the $i$-th row and $j$-th column of the matrix and $B_{in}$ be the $i$-th row of the matrix with $i, j = 0, 1, ..., n$. Then,

$$J_{ijn} = E[\frac{\partial}{\partial x_i} \ln f(X|M_l) \frac{\partial}{\partial x_j} \ln f(X|M_l)] \tag{3}$$

where $i, j = 0, 1, ..., n$; $E[\cdot]$ denotes the expected value. In this paper, we suppress the coefficient $\sigma^2$ in calculations for simplicity or equivalently, assume it is 1. Then

$$J_{ijn} = \begin{cases} d_i, & \text{i=j;} \\ -1, & \text{i}\neq\text{j} \wedge (\exists \overline{d_{ij}} \in M_l \vee \exists \overline{d_{ji}} \in M_l); \\ 0, & otherwise. \end{cases} \tag{4}$$

where $d_i$ is the number of range measurements associated with sensor $i$. Since the WSN is connected, $d_i > 0$. $J_n$ is symmetric. If $J_n$ is nonsingular, we define

$$C_n = J_n^{-1} \tag{5}$$

In the matrix $C_n$, each diagonal entry is the lower bound, i.e. the CRLB, on the variance of the corresponding sensor location estimate and thus is a metric for the sensor localization accuracy; as to the system localization accuracy in this WSN, the average of all diagonal entries, the sum of all diagonal entries, or the maximal diagonal entry can be employed as a metric. Theorem 1 provides a sufficient and necessary condition for non-singularity of $J_n$ [12].

*Theorem 1:* Given a connected WSN $\mathcal{N} = (A_m, S_n, M_l)$, its FIM $J_n$ is positive-definite if and only if $A_m \neq \phi$.

## III. Factors Influencing the CRLB

In the problem model, noises in range measurements are the only source of errors. Different range measurements will result in dramatically different effects on the CRLB. Theorem 2 discusses a range measurement between two existing sensors. Theorem 3 discusses adding a new sensor and an associated range measurement and Theorem 4 discusses adding a new anchor and an associated range measurement.

*Theorem 2:* Suppose $\mathcal{N} = (A_m, S_n, M_l)$ $(A_m \neq \phi)$ is a connected WSN and its FIM is $J_n$. If there is no range measurements between sensors $i$ and $j$ $(i < j)$ in $\mathcal{N}$, let $M'_{l+1} = M_l \bigcup \{\overline{d_{ij}}\}$ and construct a new WSN $\mathcal{N}' = (A_m, S_n, M'_{l+1})$. Let $J'_n$ be the FIM of $\mathcal{N}'$, and $C_n$ and $C'_n$ be the inverses of $J_n$ and $J'_n$. Then for every sensor $s$ $(0 \le s \le n)$, $C_{ssn} \ge C'_{ssn}$.

*Proof:* Using the method of inverting perturbed matrices, see e.g. [8], we can obtain

$$C'_n = C_n - \frac{(C_{in} - C_{jn})(C_{in} - C_{jn})^T}{1 - 2C_{ijn} + C_{iin} + C_{jjn}}$$
$$C'_{ssn} = C_{ssn} - \frac{(C_{isn} - C_{jsn})^2}{1 - 2C_{ijn} + C_{iin} + C_{jjn}} \tag{6}$$

Since $C_n$ is a covariance matrix, $|C_{ijn}| \le \sqrt{C_{iin}C_{jjn}}$. Then, $C'_{ssn} \le C_{ssn}$. ■

*Remark 1:* Theorem 2 reveals that an additional range measurement between two existing sensors will either decrease the CRLB for each sensor or keep it unchanged. In contrast, removal of a range measurement will either increase the CRLB for each sensor or keep it unchanged. As a rule of thumb, for sensors $i$ and $j$, the smaller is the difference between the number of hops from them to sensor $s$, usually the less is the difference $|C_{isn} - C_{jsn}|$, and thus the less is $C_{ssn} - C'_{ssn}$ based on (6). Therefore, a range measurement between near sensors is potentially less efficient than that between distant sensors. [*In this paper, when we say one node is close to or far away from another node, the relevant measure is the number of hops between them rather than the physical distance.*]

Next, consider the introduction of a new sensor and an associated range measurement.

*Theorem 3:* Suppose $\mathcal{N} = (A_m, S_n, M_l)$ $(A_m \neq \phi)$ is a connected WSN and its FIM is $J_n$. Let $S'_{n+1} = S_n \bigcup \{n+m+1\}$ and $M'_{l+1} = M_l \bigcup \{\overline{d_{i,n+m+1}}\}$ $(0 \leq i \leq n)$, and construct a new WSN $\mathcal{N}' = (A_m, S'_{n+1}, M'_{l+1})$. Let $J'_{n+1}$ be the FIM of $\mathcal{N}'$ and $C_n$ and $C'_{n+1}$ be the inverses of $J_n$ and $J'_{n+1}$. Then $C_n$ is identical to the leading $(n+1)\times(n+1)$ submatrix of $C'_{n+1}$ and $C'_{n+1,n+1,n+1} = C_{iin} + 1$.

*Proof:* According to the calculations of the FIM,

$$J'_{n+1} = \begin{pmatrix} J_n + e_n^{(i)} {e_n^{(i)}}^T & -e_n^{(i)} \\ -{e_n^{(i)}}^T & {e_n^{(i)}}^T e_n^{(i)} \end{pmatrix}$$

where $e_n^{(i)}$ is a column vector with $n+1$ entries which are all zeros except for the $i$-th entry being 1. We can obtain

$$\begin{aligned} {J'_{n+1}}^{-1} &= \begin{pmatrix} J_n^{-1} & \cdots \\ \cdots & 1 + {e_n^{(i)}}^T J_n^{-1} e_n^{(i)} \end{pmatrix} \\ C'_{n+1} &= \begin{pmatrix} C_n & \cdots \\ \cdots & 1 + C_{iin} \end{pmatrix} \end{aligned}$$

■

*Remark 2:* Given a WSN and its $C_n$, Theorem 3 reveals that a new sensor and an associated range measurement do not change the CRLBs for sensors in the original WSN. But if there is more than one range measurement from this new sensor, according to Theorem 2, the CRLBs in the original WSN will be either decreased or kept unchanged.

Lastly, consider a range measurement from another anchor.

*Theorem 4:* Suppose $\mathcal{N} = (A_m, S_n, M_l)$ $(A_m \neq \phi)$ is a connected WSN and its FIM is $J_n$. Let $A'_{m+1} = A_m \bigcup \{n+m+1\}$ and $M'_{l+1} = M_l \bigcup \{\overline{d_{i,n+m+1}}\}$ $(0 \leq i \leq n)$, and construct a new WSN $\mathcal{N}' = (A'_{m+1}, S_n, M'_{l+1})$. Let $J'_n$ be the FIM of $\mathcal{N}'$ and $C_n$ and $C'_n$ be the inverses of $J_n$ and $J'_n$. Then for every sensor $s$ $(0 \leq s \leq n)$, $C_{ssn} \geq C'_{ssn}$.

*Proof:* Based on the formulation of the FIM, we have

$$J'_n = J_n + e_n^{(i)} {e_n^{(i)}}^T$$

Using the *Sherman-Morrison-Woodbury formula*, see e.g. [13], we can obtain

$$\begin{aligned} C'_n &= C_n - \frac{(C_n e_n^{(i)})(C_n e_n^{(i)})^T}{1 + {e_n^{(i)}}^T C_n e_n^{(i)}} \\ C'_{ssn} &= C_{ssn} - \frac{(C_{sin})^2}{1 + C_{iin}} \end{aligned} \tag{7}$$

Because of $C_{iin} > 0$, we derive $C'_{ssn} \leq C_{ssn}$. ■

*Remark 3:* Provided that $s = i$ in (7), we obtain $C'_{iin} = \frac{C_{iin}}{1+C_{iin}} < 1$, which reveals that no matter how large $C_{iin}$ is, a new range measurement between sensor $i$ and an anchor always makes $C'_{iin}$ be less than 1.

*Remark 4:* Based on (7), we can derive

$$r_{i,s} = \frac{C_{ssn} - C'_{ssn}}{C_{ssn}} \frac{\rho_{si}^2}{1 + \frac{1}{C_{iin}}} \tag{8}$$

where $i, s = 0, ...n$, $\rho_{si}$ denotes the correlation coefficient between location errors of sensors $i$ and $s$, and $r_{i,s}$ measures the relative reduction in the CRLB for sensor $s$. The coefficients $\rho_{si}$ depend on network topologies and may require lengthy calculation for their determination. Simulations however reveal $\rho_{si}$ decreases roughly linearly as the number of hops between sensors $i$ and $s$ increases. Thus, according to (8) the influence of a new anchor on the CRLB dies off at a quadratic speed in a direction opposite to this anchor.

*Remark 5:* Given a WSN and a new anchor, the optimal placement of this anchor will maximize $\sum_{s=0}^{n}(C_{ssn} - C'_{ssn})$, which, see (7), is equivalent to deciding the value of $i$. The coefficient $|\rho_{si}|$ reaches the maximum 1 when $s = i$. Assume that $\rho_{si}$ increases monotonically with $s$ increasing from 0 to $i$ and decreases monotonically from $i$ to $n$. By observing (8), for a small $C_{iin}$, $\frac{1}{1+\frac{1}{C_{iin}}}$ is also small and a large $\rho_{si}^2$ is multiplied by a small $C_{ssn}$, which probably leads to a small sum; on the other hand, for a large $C_{iin}$, $\frac{1}{1+\frac{1}{C_{iin}}}$ is also large and a large $\rho_{si}^2$ is multiplied by a large $C_{ssn}$, which probably leads to a large sum. Thus, a new anchor should be put near the sensor with the largest CRLB. Because sensors near anchors always have small CRLBs, new anchors should not be placed near these sensors and hence anchors should not be clustered. Essentially, adding a new anchor will probably result in more than one additional range measurement. But if we assume these range measurements are taken into account one by one, based on this remark, influences of other range measurements will be trivial compared with that of the first one. Actually, which one should be the first does not matter.

*Remark 6:* It is evident that the theorem still holds even if there is a new range measurement from an existing anchor rather than from a new anchor.

## IV. Analysis in 1-D WSNs

In this section, we propose a divide and conquer method to analyze error propagation in arbitrary 1-D WSNs.

### A. Theory

We first provide a lemma that will be useful in the later analysis.

*Lemma 1:* Suppose $\mathcal{N} = (A_m, S_n, M_l)$ $(A_m \neq \phi)$ is a connected WSN and its FIM is $J_n$. Let $A'_{m+1} = A_m \bigcup \{n\}$ and $S'_{n-1} = S_n - \{n\}$, and construct a new WSN $\mathcal{N}' = (A'_{m+1}, S_{n-1}, M_l)$. Let $J'_{n-1}$ be the FIM of $\mathcal{N}'$, and $C_n$ and $C'_{n-1}$ be the inverses of $J_n$ and $J'_{n-1}$. For every sensor $s$ $(0 \leq s \leq n-1)$, $C'_{s,s,n-1} \leq C_{ssn}$.

*Proof:* Based on the formulation of the FIM, we have

$$\begin{aligned} J_n &= \begin{pmatrix} J'_{n-1} & B_{n-1} \\ B_{n-1}^T & d_n \end{pmatrix} \\ J_n^{-1} &= \begin{pmatrix} (J'_{n-1} - d_n B_{n-1} B_{n-1}^T)^{-1} & \cdots \\ \cdots & \cdots \end{pmatrix} \end{aligned}$$

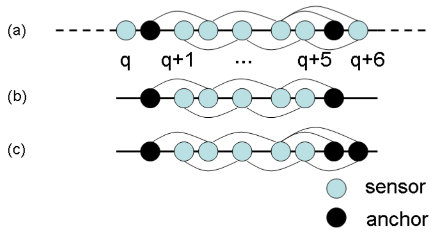


Fig. 1. (a) A WSN (both the curves connecting two nodes and the straight lines between two adjacent nodes represent range measurements) and a sensor subset $S_p = \{q+1, q+2, q+3, q+4, q+5\}$. (b) Upper bound network associated with $S_p$. (c) Lower bound network associated with $S_p$.

where $B_{n-1}$ is a column vector consisting of the leading $n$ entries in the column vector $J_{nn}$. Suppose the leading $n \times n$ submatrix of $C_n = (J_n)^{-1}$ is $V_{n-1}^{-1}$, which is positive-definite since $C_n$ is positive-definite. Then,

$$\begin{aligned} V_{n-1} &= J'_{n-1} - d_n B_{n-1} B_{n-1}^T \\ {J'_{n-1}}^{-1} &= V_{n-1}^{-1} - \frac{d_n (V_{n-1}^{-1} B_{n-1})(V_{n-1}^{-1} B_{n-1})^T}{1 + d_n B_{n-1}^T V_{n-1}^{-1} B_{n-1}} \\ C'_{s,s,n-1} &= V_{s,s,n-1}^{-1} - \frac{d_n ((V_{s,n-1}^{-1})^T B_{n-1})^2}{1 + d_n B_{n-1}^T V_{n-1}^{-1} B_{n-1}} \\ B_{n-1}^T V_{n-1}^{-1} B_{n-1} &= \begin{pmatrix} B_{n-1}^T & 0 \end{pmatrix} C_n \begin{pmatrix} B_{n-1} \\ 0 \end{pmatrix} \end{aligned}$$

Since $B_{n-1}^T V_{n-1}^{-1} B_{n-1} > 0$ and $d_n > 0$, we can obtain $C'_{s,s,n-1} \le V_{s,s,n-1}^{-1}$ and $C'_{s,s,n-1} \le C_{ssn}$. ■

This lemma reveals that replacing a sensor by an anchor and keeping all range measurements unchanged decrease the CRLBs or keep them unchanged. Now, to propose the divide and conquer method, we introduce the concepts of *upper bound network* and *lower bound network*.

*Definition 1:* Suppose $\mathcal{N} = (A_m, S_n, M_l)$ is a connected WSN. Given a subset $S_p \subseteq S_n$, define two networks:

1) Upper bound network $\mathcal{N}^{(u)} = (A_r, S_s, M_t)$, where $S_s = S_p$, $A_r = \{i \mid \exists \overline{d_{ij}} \in M_l, i \in A_m \text{ and } j \in S_s\}$ and $M_t = \{\overline{d_{ij}} \mid \exists \overline{d_{ij}} \in M_l, i, j \in S_s \bigcup A_r\}$;
2) Lower bound network $\mathcal{N}^{(l)} = (A_u, S_v, M_w)$, where $S_v = S_p$, $A_u = \{i \mid (\exists \overline{d_{ij}} \in M_l) \vee (\exists \overline{d_{ji}} \in M_l), i \in (S_n - S_v) \bigcup A_r \text{ and } j \in S_v\}$ and $M_w = \{\overline{d_{ij}} \mid \exists \overline{d_{ij}} \in M_l, i \in S_v\}$;

According to Definition 1, $\mathcal{N}^{(u)}$ is constructed by eliminating sensors not in $S_p$, anchors not directly connected to sensors in $S_p$ and all range measurements associated with these eliminated nodes; $\mathcal{N}^{(l)}$ is constructed by the same operations as in $\mathcal{N}^{(u)}$ after replacing sensors not in $S_p$ but directly connected to sensors in $S_p$ by new anchors, as illustrated in Fig. 1. Whether $\mathcal{N}^{(u)}$ and $\mathcal{N}^{(l)}$ are connected depends on $S_p$, and however if $\mathcal{N}^{(u)}$ is connected, $\mathcal{N}^{(l)}$ must be connected.

The following theorem describes the relationships between the CRLBs for sensor set $S_p$ in three different networks.

*Theorem 5:* Given a connected WSN $\mathcal{N} = (A_m, S_n, M_l)$ and a subset $S_p \subseteq S_n$, construct the upper bound network $\mathcal{N}^{(u)} = (A_r, S_s, M_t)$ and the lower bound network $\mathcal{N}^{(l)} = (A_u, S_v, M_w)$. If $\mathcal{N}^{(u)}$ is connected and $A_r \neq \phi$, let $c_i$, $c_i^{(u)}$ and $c_i^{(l)}$ be the CRLBs for sensor $i$ in $S_p$ in the three networks respectively. Then, $c_i^{(l)} \le c_i \le c_i^{(u)}$.

*Proof:* Without loss of generality, let $S_p = \{0, 1, ..., p\}$. Define the FIMs of the three networks as $J_n, J_p^{(u)}$ and $J_p^{(l)}$ respectively. Because both $\mathcal{N}$ and $\mathcal{N}^{(u)}$ are connected and $A_r \neq \phi$, the three FIMs are all positive-definite. Then define $C_n, C_p^{(u)}$ and $C_p^{(l)}$ for their inverses respectively and $C_p$ for the leading $(p+1) \times (p+1)$ submatrix of $C_n$.

In $\mathcal{N}$, replace $q$ sensors not in $S_p$ but directly connected to sensors in $S_p$ by new anchors and the new FIM is

$$J'_{n-q} = \begin{pmatrix} J_p^{(l)} & \mathbf{0} \\ \mathbf{0} & \cdots \end{pmatrix}$$

Because this replacement neither empties the anchor set nor disconnects the WSN, $J'_{n-q}$ is positive-definite.

$${J'_{n-q}}^{-1} = \begin{pmatrix} {J_p^{(l)}}^{-1} & \mathbf{0} \\ \mathbf{0} & \cdots \end{pmatrix}$$

From Lemma 1, every diagonal entry in $C_p^{(l)}$ is less than or equal to the corresponding entry in $C_n$. Thus, $c_i^{(l)} \le c_i$.

Secondly, we can transform $\mathcal{N}^{(u)}$ into $\mathcal{N}$ by adding sensors, anchors and range measurements. According to Theorem 2, 3 and 4, all these operations will not increase the CRLBs for sensors in $S_p$. Obviously, $c_i \le c_i^{(u)}$. ■

The subset $S_p$ plus associated range measurements forms a segment of $\mathcal{N}$ and based on Theorem 5, the CRLBs in $S_p$ can be bounded by using two additional networks associated with $S_p$. If the bounds are tight, studying error propagation in $\mathcal{N}$ is converted into studying the associated bounds. And we shall introduce a technique to generate tight bounds.

### B. Partitioning Networks and Simulations

We use anchors as boundaries to partition a connected WSN $\mathcal{N}$ into segments. For each segment, its associated upper bound network $\mathcal{N}^{(u)}$ and lower bound network $\mathcal{N}^{(l)}$ have two categories of topologies, as illustrated in Fig. 2: (a) anchors are at both the leftmost side and the rightmost side; (b) an anchor(s) is at the leftmost side or the rightmost side but not both. The latter category is rare and only occurs when there are no anchors at the leftmost side or the rightmost side of $\mathcal{N}$. Next, define the concept of unit network.

*Definition 2:* A 1-D connected WSN is a unit network if and only if all anchors are placed at the leftmost side and/or the rightmost side of the WSN.

Obviously, both $\mathcal{N}^{(u)}$ and $\mathcal{N}^{(l)}$ are unit networks. $\mathcal{N}^{(u)}$ is constructed by removing range measurements between sensors in different segments. We can transfer $\mathcal{N}^{(u)}$ to $\mathcal{N}^{(l)}$ by placing (maybe zero) new anchors at the leftmost side and/or the rightmost side and range measurements associated with these anchors. According to section III, these modifications have little influence on reducing CRLBs for sensors in $\mathcal{N}^{(u)}$ and consequently, the CRLBs in $\mathcal{N}^{(u)}$ are close to those in $\mathcal{N}^{(l)}$. Hence, the bounds are tight.

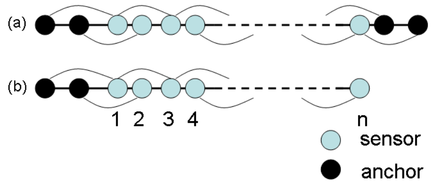


Fig. 2. (a) A unit network with anchors at the leftmost side and the rightmost side. (b) A unit network with anchors at the leftmost side.

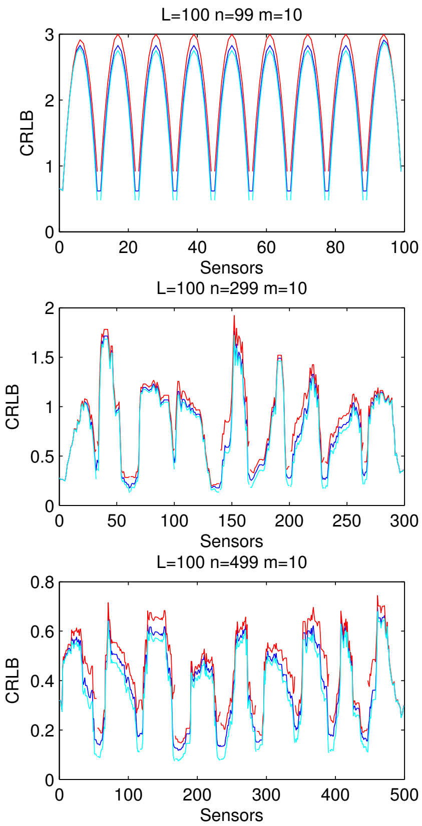


Fig. 3. The CRLBs in instance networks (blue), the upper bound networks (red), and the lower bound networks (cyan).

Simulations were conducted in Matlab to verify the effectiveness of this partitioning strategy. In the simulations, $n+1$ sensors are randomly distributed in the range $[0, L]$, $m$ anchors are equally spaced with two of them at $0$ and $L$ respectively, and the resulting networks are connected. Anchors partition networks into segments and a lower bound network and an upper bound network are easily obtained for each segment. The CRLBs in these bounding networks and the original networks are plotted in Fig. 3 and the CRLB for each sensor in the original networks is tightly bounded by those in its associated upper and lower bound networks, which can be used to approximate the original networks.

More importantly, in a WSN removing all range measurements between sensors belonging to different segments results in a new WSN which is in fact the concatenation of all upper bound networks associated with the original WSN and can be used to approximate the original WSN in terms of the CRLB. In other words, the range measurements between sensors belonging to different segments are trivial for they have little influence on the CRLBs.

### C. Properties of Error Propagation in Unit Networks

As the analysis of error propagation in WSNs can be transformed to analysis in a certain number of unit networks, it is valuable to study error propagation in unit networks.

One type of regular unit network has the number of range measurements associated with each sensor, or the node degree in the associated graph, is identical and thus the FIMs are symmetric band Toeplitz matrices. *Analytic formulas* for the CRLBs in such regular unit networks have been obtained in [12]. The following theorem establishes the relation between regular and irregular unit networks in terms of the CRLB.

*Theorem 6:* $\mathcal{N} = (A_m, S_n, M_l)$ $(A_m \neq \phi)$ is a connected unit network. Let $d_{max}$ and $d_{min}$ be the maximal and minimal node degrees in $\mathcal{N}$. Construct two regular unit networks $\mathcal{N}_1$ and $\mathcal{N}_2$ having $n+1$ sensors but different node degrees $k_1$ and $k_2$ $(k_1 \leq k_2)$ respectively. Suppose $C_n$, $C_n^{k_1}$ and $C_n^{k_2}$ are the inverses of the FIMs for the three unit networks. If $k_1 \leq d_{min} \leq d_{max} \leq k_2$, for every sensor $s$ $(0 \leq s \leq n)$, $C_{ssn}^{k_1} \geq C_{ssn} \geq C_{ssn}^{k_2}$.

*Proof:* If $k_1 \leq d_{min} \leq d_{max} \leq k_2$, $\mathcal{N}_1$ can be transferred into $\mathcal{N}$ by adding range measurements between sensors or between sensors and anchors. Based on Theorem 2 and 4, these operations do not decrease the CRLBs and thus $C_{ssn}^{k_1} \geq C_{ssn}$. Similarly, we have $C_{ssn} \geq C_{ssn}^{k_2}$. ■

Theorem 6 reveals that in a unit network, according to $d_{min}$ and $d_{max}$, the CRLB for each sensor can be analytically bounded by using regular unit networks. When the difference between $d_{min}$ and $d_{max}$ is small, the bounds are tight; otherwise, the bounds are useless. If sensors are randomly distributed in a unit network, the difference can always be large. For this case, let the mean value and standard deviation of the node degree be $x$ and $\delta$ respectively. Then one could postulate that an approximation to performance would follow from a regular unit network calculation in which the parameters $d_{min}$ and $d_{max}$ are approximated by $\overline{d_{min}} = x - \delta$ and $\overline{d_{max}} = x + \delta$ respectively.

We simulate in Matlab for this approximate method by generating four instances of WSNs where anchors are at both the leftmost side and the rightmost side. $L = 100$ and $n$ varies in the different instances. As shown in Fig. 4, the upper and lower bounds by using $\overline{d_{min}}$ and $\overline{d_{max}}$ are much tighter than using $d_{min}$ and $d_{max}$. Though the curves corresponding to the CRLBs in the instances are not as smoothly parabolic as in regular unit networks, they display some similar properties, such as increasing at a quadratic speed with the maximum occurring around the middle sensor of the network. Moreover, the maximal CRLB in the upper bound network is an approximation to the maximal CRLB in the associated original network.

### D. Further Directions

We note further directions for investigation based on aforementioned analysis.

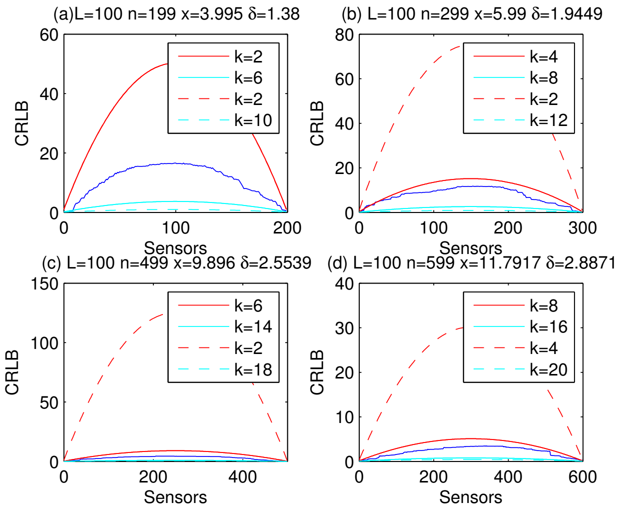


Fig. 4. Simulations for bounding CRLB in unit networks with randomly distributed sensors by regular unit networks. The blue curve denotes the CRLB in the unit networks, the red solid curve and the cyan solid curve denote the upper bound and lower bound by using $\overline{d_{min}}$ and $\overline{d_{max}}$, and the dashed curves denote bounds by using exact $d_{min}$ and $d_{max}$. Note that the red solid curve and the red dashed curve overlap in (a).

*1) The connection between system localization accuracies and deployment parameters in WSNs:* This connection is critical in WSNs, because it can indicate how to deploy a WSN to achieve a certain system localization accuracy. Given the often random nature of WSNs in practice, it is too hard to find an exact solution, but it makes sense to seek rules of thumb for design purposes.

Suppose we use the maximal CRLB in a WSN to measure the system localization accuracy and aim to guarantee it to be less than a threshold. Suppose sensors are randomly distributed in this WSN. As the WSN can be approximated with the concatenation of a certain number of unit networks based on subsection IV-B, we only need to require the maximal CRLB in every unit network to be less than this threshold. Because these unit networks have common deployment parameters as the WSN does, analyzing the maximal CRLB in these unit networks depends on the deployment parameters and thus system localization accuracies are related with the deployment parameters. However, further effort is necessary to implement and verify this idea.

*2) The placement of anchors:* In our model, given a WSN in which sensors have been deployed, the optimal placement of anchors should minimize the sum of CRLBs for all sensors, which is intrinsically an optimization problem. Here we give a preliminary discussion.

According to Remark 5, a new anchor should be placed near the sensor with the largest CRLB in a WSN to efficiently reduce the sum of CRLBs. If sensors are randomly distributed in a WSN and two anchors have been placed at the leftmost and rightmost points respectively, based on subsection IV-C the maximum of the CRLBs will occur around the middle sensor which is also around the middle of the WSN, and therefore a new anchor should be placed in the middle of the WSN. By iteratively applying this strategy, anchors are finally equally spaced in the WSN. Hence, we conjecture that distributing anchors with equal spacing should be optimal in WSNs with randomly distributed sensors.

## V. CONCLUSIONS

In this paper, we investigated the error propagation problem in 1-D WSNs as measured by the CRLB in an algorithm-independent manner. We theoretically proved the influences of several factors on the CRLB and proposed a divide and conquer method to simplify the analysis of error propagation in WSNs with randomly distributed sensors. Besides the further directions proposed in subsection IV-D, how different sensor distributions effect error propagation is also an important problem. However, the study of error propagation in 2D WSNs offers more practical benefits.

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