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Verifying nonlinear controllers for stability utilizing closed-loop noisy data

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***Abstract*— A framework for addressing a potential instability problem in adaptive control and iterative identification and controller design algorithms is proposed. Suppose an unknown plant is stabilized by a known controller, some knowledge of this stable closed-loop system is available, and the use of a new controller to replace the current stabilizing controller becomes imminent. Our analysis results assume that the 'unknown' plant and the controllers are all nonlinear. We further develop a data-based test which utilizes a limited amount of experimental data from an existing stable closed-loop (a plant in connection with a linear stabilizing controller) for verifying that the introduction of a new *nonlinear* controller will stabilize the unknown plant.**

I. Introduction

In many adaptive control methodologies, especially in iterative identification and controller design algorithms, switching may occur from an existing controller to a newly designed controller [1], [2], [3]. The new controller is determined using closed-loop measurements reflecting the existing controller, usually contaminated with noise. Assuring stability and acceptable performance of the closed-loop with the new controller prior to the actual switching to the new controller is extremely desirable. However, many contemporary adaptive control design methodologies do not explicitly rule out the possibility of placing a destabilizing controller in the closed-loop [4]; the core of the difficulty is that closed-loop measurements taken with the existing controller may not constitute reliable information to draw conclusions about a new controller, if that new controller is significantly different to the existing controller, and even if with the existing controller an accurate (though not perfect) closed-loop model is available. Iterative identification and controller redesign can even lead to the insertion of a destabilizing controller [4].

The problem of making a stability prediction becomes even harder when any combination of the plant and/or controllers are nonlinear. This is partly due to the fact that there exist fewer tools for analyzing nonlinear systems. In this context, our proposed results not only advance the existing results for the LTI plant and controllers in [5], but also provide additional analysis tools for nonlinear systems. In particular, we extend the applicability of the kernel representation of a nonlinear system as a generalization for the existing linear time-invariant (LTI) results for controller update in an adaptive control setting.

We build on our analysis results of Section III to propose data-based tests in Section IV for verifying that the introduction of a new known *nonlinear* controller will stabilize an unknown LTI plant. The verification test uses a limited amount of noisy input-output experimental data obtained from the plant connected to an existing known stabilizing LTI controller. The scenario of switching from a linear to a nonlinear controller is advocated when one seeks to improve some aspects of closed-loop performance without sacrificing some other aspects. For example, a nonlinear controller is used in place of a linear controller in [6] to achieve a faster rise-time without increasing the percentage overshoot.

The structure of the paper is as follows. Section II collects the required definitions and notations from the relevant literature. In Section III, we will elucidate the problem of concern and present our nonlinear analysis results utilizing kernel representations. This leads to introduction of our data-based verification test in Section IV. Section V contains concluding remarks and future research directions.

II. Preliminaries

A. Signal and operator spaces

Let $\mathscr{L}_2^m[0,\infty)$ (in short $\mathscr{L}_2^m$ or $\mathscr{L}_2$) denote a vector space of $\mathbb{R}^m$ valued square integrable functions with norm defined by $\|f\| := (\int_0^\infty f^T f\ dt)^{1/2}$. Define a truncation operator, $\mathcal{T}_T$, on the vector space of functions mapping from $\mathbb{R}$ to $\mathbb{R}^m$ by $\mathcal{T}_T f(t) = f(t)$ if $t \leq T$, $\mathcal{T}_T f(t) = 0$ if $t > T$. Let $\mathscr{L}_{2e}^m[0,\infty)$ (in short $\mathscr{L}_{2e}^m$ or $\mathscr{L}_{2e}$) denote the extended space of functions $f : \mathbb{R}^+ \to \mathbb{R}^m$ satisfying $\mathcal{T}_T f \in \mathscr{L}_2,\ \forall T > 0$. Analogously, let $\mathscr{H}_2$ denote a vector space of matrix-valued functions $F(s)$ analytic in the open right-half plane such that $\|F\| := \sup_{\sigma>0}(\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\sigma + j\omega)|^2\ d\omega)^{1/2} < \infty$. Also, let $\mathscr{H}_\infty$ denote the space of bounded functions in the open right-half plane such that $\|F\|_\infty := \sup_{\omega\in\mathbb{R}} \bar{\sigma}[F(j\omega)] < \infty$, where $\bar{\sigma}(F)$ denotes the largest singular value of $F(s)$. We shall denote the real rational subspace of $\mathscr{H}_2$ (resp. $\mathscr{H}_\infty$) by $\mathscr{RH}_2$ (resp. $\mathscr{RH}_\infty$). Since $\mathscr{L}_2$ and $\mathscr{H}_2$ form an isomorphism through the Laplace transform (the Parseval's relations), $f(t) \in \mathscr{L}_2$ and $F(s) \in \mathscr{H}_2$ will be used interchangeably throughout this paper.

Consider an operator $\Sigma^x : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$ with an initial condition $x \in \mathcal{X}_\Sigma \subset \mathbb{R}^n$.

Definition 1: The operator Σ^x is said to be ***causal*** if $\Sigma^x(f) \in \mathscr{L}_{2e}^k$ is uniquely determined for $\forall f \in \mathscr{L}_{2e}^m$ and $\forall x \in \mathcal{X}_\Sigma$, and $\mathcal{T}_T\Sigma^x\mathcal{T}_T = \mathcal{T}_T\Sigma^x$ holds for $\forall T > 0$ and $\forall x \in \mathcal{X}_\Sigma$.

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Definition 2: The operator is said to be ***(causally) invertible*** if it is causal, $m \equiv k$ holds and there exists a causal operator $(\Sigma^x)^{-1} : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^m$ (also denoted as $(\Sigma^{-1})^x$ or Σ^{-1}) such that $\Sigma^x(\Sigma^x)^{-1} = (\Sigma^x)^{-1}\Sigma^x = I$ holds for $\forall x \in \mathcal{X}_\Sigma$, where I denotes the identity operator.

Definition 3: The operator is said to be ***bounded*** if it is causal and there exists a finite constant γ and a scalar non-negative function ϕ with $\phi(0) = 0$ such that $\|\Sigma^x(u)\|_p \leq \gamma\|u\|_p + \phi(x), \forall u \in \mathscr{L}_2^m, \forall x \in \mathcal{X}_\Sigma$. The minimum value of γ which satisfies this inequality is called the gain, denoted by $\|\Sigma\|_{p_i}$.

Definition 4: The operator is said to be ***weakly Lipschitz*** (or weakly Lipschitz continuous) if it is causal and its Lipschitz semi-norm

$$\|\mathcal{T}_T\Sigma^x\|_L := \sup_{\substack{u,\nu\in\mathscr{L}_{2e}^m \\ \mathcal{T}_T u\neq\mathcal{T}_T\nu}} \frac{\|\mathcal{T}_T\Sigma^x u - \mathcal{T}_T\Sigma^x\nu\|}{\|\mathcal{T}_T u - \mathcal{T}_T\nu\|} \tag{1}$$

is finite for every $T > 0$ and $x \in \mathcal{X}_\Sigma$.

Definition 5: The operator Σ^x is said to be ***smoothing*** if it is weakly Lipschitz and for every $T > 0$, $\gamma > 0$ and $x \in \mathcal{X}_\Sigma$ there exists $t_1 = t_1(T, \gamma, x) \in (0, T)$ such that $\|\mathcal{T}_{t+t_1}(\Sigma^x\mathcal{T}_{t+t_1} - \Sigma^x\mathcal{T}_t)\|_L \leq \gamma$ holds for $\forall t \in [0, T - t_1]$.

Remark 6: [7] Some powerful results are:

- The sum (or cascade) of two weakly Lipschitz (resp. smoothing) operators is also weakly Lipschitz (resp. smoothing);
- AB is smoothing if A is smoothing and B is weakly Lipschitz; BA however is not necessarily smoothing;
- $[A, B]$ is well-posed if A is smoothing and B is weakly Lipschitz.

Lemma 7: Let $A = kI + B$ be a linear operator, where k is a constant, I is the identity and B is a smoothing operator. If C is smoothing, then AC is smoothing.

Proof: $AC = (kI + B)C = kC + BC$. As C is smoothing, kC is also smoothing. With B being smoothing, BC is also smoothing. The sum of two smoothing operators is again smoothing. ■

Definition 8: The operator $\Sigma_w^x : \mathscr{L}_{2e}^m \Rightarrow \mathscr{L}_{2e}^k$ is said to be ***parameterized*** with $w \in \mathscr{L}_{2e}^l$ if there exists an associated operator $\Sigma^x : \mathscr{L}_{2e}^l \times \mathscr{L}_{2e}^m \Rightarrow \mathscr{L}_{2e}^k$ such that $\Sigma_w^x(u) = \Sigma^x(w, u), \forall u \in \mathscr{L}_{2e}^m, \forall w \in \mathscr{L}_{2e}^l, \forall x \in \mathcal{X}_{\Sigma_w}$.

Definition 9: A parameterized operator Σ_w^x is said to be ***parametrically linearly bounded*** if there exists a finite constant γ and a scalar function ϕ with $\phi(0) = 0$ such that $\|\Sigma_w^x(u)\| \leq \gamma\|\binom{w}{u}\| + \phi(x)$ $\forall u \in \mathscr{L}_{2e}^m$, $\forall w \in \mathscr{L}_{2e}^l$, $\forall x \in \mathcal{X}_\Sigma$. The minimum value of γ is called the parametric gain and denoted by $\|\Sigma_w\|_{p_i}$.

B. Kernel representations

As a generalization of left fractional descriptions of LTI systems, we consider *kernel representations* from eg. [8].

Definition 10: Consider a causal operator $P : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$ with an initial condition space $\mathcal{X}_P$. Then a causal operator $R_P^{x_P} : \mathscr{L}_{2e}^m \times \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$, $\forall x_P \in \mathcal{X}_P$ is called a ***kernel representation*** of P if for $\forall x_P \in \mathcal{X}_P$ and $\forall u \in \mathscr{L}_{2e}^m$, $y = P^{x_P}u \Leftrightarrow R_P^{x_P}(u, y) = 0$ holds with $y \in \mathscr{L}_{2e}^k$.

Definition 11: A kernel operator $R_P^{x_P}$ is ***well-defined*** if there exists the causal operator $(R_P^{x_P})^\# : \mathscr{L}_{2e}^m \times \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$ such that $y = (R_P^{x_P})^\#(u, z) \Leftrightarrow R_P^{x_P}(u, y) = z, \forall x_P \in \mathcal{X}_P$, $\forall u \in \mathscr{L}_{2e}^m$ and $y, z \in \mathscr{L}_{2e}^k$.

All kernel representations are assumed to be well-defined.

Definition 12: A bounded operator $R_\Sigma^{x_P} : \mathscr{L}_{2e}^m \times \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$ is ***coprime*** if there exists a bounded operator $M^{x_P} : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^m \times \mathscr{L}_{2e}^k$ such that $R_\Sigma^{x_P} M^{x_P} = I$ $\forall x_P \in \mathcal{X}_\Sigma$.

The feedback configuration $[P, C]$ in the kernel representation is shown in Fig. 1.

Definition 13: Consider $P : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$ with $\mathcal{X}_P$ and $C : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^m$ with $\mathcal{X}_C$. Suppose we have kernel representations of P and C as $R_P : \mathscr{L}_{2e}^m \times \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$ and $R_C : \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^m$, respectively. If R_P and R_C are interconnected to form a feedback loop as shown in Fig. 1, then a ***closed-loop kernel representation*** $R_{[P,C]} : \mathscr{L}_{2e}^m \times \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^m$ is defined as

$$(z_P, z_C) := R_{[P,C]}^{(x_p,x_c)}(u, y) = \begin{pmatrix} R_P(u, y) \\ R_C(y, u) \end{pmatrix}, \tag{2}$$

for $\forall(x_P, x_C) \in \mathcal{X}_{PC}$, where $\mathcal{X}_{PC} := \mathcal{X}_P \times \mathcal{X}_C$.

Definition 14: [9] $[P, C]$ with a weakly Lipschitz kernel representation $R_{[P,C]} : (u, y) \mapsto (z_P, z_C)$ of Fig. 1 is ***null well-posed*** if $\forall(x_P, x_C) \in \mathcal{X}_{PC}$, $R_{[P,C]}^{-1} : (z_P, z_C) \mapsto (u, y)$ exists and it is weakly Lipschitz. Also, $[P, C]$ is ***null internally stable*** if it is null well-posed and $R_{[P,C]}^{-1}$ is bounded.

C. A modified version of Small Gain Theorem

We shall present a modified version of the small gain theorem, originally discussed in [10], for a general operator P and a parameterized operator C_d, as shown in Fig. 2.

Theorem 15: Consider the system depicted in Fig. 2 with an operator $P : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$ and a parameterized operator $C_d : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^m$ with $d \in \mathscr{L}_{2e}$. Let us define $w := u - C_d(e)$ and $r := e - P(u)$, where $e := (y + r)$. Suppose P is bounded and C_d is parametrically linearly bounded. If $\|P\|_{p_i}\|C_d\|_{p_i} < 1$, then the mapping $H_d : \binom{r}{w} \mapsto \binom{y}{u}$ is parametrically linearly bounded.

Proof: The proof is very similar to the original proof in [10], but one shall instead use the notion of parameterically linearly boundedness in Definition 8 for C_d. ■

III. Problem Set-up and Analysis Results

Given a stable closed-loop $[P, C_0]$ with an unknown P and a known C_0, how can one verify if the introduction of a new known controller C_1 will stabilize the unknown P in advance?

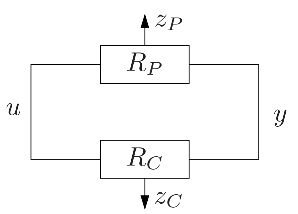

Fig. 1. Kernel configuration $[P, C]$.

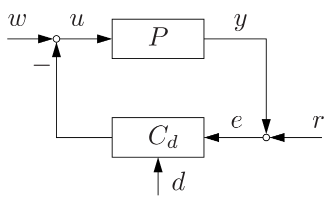

Fig. 2. Modified feedback setting.

For the LTI case, the aforementioned problem has well-established solutions [5], [11], [12]. However, for the nonlinear case the soultion is not straightforward. We shall extend the LTI analysis results of [5], [11], [12] to include the cases where the plant and/or controllers are nonlinear. In the sequel, we develop analysis tools utilising kernel representations as left fractional descriptions are generally absent for nonlinear systems.

Lemma 16: Consider a closed-loop kernel operator $R_{[P,C]}$ as in (2) with $R_P : \mathscr{L}_{2e}^m \times \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$ and $R_C : \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^m$ as shown in Fig. 1. Then $R_{[P,C]}$ is weakly Lipschitz (resp. bounded) ***iff*** R_P and R_C are both weakly Lipschitz (resp. bounded).

Proof: Given R_P and R_C are weakly Lipschitz (resp. bounded), by Definition 4 (resp. Definition 3), they are causal and their Lipschitz norms (resp. gains) are finite. Since $R_{[P,C]}$ is simply an interconnected collection of R_P and R_C, it is also causal and elementary properties of norms can show that its Lipschitz norm (resp. gain) is finite. Similar arguments hold for the other direction of the proof. ■

One can construct a mapping from $R_{[P,C_I]}$ to $R_{[P,C_J]}$, where C_I and C_J are two different controllers used with P.

Lemma 17: Let R_P, R_{C_I} and R_{C_J} be bounded and weakly Lipschitz kernel representations for the operators P, C_I and C_J, respectively. Suppose $R_{[P,C_I]}$ and $R_{[P,C_J]}$ are kernel representations of $[P,C_I]$ and $[P,C_J]$, respectively. Suppose $[P,C_I]$ is null internally stable; then one can define

$$Q_{C_I}^{C_J} : \mathcal{Z}_{PC_I} \to \mathcal{Z}_{PC_J} := R_{[P,C_J]}^{x_{PC_J}} \circ [R_{[P,C_I]}^{x_{PC_I}}]^{-1}, \quad (3)$$

where $Q_{C_I}^{C_J}(z_P, z_{C_I}) = ([Q_{C_I}^{C_J}]_1(z_P, z_{C_I}), [Q_{C_I}^{C_J}]_2(z_P, z_{C_I}))$, and there holds $[Q_{C_I}^{C_J}]_1(z_P, z_{C_I}) = z_P \in \mathcal{Z}_P$.

Proof: For an external input $(z_P, z_{C_I}) \in \mathcal{Z}_{PC_I}$, $[P,C_I]$ has the plant input u and output y related by

$$R_{[P,C_I]}^{x_{PC_I}}(u,y) = (z_P, z_{C_I}), \quad (4)$$

which includes $R_P^{x_P}(u,y) = z_P$. Since it is given that $[P,C_I]$ is null internally stable (ie. $[R_{[P,C_I]}^{x_{PC_I}}]^{-1}$ exists, is weakly Lipschitz and bounded), we have the inverse relationship of (4), $[R_{[P,C_I]}^{x_{PC_I}}]^{-1}(z_P, z_{C_I}) = (u,y)$, such that $R_{[P,C_I]}^{x_{PC_I}} \circ [R_{[P,C_I]}^{x_{PC_I}}]^{-1} = [R_{[P,C_I]}^{x_{PC_I}}]^{-1} \circ R_{[P,C_I]}^{x_{PC_I}} = I$. When $R_P^{x_P}(u,y) = z_P$ and $R_{C_J}^{x_{C_J}}(y,u) = z_{C_J}$ are connected as an interconnection, it becomes $R_{[P,C_J]}^{x_{PC_J}}(u,y) = (z_P, z_{C_J})$. Hence, we can define $Q_{C_I}^{C_J}$ as shown in (3)

$$\begin{aligned} Q_{C_I}^{C_J}(z_P, z_{C_I}) &= R_{[P,C_J]}^{x_{PC_J}} \circ [R_{[P,C_I]}^{x_{PC_I}}]^{-1}(z_P, z_{C_I}) \\ &= R_{[P,C_J]}^{x_{PC_J}}(u,y) = (z_P, z_{C_J}). \quad (5) \end{aligned}$$

Thus $[Q_{C_I}^{C_J}]_1(z_P, z_{C_I}) = z_P$. ■

Lemma 18: The projection, Proj : $\mathscr{L}_{2e}^m \times \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^m$, defined as $\text{Proj}(a,b) = a$, is weakly Lipschitz and bounded.

Proof: The proof is straightforward. ■

Now, we shall introduce an experimental setting for the nonlinear plant and the two controllers. Let R_P and R_{C_0} be the kernel representations of P and C_0, respectively, where $R_P(u,y) = w$ and $R_{C_0}(y,u) = r$. If we assume that $[P,C_0]$ is null internally stable, we have a bounded operator, $R_{[P,C_0]}^{-1} : (w,r) \mapsto (u,y)$. Then, one can attach $R_{C_1} : (y,u) \mapsto z$, the kernel operator of C_1, to $R_{[P,C_0]}^{-1}$ as shown in Fig. 3.

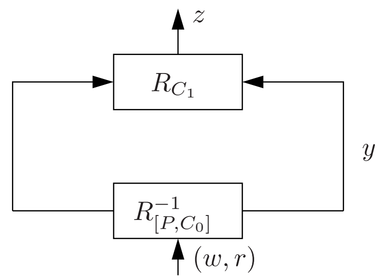

Fig. 3. Experiment setting for nonlinear case in Kernel Representation.

Theorem 19: Let R_P, R_{C_0} and R_{C_1} be bounded and weakly Lipschitz kernel representations for the operators P, C_0 and C_1, respectively. Suppose $R_{[P,C_0]}$ and $R_{[P,C_1]}$ are kernel representations of $[P,C_0]$ and $[P,C_1]$, respectively, and assume $[P,C_0]$ is null internally stable (ie. $[R_{[P,C_0]}^{x_{PC_0}}]^{-1}$ exists, is weakly Lipschitz and bounded). Then one can define a family of mappings $T_w : r \in \mathcal{Z}_{C_0} \mapsto z \in \mathcal{Z}_{C_1}$ parameterized by w

$$T_w(r) := R_{C_1}^{x_{C_1}} \circ [R_{[P,C_0]}^{x_{PC_0}}]^{-1}(w,r) \quad (6)$$

as shown in Fig. 3. Then the following are equivalent:

(a) $[P,C_1]$ is null internally stable;

(b) $T_w^{-1} : z \in \mathcal{Z}_{C_1} \mapsto r \in \mathcal{Z}_{C_0}$ exists, is weakly Lipschitz and parametrically bounded.

Proof: Since $[P,C_0]$ is assumed to be null internally stable, Lemma 17 gives

$$Q_{C_0}^{C_1} : \mathcal{Z}_{PC_0} \to \mathcal{Z}_{PC_1} := R_{[P,C_1]}^{x_{PC_1}} \circ [R_{[P,C_0]}^{x_{PC_0}}]^{-1}, \quad (7)$$

where

$$[Q_{C_0}^{C_1}]_1(w,r) = w. \quad (8)$$

One should note that

$$T_w(r) \equiv [Q_{C_0}^{C_1}]_2(w,r). \quad (9)$$

(a ⇒ b) Now suppose $[P,C_1]$ is null internally stable (ie. by Definition 14, $[R_{[P,C_1]}^{x_{PC_1}}]^{-1}$ exists, is weakly Lipschitz and bounded). By using Lemma 17 again, one can find

$$Q_{C_1}^{C_0} : \mathcal{Z}_{PC_1} \to \mathcal{Z}_{PC_0} := R_{[P,C_0]}^{x_{PC_0}} \circ [R_{[P,C_1]}^{x_{PC_1}}]^{-1}, \quad (10)$$

where

$$[Q_{C_1}^{C_0}]_1(w,z) = w. \quad (11)$$

Note that $Q_{C_1}^{C_0}$ is, in fact, the inverse of $Q_{C_0}^{C_1}$. Now define

$$S_w(z) := [Q_{C_1}^{C_0}]_2(w,z) \quad (12)$$

and show that it is the inverse of T_w for arbitrary but fixed w. First, observe that

$$\begin{aligned} &T_w \circ S_w(z) = T_w \circ [Q_{C_1}^{C_0}]_2(w,z) \text{ via (12)} \\ &= [Q_{C_0}^{C_1}]_2 \left(w, [Q_{C_1}^{C_0}]_2(w,z)\right) \text{ via (9)} \\ &= [Q_{C_0}^{C_1}]_2 \left([Q_{C_1}^{C_0}]_1(w,z), [Q_{C_1}^{C_0}]_2(w,z)\right) \text{ via (11)} \\ &= [Q_{C_0}^{C_1}]_2 \circ Q_{C_1}^{C_0}(w,z) = R_{C_1}^{x_{C_1}} \circ [R_{[P,C_1]}^{x_{PC_1}}]^{-1}(w,z) = z. \quad (13) \end{aligned}$$

Second, note that

$$
\begin{aligned}
&S_w \circ T_w(r) = S_w \circ [Q_{C_0}^{C_1}]_2(w, r) \text{ via (9)}\\
&= [Q_{C_1}^{C_0}]_2 \left(w, [Q_{C_0}^{C_1}]_2(w, r)\right) \text{ via (12)}\\
&= [Q_{C_1}^{C_0}]_2 \left([Q_{C_0}^{C_1}]_1(w, r), [Q_{C_0}^{C_1}]_2(w, r)\right) \text{ via (8)}\\
&= [Q_{C_1}^{C_0}]_2 \circ Q_{C_0}^{C_1}(w, r) = R_{C_0}^{x_{C_0}} \circ [R_{[P,C_0]}^{x_{PC_0}}]^{-1}(w, r) = r. \quad (14)
\end{aligned}
$$

Hence we have that

$$
\begin{aligned}
T_w^{-1}(z) := S_w(z) &= [Q_{C_1}^{C_0}]_2(w, z)\\
&= R_{C_0}^{x_{C_0}} \circ [R_{[P,C_1]}^{x_{PC_1}}]^{-1}(w, z) \quad (15)
\end{aligned}
$$

exists, and T_w^{-1} is weakly Lipschitz since $R_{C_0}^{x_{C_0}}$ and $[R_{[P,C_1]}^{x_{PC_1}}]^{-1}$ are weakly Lipschitz, and the cascade is also weakly Lipschitz (See Remark 6). Furthermore, T_w^{-1} is parametrically bounded since the cascade of two bounded operators is bounded.

(a $\Leftarrow$ b) Suppose that for fixed w, there exists $T_w^{-1} : z \in \mathcal{Z}_{C_1} \mapsto r \in \mathcal{Z}_{C_0}$ such that

$$
T_w \circ T_w^{-1} = T_w^{-1} \circ T_w = I, \quad (16)
$$

with T_w^{-1} weakly Lipschitz and parametrically bounded.

If we define

$$
W(w, z) := (w, T_w^{-1}(z)) = (w, r), \quad (17)
$$

then

$$
\begin{aligned}
W \circ Q_{C_0}^{C_1}(w, r) &= W\left([Q_{C_0}^{C_1}]_1(w, r), [Q_{C_0}^{C_1}]_2(w, r)\right) \text{ via (7)}\\
&= \left([Q_{C_0}^{C_1}]_1(w, r), T_w^{-1} \circ [Q_{C_0}^{C_1}]_2(w, r)\right) \text{ via (17)}\\
&= \left(w, T_w^{-1} \circ [Q_{C_0}^{C_1}]_2(w, r)\right) \text{ via (8)}\\
&= \left(w, T_w^{-1} \circ T_w(r)\right) = (w, r) \quad \text{via (16).}
\end{aligned}
$$

Similarly, we also have

$$
\begin{aligned}
Q_{C_0}^{C_1} \circ W(w, z) &= Q_{C_0}^{C_1}(w, r) \text{ via (17)}\\
&= \left([Q_{C_0}^{C_1}]_1(w, r), [Q_{C_0}^{C_1}]_2(w, r)\right) \text{ via (7)}\\
&= \left(w, [Q_{C_0}^{C_1}]_2(w, r)\right) \text{ via (8)}\\
&= (w, T_w(r)) = (w, z).
\end{aligned}
$$

Hence, W is the inverse of $Q_{C_0}^{C_1}$ (ie. $W(w, z) = Q_{C_1}^{C_0}(w, z)$). This implies that $Q_{C_1}^{C_0}$ exists for all input (w, z) and since

$$
Q_{C_1}^{C_0} : \mathcal{Z}_{PC_1} \to \mathcal{Z}_{PC_0} := R_{[P,C_0]}^{x_{PC_0}} \circ [R_{[P,C_1]}^{x_{PC_1}}]^{-1}, \quad (18)
$$

one can readily conclude that $[R_{[P,C_1]}^{x_{PC_1}}]^{-1}$ exists and $T_w^{-1}(z) = R_{C_0}^{x_{C_0}} \circ [R_{[P,C_1]}^{x_{PC_1}}]^{-1}(w, z)$. To finish off, we need to show that $[R_{[P,C_1]}^{x_{PC_1}}]^{-1}$ is weakly Lipschitz and bounded.

Observe first that $W_1(w, z) = w$ is a projection operator and by Lemma 18, W_1 is weakly Lipschitz and bounded. Since $W_2(w, z) = T_w^{-1}(z)$ is weakly Lipschitz and parametrically bounded by hypothesis, $Q_{C_1}^{C_0} = W$ is weakly Lipschitz and bounded (as each component of $W(w, z) = (W_1(w, z), W_2(w, z))$ is weakly Lipschitz and bounded).

Since $[P, C_0]$ is assumed to be null internally stable (ie. $[R_{[P,C_0]}^{x_{PC_0}}]^{-1}$ exists and weakly Lipschitz and bounded), one can conclude $[R_{[P,C_1]}^{x_{PC_1}}]^{-1} = [R_{[P,C_0]}^{x_{PC_0}}]^{-1} \circ Q_{C_1}^{C_0}$ is weakly Lipschitz (Remark 6) and bounded (as the cascade of two bounded operators is also bounded). Given Definition 14, $[P, C_1]$ is null internally stable. ∎

If P was known, our Theorem 19 would provide a method to analyze C_1 before it is inserted into a stable closed-loop $[P, C_0]$. However, we want to verify whether the new C_1 will stabilize P using only a *limited* data collected from $[P, C_0]$, where P is unknown. This is explored in the next section. Before that, the following lemma connects the LTI results in [5], [11], [12] to Theorem 19.

Lemma 20: Suppose the hypotheses of Theorem 19 hold and consider the setting in Fig. 3. If we assume P, C_0 and C_1 are all LTI and define $z := T_w(r)$ and $\tilde{z} := T_0(r)$, then we have $T_0^{-1} : r \mapsto \tilde{z}$ exists *iff* $T_w^{-1} : r \mapsto z$ exists.

Proof: Since P, C_0 and C_1 are all LTI, we have

$$
\begin{aligned}
T_w(r) &= R_{C_1}^{x_{C_1}} \circ [R_{[P,C_0]}^{x_{PC_0}}]^{-1}(w, r)\\
&= R_{C_1}^{x_{C_1}} \circ [R_{[P,C_0]}^{x_{PC_0}}]^{-1}(0, r) + R_{C_1} \circ [R_{[P,C_0]}]^{-1}(w, 0)\\
&= T_0(r) + T_w(0). \quad (19)
\end{aligned}
$$

Note that the initial condition is considered on the term $T_0(r)$. One should also notice that for a fixed w, $T_w(0)$ can be regarded as a constant for all r.

($\Rightarrow$) Suppose T_0^{-1} exists (ie. $T_0(r) = \tilde{z} \Leftrightarrow T_0^{-1}(\tilde{z}) = r$). Since $z = \tilde{z} + T_w(0)$ (or $\tilde{z} = z - T_w(0)$) , we have

$$
T_0(r) = z - T_w(0) \quad \Leftrightarrow \quad r = T_0^{-1}(z - T_w(0)) \quad (20)
$$

If we define $S_w(z) := T_0^{-1}(z - T_w(0))$, then we have

$$
\begin{aligned}
T_w(S_w(z)) &= T_w(T_0^{-1}(z - T_w(0))) = T_w(r) = z,\\
S_w(T_w(r)) &= T_0^{-1}(z - T_w(0)) = T_0^{-1}(\tilde{z}) = r.
\end{aligned}
$$

Hence S_w is, in fact the inverse of T_w, that is

$$
T_w^{-1}(z) := T_0^{-1}(z - T_w(0)). \quad (21)
$$

($\Leftarrow$) If T_w^{-1} exists for all w, then T_0^{-1} also exists as it is a special case with $w = 0$. ∎

IV. Proposed Data-based Stability Tests

We aim to build on the existing LTI results of [5], [11], [12] to develop a generalized data-based test for the nonlinear plant and controllers utilizing Theorem 19; this is not straightforward. Also, the kernel representation lacks stability analysis tools that can be readily applied. Thus, we shall reformulate the problem of interest. Let us assume that C_1 is nonlinear and has a particular structure $C_1 = C_1^L + C_1^{NL}$, where C_1^L denotes the linear part and C_1^{NL} denotes the nonlinear part. Additionally, $[P, C_0]$ is assumed to be stable, where the components P and C_0 are LTI. We will treat this nonlinear C_1 using our modified small gain theorem.

Let $\mathbf{C}_0 = \tilde{V}_0^{-1}\tilde{U}_0$ be a left coprime factorization over $\mathscr{RH}_\infty$. Since we assume that $C_1 = C_1^L + C_1^{NL}$, one can express the LTI part, $\mathbf{C}_1^L$, using a left coprime factorization over $\mathscr{RH}_\infty$ as $\mathbf{C}_1^L := (\tilde{V}_1^L)^{-1}\tilde{U}_1^L$. Since

$$C_1 = (\tilde{V}_1^L)^{-1}\tilde{U}_1^L + C_1^{NL} = (\tilde{V}_1^L)^{-1}(\tilde{U}_1^L + \tilde{V}_1^L C_1^{NL}), \quad (22)$$

by Definition 10 one can express a kernel operator for C_1 as

$$R_{C_1}(y,u) = \begin{bmatrix} -(\tilde{U}_1^L + \tilde{V}_1^L C_1^{NL}) & \tilde{V}_1^L \end{bmatrix} \begin{pmatrix} y \\ u \end{pmatrix}. \quad (23)$$

Furthermore, we separate this into $R_{C_1} = R_{C_1}^L + R_{C_1}^{NL}$, where

$$R_{C_1}^L(y,u) := [-\tilde{U}_1^L \ \tilde{V}_1^L] \begin{pmatrix} y \\ u \end{pmatrix} \quad (24)$$

$$R_{C_1}^{NL}(y,u) := -\tilde{V}_1^L C_1^{NL}(y). \quad (25)$$

One can build an experiment setting as shown in Fig. 4. From Fig. 4, we define $T_w : r \mapsto z$ as in (6)

$$\begin{aligned} z &= T_w(r) = R_{C_1} \circ R_{[P,C_0]}^{-1}(w,r) \\ &= [-\tilde{U}_1^L \quad \tilde{V}_1^L] \circ R_{[P,C_0]}^{-1}(w,r) + R_{C_1}^{NL} \circ R_{[P,C_0]}^{-1}(w,r) \\ &= (T_w^L + T_w^{NL})r \end{aligned} \quad (26)$$

where $z_1 := T_w^L(r) = [-\tilde{U}_1^L \quad \tilde{V}_1^L] \circ R_{[P,C_0]}^{-1}(w,r)$ and $z_2 := T_w^{NL}(r) = R_{C_1}^{NL} \circ R_{[P,C_0]}^{-1}(w,r)$ providing $z = z_1 + z_2$. We shall now show how to use the small gain theorem, discussed in Section II-C, to provide a sufficient condition for the internal stability of $[P, C_1]$.

Theorem 21: Let $[P, C_0]$ be null internally stable. Let $C_0 = \tilde{V}_0^{-1}\tilde{U}_0$ be a left coprime factorization over $\mathscr{RH}_\infty$. Suppose $C_1 = C_1^L + C_1^{NL}$, where the LTI part, $C_1^L = (\tilde{V}_1^L)^{-1}\tilde{U}_1^L$, is a left coprime factorization over $\mathscr{RH}_\infty$ and the nonlinear part, C_1^{NL}, has a kernel representation $R_{C_1}^{NL}$ as (25). Suppose $R_{[P,C_0]}$ and $R_{[P,C_1^L]}$ are kernel representations of $[P, C_0]$ and $[P, C_1^L]$, respectively. Consider the configuration in Fig. 4 and define the mappings $T_w^L : r \mapsto z_1$ and $T_w^{NL} : r \mapsto z_2$ to be as in (26). If the following hold:
(a) $[P, C_1^L]$ is internally stable;
(b) $R_{C_1}^{NL}$ is smoothing;
(c) T_w^{NL} is parametrically linearly bounded;
(d) $\|T_w^{NL}\|_{2_i} \|(T_0^L)^{-1}\|_{2_i} < 1$,
then $[P, C_1]$ is null internally stable.

Proof: Since $[P, C_0]$ is null internally stable, the condition (a) provides that $(T_w^L)^{-1} : z_1 \mapsto r$ exists, is weakly Lipschitz and parametrically linearly bounded via Theorem 19 (ie. $r = (T_w^L)^{-1} z_1$). In Fig. 4,

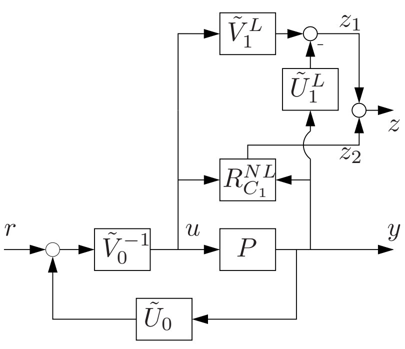

Fig. 4. Nonlinear experiment setting.

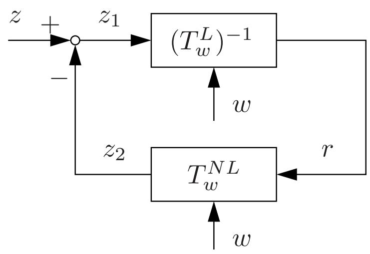

Fig. 5. Feedback interconnection of $[(T_w^L)^{-1}, T_w^{NL}]$.

$$z_2 = T_w^{NL}(r) = T_w^{NL} \circ (T_w^L)^{-1} z_1. \quad (27)$$

Since $z = z_1 + z_2$, we have

$$z = z_1 + [T_w^{NL} \circ (T_w^L)^{-1}] z_1 = [I + T_w^{NL} \circ (T_w^L)^{-1}] z_1. \quad (28)$$

This mapping can be considered as the standard closed-loop $[(T_w^L)^{-1}, T_w^{NL}]$ with an external input z with one parameter w as shown in Fig. 5. Since $[P, C_0]$ is null internally stable, $R_{[P,C_0]}^{-1}$ is weakly Lipschitz. Together with $R_{C_1}^{NL}$ being smoothing (condition (c)), $T_w^{NL} = R_{C_1}^{NL} \circ R_{[P,C_0]}^{-1}(w,r)$, given in (26), is also smoothing via Remark 6. Since we have showed that $(T_w^L)^{-1}$ is weakly Lipschitz at the beginning of this proof, we conclude that $[(T_0^L)^{-1}, T_w^{NL}]$ is null well-posed (Remark 6).

Hence, the inverse mapping of (28) exists as

$$\begin{aligned} & z_1 = [I + T_w^{NL} \circ (T_w^L)^{-1}]^{-1} z \\ & \Leftrightarrow z - z_2 = [I + T_w^{NL} \circ (T_w^L)^{-1}]^{-1} z \\ & \Leftrightarrow z_2 = z - [I + T_w^{NL} \circ (T_w^L)^{-1}]^{-1} z \\ & \Leftrightarrow z_2 = [T_w^{NL} \circ (T_w^L)^{-1}][I + T_w^{NL} \circ (T_w^L)^{-1}]^{-1} z. \end{aligned} \quad (29)$$

Here, the definition of $T_w^L : r \mapsto z_1$ is analogous to the definition of $T_w : r \mapsto z$ in Lemma 17 as we have C_1^L, instead of C_1. Via (21), we know that

$$(T_w^L)^{-1}(z_1) := (T_0^L)^{-1}(z_1 - (T_w^L)(0)), \quad (30)$$

where $T_w^L (T_w^L)^{-1} = (T_w^L)^{-1} T_w^L = I$. Hence, we have that

$$\begin{aligned} r &= (T_w^L)^{-1}(z_1) = (T_0^L)^{-1}(z_1 - (T_w^L)(0)) \\ &= (T_0^L)^{-1}(z_1) - (T_0^L)^{-1}(T_w^L)(0) \\ &= (T_0^L)^{-1}(z_1) + (T_0^L)^{-1}(0 - (T_w^L)(0)) \\ &= (T_0^L)^{-1}(z_1) + (T_w^L)^{-1}(0). \end{aligned} \quad (31)$$

Since $(T_w^L)^{-1}$ is parametrically linearly bounded (as shown in the beginning of this proof), $\|(T_w^L)^{-1}(0)\|_{2_i}$ has an upper bound with bounded w. We shall consider z and w as (bounded) external disturbances to the feedback connection and focus our attention on the loop gain $T_w^{NL}(T_0^L)^{-1}$ (ie. we consider a closed-loop $[(T_0^L)^{-1}, T_w^{NL}]$ with two external inputs, one being z and the other being $(T_w^L)^{-1}(0)$, with one parameter w, which is shown in Fig. 6).

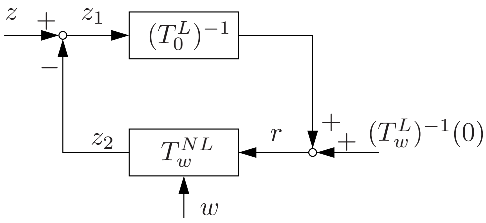

Fig. 6. Feedback interconnection of $[(T_0^L)^{-1}, T_w^{NL}]$.

From $[(T_0^L)^{-1}, T_w^{NL}]$, following conditions are satisfied:

- $(T_0^L)^{-1}$ is bounded (shown in the beginning of this proof, $(T_w^L)^{-1}$ is parametrically linearly bounded);
- T_w^{NL} is parametrically linearly bounded (condition (c));
- $\|T_w^{NL}\|_{2_i}\|(T_0^L)^{-1}\|_{2_i} < 1$ (condition (d)).

Theorem 15 verifies that all internal signals of $[(T_0^L)^{-1}, T_w^{NL}]$ are bounded if all external signals are bounded. This implies that $(T_w)^{-1} : z \mapsto r$ exists and is bounded for any bounded external input of z and w. By Theorem 19, we conclude that $[P, C_1]$ is internally stable. ■

To verify the conditions in the above-stated theorem, we shall discuss data-based experimental tests in the sequel. First, we verify the internal stability of $[P, C_1^L]$. Given that $[P, C_0]$ is assumed to be null internally stable, this can be verified using a stability test as in [5].

Second, we need to check the smoothing condition of $R_{C_1}^{NL}$. As shown in (25), $R_{C_1}^{NL}$ is simply a cascade of two operators. Note that since $\tilde{V}_1^L$ is a linear and proper transfer function, it can be expressed as $kI+B$, where k is a constant, I is the identity and B is a smoothing operator. Furthermore, C_1^{NL} can be restricted to be smoothing at the design stage. Hence, $R_{C_1}^{NL}$ will be smoothing via Lemma 7.

Third, one needs to verify whether T_w^{NL} is parametrically linearly bounded. Here, $T_w^{NL}(r) = R_{C_1}^{NL} \circ R_{[P,C_0]}^{-1}(w, r)$ as in (26), where the second term, $R_{[P,C_0]}^{-1}$, is given to be bounded (since $[P, C_0]$ is null internally stable). As discussed previously, $R_{C_1}^{NL}$ is a cascade of a stable operator $\tilde{V}_1^L$ and a nonlinear operator C_1^{NL}. Hence, if C_1^{NL} is bounded, then T_w^{NL} will be parametrically linearly bounded, which can be also restricted at the controller design stage. Furthermore, one can found an upper bound of parametric gain as

$$\|T_w^{NL}\|_{2_i} \leq \|\tilde{V}_1^L\|_{2_i}\|C_1^{NL}\|_{2_i}\|R_{[P,C_0]}^{-1}\|_{2_i}. \tag{32}$$

Here, the exact value of $\|\tilde{V}_1^L\|_{2_i}$ can be computed numerically as $\tilde{V}_1^L$ is known and an upper bound of $\|C_1^{NL}\|_{2_i}$ can be given from the controller design stage. Furthermore, an upper bound of $\|R_{[P,C_0]}^{-1}\|_{2_i}$ can be determined experimentally. Recall that $R_{[P,C_0]}^{-1}$ is a LTI mapping from $r \in \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^m$ to $y \in \mathscr{L}_{2e}^m \times \mathscr{L}_{2e}^k$ (as in Fig. 4) and we know $\|R_{[P,C_0]}^{-1}\|_{2_i}$ is equivalent to $\sup_{\omega\in\mathbb{R}} \bar{\sigma}[R_{[P,C_0]}^{-1}(j\omega)]$, which can be determined experimentally. For computing $\|(T_0^L)^{-1}\|_{2_i}$, one should notice that by adding $(\tilde{U}_1^L - \tilde{U}_0)y$ to r and $[(\tilde{V}_1^L)^{-1} - \tilde{V}_0^{-1}]r$ to u, the closed-loop interconnection $[P, C_0]$ becomes $[P, C_1^L]$. Since we have already verified that $[P, C_1^L]$ is internally stable, the modified closed-loop will be also internally stable. From this setting, one can add $\tilde{V}_0$ and $\tilde{U}_0$ to make the mapping $(T_0^L)^{-1} : z_1 \mapsto r$, hence $\|(T_0^L)^{-1}\|_{2_i}$, or at least an upper bound, can be determined experimentally. Given that the above measurements satisfy all conditions of Theorem 21, we can ensure that $[P, C_1]$ is null internally stable. The versatility of proposed results are demonstrated via numerical examples in [13].

V. Conclusions

We have presented a nonlinear extension of the LTI data-based tests in [5], [11], [12]. This particular test verifies whether the introduction of C_1 will stabilize the unknown (or partially known) plant using only limited noisy data collected from the initial closed-loop. Using the kernel representations and the small gain theorem, we achieved an additional data-based test to guarantee internal stability of $[P, C_1]$ when C_1 has a special structure.

Our current research focuses on removing the restriction used for our additional data-based test. It is desired to have such experimental tests with even allowing P and/or C_0 allowed to be nonlinear as well as C_1.

Acknowledgment

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