

# Error Propagation in Sensor Network Localization with Regular Topologies

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**Abstract**—Location information for sensors in wireless sensor networks (WSNs) is essential to many tasks. In the presence of noise, locations must be estimated and thus the errors are unavoidable. Moreover, the errors can propagate (i.e. increase) as sensors progressively more distant from anchors are localized. Understanding the rules governing error propagation is quite helpful to deploying WSNs and improving performances of localization systems. In this paper, we investigate error propagation measured by the Cramér-Rao Lower Bound (CRLB) in a type of regular 1-Dimensional WSNs whose Fisher Information Matrices are symmetric band Toeplitz matrices. Approximate analytic formulas for the CRLBs in the regular and almost regular WSNs are derived, and properties of error propagation are also obtained. In addition, we derive a magic number relating to the number of range measurements, which indicates a turning point as to system localization accuracies.

## I. INTRODUCTION

In wireless sensor networks (WSNs), knowledge of sensor locations can be used to report the geographic origin of events, to assist in target tracking, to achieve geographic aware routing, and to evaluate their coverage. Therefore, considerable effort has been invested in the development of localization systems [1]–[4]. In the literature, most studies assume the existence of a small fraction of anchor nodes, often termed simply anchors, whose locations are a priori known, and the remaining nodes are termed non-anchor nodes, sensor nodes or simply sensors, which are to be localized under the assistance of anchors. However, in the presence of noise, only location estimates can be derived and consequently the errors have received considerable attention in the literature [5]–[8].

A preliminary network-wide issue relevant to the understanding and dimension of errors is error propagation. If not every sensor can refer to sufficient anchors to localize itself, already localized sensors must be used as pseudo-anchors to help their unlocalized neighboring sensors become localized. Error propagation arises during this procedure in the sense that the errors in pseudo-anchor locations are propagated into locations of later localized sensors. Error propagation was addressed in several localization systems and some strategies were proposed to mitigate error propagation as well [9]–[11]. These papers were concerned with local areas (i.e. neighborhood of sensors). However, the overall properties of error propagation (for example rate of growth in terms of anchor density, hop counts to anchors, and connectivity) are still unknown.

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In this paper, we attempt to analyze error propagation measured by the Cramér-Rao Lower Bound (CRLB) in a type of regular 1-Dimensional (1-D) WSN that is defined as: (1) each sensor receives  $h$  ( $h$  is a constant positive integer) range measurements from its left neighboring nodes and another  $h$  from its right neighboring nodes; (2)  $h$  anchors are placed at the leftmost side and another  $h$  at the rightmost side. Since the CRLB gives a lower bound on the error covariance matrix for an unbiased estimate of certain parameters, it is widely used to evaluate the fundamental difficulty of an estimation problem and has become a powerful tool in the error analysis of sensor network localization. It turns out that calculating the CRLBs in these regular 1-D WSNs amounts to inverting symmetric band<sup>1</sup> Toeplitz<sup>2</sup> matrices. The inverse of a Toeplitz matrix is required in various areas of application of statistical, stochastic control and communication theory, and has been widely studied [12]–[14]. We derive closed-form approximate analytic formulas for the CRLBs so as to achieve some properties of error propagation. Additionally, in view of system localization accuracies, a “magic number” is obtained for the desirable number of range measurements available to a sensor. Note that the physical embodiment of a 1-D WSN does not necessarily involve an ideal straight line. It might involve an irregular boundary of a region, or a coastline for example, as long as the curvature is small.

The remainder of this paper is organized as follows. Section II provides the formulation of the CRLB in 1-D WSNs with noisy range measurements. Section III studies the regular 1-D WSNs where the number of range measurements associated with each sensor is 4 and the regular WSNs with other numbers of range measurements are investigated in Section IV. Section V concludes the paper and sheds light on future work.

## II. FORMULATION OF THE CRLB

We indicate in this section how to formulate the CRLB in 1-D WSNs.

### A. The Problem Model of 1-D WSNs

We first introduce the problem model. Define a 1-D WSN  $\mathcal{N}$  to be a triple  $(A_m, S_n, M_l)$ , where  $A_m$  denotes a set with  $m$  anchors,  $S_n$  with  $n + 1$  sensors and  $M_l$  with  $l$  range measurements. Assume that

<sup>1</sup>In matrix theory, a band matrix is a sparse matrix, whose non-zero entries are confined to a diagonal band, comprising the main diagonal and zero or more diagonals on either side.

<sup>2</sup>In linear algebra, a Toeplitz matrix, named after Otto Toeplitz, is a matrix in which each descending diagonal from left to right has identical entries.

- all nodes, including both anchors and sensors, are deployed along a straight line;
- the ranging model is the unit disk model and range measurements are symmetric (“symmetric” means if node  $i$  has the possibly noisy measurement between itself and node  $j$ , node  $j$  is assumed to have the same measurement);
- anchor locations are known precisely by sensors within the ranging radius, and all sensor locations are unknown;
- noises in range measurements are independent additive Gaussian with zero mean and standard deviation  $\sigma$ ;

Construct a graph  $G = (V, E)$  for  $\mathcal{N}$ , where  $V = A_m \cup S_n$  and  $E = \{e_{ij} \mid i < j \text{ and the range measurement between two nodes corresponding to vertices } i \text{ and } j \text{ exists in } M_l\}$ . Because a disconnected  $G$  can be regarded as a group of connected graphs and can be studied separately in terms of connected graphs, we assume that *in this paper*  $G$  is connected, and say  $\mathcal{N}$  is connected for simplicity. Obviously, in a graph associated with a regular WSN, all vertices corresponding to sensors have the same node degree, viz  $2h$ . Note thought that the problem model includes nonregular WSNs.

### B. The CRLB

Suppose a 1-D WSN  $\mathcal{N} = (A_m, S_n, M_l)$  conforms to the problem model. Given sufficient range measurements, all sensor locations can be estimated through localization algorithms. In order to derive the CRLB in  $\mathcal{N}$ , it is convenient to calculate its Fisher Information Matrix (FIM) first. Define the following notation:

- $n + 1$  sensors are labeled as  $0, 1, 2, \dots, n$  in order and starting with the leftmost one, and  $S_n = \{0, 1, \dots, n\}$ ;
- $m$  anchors are labeled as  $n + 1, n + 2, \dots, n + m$  and  $A_m = \{n + 1, n + 2, \dots, n + m\}$ ;
- the true location of  $i$  ( $0 \leq i \leq n + m$ ) is  $x_i$ ;  $X = \{x_0, x_1, \dots, x_n\}$ ;
- the true and noisy range measurements between  $i$  and  $j$  ( $i < j$ ) are  $d_{ij}$  and  $\overline{d}_{ij}$  respectively;  $M_l = \{\overline{d}_{ij} \mid i < j (i, j \in A_m \cup S_n) \text{ and } \overline{d}_{ij} \text{ is the range measurement between } i \text{ and } j\}$ ;
- the probability density function of  $\overline{d}_{ij}$  is  $p_{ij}$ ;
- the FIM is  $J_n$ , an  $(n + 1) \times (n + 1)$  matrix;

In terms of estimation terminologies, suppose  $X$  is the set of parameters to be estimated and  $M_l$  is the set of observations. Because of the independent Gaussian noises, the *logarithmic* likelihood function is

$$\ln f(X|M_l) = \sum_{\overline{d}_{ij} \in M_l} \ln p_{ij} \quad (1)$$

$$p_{ij} = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(\overline{d}_{ij} - |x_i - x_j|)^2}{2\sigma^2}\right\} \quad (2)$$

where  $i, j = 0, 1, \dots, n + m$ . Given a square matrix  $B_n$  with order  $n + 1$ , let  $B_{ijn}$  be the entry in the  $i$ -th row and  $j$ -th column of the matrix and  $B_{in}$  be the  $i$ -th row of the matrix with  $i, j = 0, 1, \dots, n$ . Then,

$$J_{ijn} = E\left[\frac{\partial}{\partial x_i} \ln f(X|M_l) \frac{\partial}{\partial x_j} \ln f(X|M_l)\right] \quad (3)$$

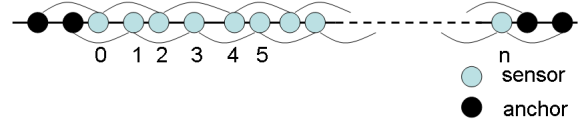


Fig. 1. A 1-D regular WSN with  $n + 1$  sensors and  $k = 4$ .

where  $i, j = 0, 1, \dots, n$ ;  $E[\cdot]$  denotes the expected value. In this paper, we suppress the coefficient  $\sigma^2$  in calculations for simplicity or equivalently, assume it is 1. Then

$$J_{ijn} = \begin{cases} d_i, & i=j; \\ -1, & i \neq j \wedge (\exists \overline{d}_{ij} \in M_l \vee \exists \overline{d}_{ji} \in M_l); \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

where  $d_i$  is the number of range measurements associated with sensor  $i$ . Obviously,  $J_n$  is symmetric. Since the network is connected,  $d_i > 0$ . If  $J_n$  is nonsingular, we define

$$C_n = J_n^{-1} \quad (5)$$

Each diagonal entry of  $C_n$  is the lower bound, i.e. the CRLB, on the variance of the corresponding sensor location estimate, and thus is a metric for the sensor localization accuracy; as to the system localization accuracy in this WSN, the average of all diagonal entries, the sum of all diagonal entries, or the maximal diagonal entry can be employed as a metric.

Theorem 1 provides the sufficient and necessary condition for non-singularity of the FIM.

*Theorem 1:* Given a connected WSN  $\mathcal{N} = (A_m, S_n, M_l)$ , its FIM  $J_n$  is positive-definite if and only if  $A_m \neq \phi$ .

*Proof:* See Appendix V-A. ■

### III. ANALYSIS IN REGULAR WSNs WITH THE NODE DEGREE 4

According to the definition of the regular WSNs, the number of range measurements or the node degree associated with each sensor, denoted  $k$ , is equal to  $2h$ , a positive even integer. For instance, given a regular WSN as illustrated in Fig. 1, based on (4) its FIM is

$$J_n = \begin{bmatrix} 4 & -1 & -1 & 0 & & 0 \\ -1 & 4 & -1 & -1 & & 0 \\ -1 & -1 & 4 & -1 & & 0 \\ 0 & -1 & -1 & 4 & & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & & 4 \end{bmatrix}$$

which is a symmetric band Toeplitz matrix. It is evident that the FIM in a regular WSN can be defined by two parameters

- $n + 1$ : the order of the matrix and the number of sensors;
- $k$ : the value of entries on the main diagonal, the number of non-zero off-diagonals, the number of range measurements associated with each sensor, and the node degree in its graph.

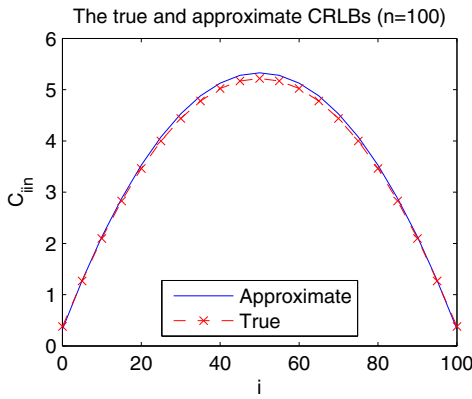


Fig. 2. The true and approximate CRLBs in a regular WSN with  $n = 100$  and  $k = 4$ .

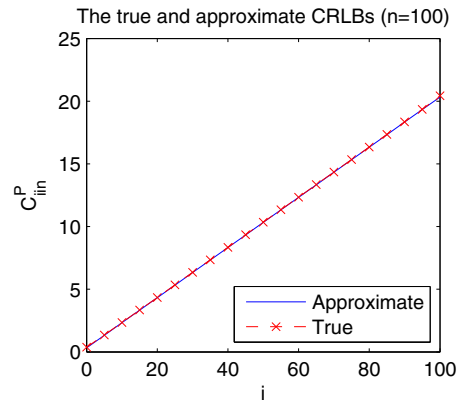


Fig. 4. The true and approximate CRLBs in WSNs, as in Fig. 3. The two lines overlap.

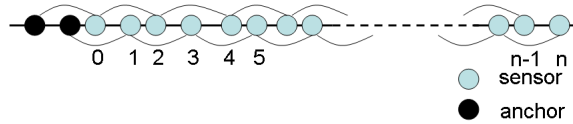


Fig. 3. A 1-D WSN with two anchors at the leftmost side 4 range measurements with each sensor (except for the two rightmost sensors).

Because each regular WSN is connected and must contain anchors, in the light of Theorem 1, its FIM is invertible and thus the CRLB exists. Since a regular WSN with  $k = 2$  is too simple, we begin with  $k = 4$  in this section and the regular WSNs with other values of  $k$  will be analyzed in the next section. Due to the limitation of the pages of this paper, we do not detail calculations. Interested readers are invited to contact the authors for more details.

#### A. The CRLB when $k = 4$

Trench [14] proposed explicit formulas to reconstruct the inverse of a Toeplitz matrix. From the Trench formulas, we can obtain

$$C_{ii}^{lin} \approx -\frac{0.200008}{n}i^2 + 0.200008i + 0.327887 \quad (6)$$

Given a regular WSN with  $n = 100$  and  $k = 4$ , we plot both the true and the approximate CRLBs according to (6) in Fig. 2 which shows that the approximate values from this formula are quite close to the exact ones.

#### B. Removing the anchors at the rightmost side

If the two anchors at the rightmost side are removed from the regular WSN shown in Fig. 1, a new WSN is derived as illustrated in Fig. 3. Let  $J_n^P$  be the FIM for this new WSN

and  $C_n^P = (J_n^P)^{-1}$ .

$$J_n^P = \begin{bmatrix} 4 & -1 & -1 & & 0 & 0 & 0 \\ -1 & 4 & -1 & & 0 & 0 & 0 \\ -1 & -1 & 4 & & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & & 4 & -1 & -1 \\ 0 & 0 & 0 & & -1 & 3 & -1 \\ 0 & 0 & 0 & & -1 & -1 & 2 \end{bmatrix}$$

Also by using the Trench formulas [14], we can obtain

$$C_{ii}^P \approx 0.200008i + 0.327887 \quad (7)$$

Again, given a WSN as illustrated in Fig. 3 with  $n = 100$ , we plot the true CRLB and the approximate CRLB based on (7) in Fig. 4. As can be seen, the two lines are almost identical, which reveals that the approximate CRLBs from the formula are extremely close to the exact ones. Note that due to the symmetry, the analysis will be the same if the anchors at the leftmost side are removed.

#### C. Properties of Error Propagation

In the first place, Fig. 2 and 4 reveal that the error is propagated, or increases, along the directions opposite to anchors: in regular WSNs like Fig. 1, the error is propagated from the two ends towards the middle of WSNs in a quadratic manner; in WSNs like Fig. 3, the error is propagated from left to right in a linear manner.

Secondly, the absolute value of the first partial derivative of  $C_{ii}^{lin}$  with respect to  $i$  indicates the speed of error propagation at sensor  $i$ . The absolute derivative in WSNs like Fig. 3 is 0.200008, which reflects that the error propagation speed is constant. While the absolute derivative in regular WSNs like Fig. 1 is  $|0.200008(1 - \frac{2i}{n})|$ , which reflects that the speed is less than or equal to that in WSNs like Fig. 3 and the closer to anchors is a sensor, the larger is the associated speed of error propagation. The reason is that sensors in regular WSNs like Fig. 1 can access anchors in both directions but those in WSNs like Fig. 3 can only access anchors in one direction.

Moreover, as the maximal CRLB can be used to measure the system localization accuracy in a WSN, based on (6) we can obtain the formula for the summit of the parabola:  $0.050002n + 0.327887$ . According to the parity of  $n$ , the maximal CRLB may be equal to or a bit less than the summit. However, it is reasonable to use  $0.050002n + 0.327887$  to approach the maximal CRLB in a regular WSN with  $n + 1$  sensors and  $k = 4$ . Hence, the maximal CRLB increases linearly with  $n$ .

By comparing (6) and (7), the quadratic term  $-\frac{0.200008}{n}i^2$  embodies the influence of the removed anchors on the CRLB for sensor  $i$ . When  $i$  is small, which means that the sensor  $i$  is far away from the removed anchors, the CRLB for sensor  $i$  is slightly affected by the anchors; on the other hand, when  $i$  is large, which means that the sensor  $i$  is close to the removed anchors, the CRLB for sensor  $i$  is greatly affected.

#### IV. INFLUENCE OF THE NODE DEGREES

In this section, we shall consider the influence of the different node degrees in regular 1-D WSNs on error propagation.

##### A. The CRLB with respect to different $k$

The key step of applying the Trench formulas [14] is to obtain the formula for the first row of the inverse of a FIM, which is determined by the determinants of two  $k \times k$  matrices. Each determinant involves  $k!$  terms. When  $k \geq 8$ , there are too many terms to be reduced so that it is almost impossible to obtain analytic formulas. But for the extreme case of  $k = 2n$ , a trick can be applied. Note that in a WSN,  $k = 2n$  means that  $2n$  anchors are required, which is certainly impractical when  $n \gg 1$ , and this is only used to introduce a lower bound on the CRLB with respect to different  $k$ . Therefore, we shall show the inverses of the FIMs with three different  $k$ : 2, 6 and  $2n$ . Define  $J_n^{(k)}$  to be the FIM in a regular WSN parameterized by  $n$  and  $k$  and  $C_n^{(k)} = [J_n^{(k)}]^{-1}$ .

When  $k = 2, 6$ , based on the Trench formulas [14], we can derive

$$\begin{aligned} C_{ii}^{(2)} &= -\frac{1}{n+2}i^2 + \frac{n}{n+2}i + \frac{n+1}{n+2} \\ C_{ii}^{(6)} &\approx -\frac{0.0714289}{n}i^2 + 0.0714289i + 0.191251 \end{aligned} \quad (8)$$

When  $k = 2n$ , the FIM is

$$J_n^{(2n)} = \begin{bmatrix} 2n & -1 & -1 & & -1 \\ -1 & 2n & -1 & & -1 \\ -1 & -1 & 2n & & -1 \\ & & & \ddots & \\ -1 & -1 & -1 & & 2n \end{bmatrix}$$

The trick is to recognize this is a circulant matrix. Its inverse is then necessarily a circulant matrix. In fact, if one constructs a  $(n+1) \times (n+1)$  matrix  $H_n$  all diagonal entries of which are  $\frac{n+1}{n(2n+1)}$  and all off-diagonal entries of which are  $\frac{1}{n(2n+1)}$ , one easily checks  $J_n^{(2n)}H_n$  is equal to the identity matrix and thus

$$C_{ii}^{(2n)} = \frac{n+1}{n(n+2)} \quad (10)$$

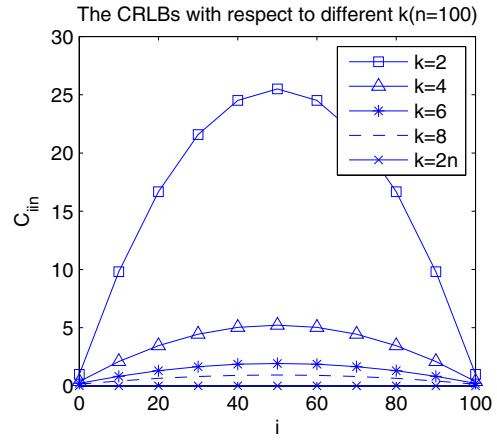


Fig. 5. The CRLBs in regular WSNs with respect to different  $k$ .

According to (6), (8), (9) and (10), we plot the CRLBs in regular WSNs with  $n = 100$  and different  $k$ , as illustrated in Fig. 5. Note that the curve of  $k = 8$  is plotted through simulations. As can be seen from Fig. 5, when  $k = 2, 4, 6$  the overall varying trends of the CRLBs are obviously concave parabolic, but when  $k = 8, 2n$  the CRLBs approach to zeros and the curves are almost straight. Moreover, the larger is  $k$ , the smaller is the CRLB for a sensor  $i$  as well as the slower is error propagated, which is also supported by Theorem 2 no matter how large  $n$  is.

*Theorem 2:* Suppose  $\mathcal{R}$  is a regular WSN with  $n+1$  sensors and the node degree  $k$  (a positive even integer). Let  $J_n^{(k)}$  be the FIM in  $\mathcal{R}$  and  $C_n^{(k)} = [J_n^{(k)}]^{-1}$ . Given two positive even integers  $k_1$  and  $k_2$  ( $k_1 > k_2$ ), for each sensor  $s$ ,  $C_{ssn}^{(k_1)} \leq C_{ssn}^{(k_2)}$ .

*Proof:* See Appendix V-B. ■

Based on Theorem 2, we know that when  $6 < k < 2n$  all the curves must lie between the curve of  $k = 6$  and the curve of  $k = 2n$ .

##### B. The Magic number

Define the average CRLB in a network

$$AC^{(k,n)} = \frac{1}{n+1} \sum_{i=0}^n C_{ii}^{(k)} \quad (11)$$

to measure the system localization accuracy. Because  $AC^{(k,n)}$  depends on both  $k$  and  $n$ , with the dependence on  $n$  being typically affine it is not straightforward to isolate the influence of  $k$ . Therefore, define the Normalized Average CRLB

$$NAC^{(k,n)} = \frac{1}{n} AC^{(k,n)} \quad (12)$$

and calculate the approximate  $NAC^{(k,n)}$  to observe better the influence of  $k$  on system localization accuracies. Table I shows the  $NAC^{(k,n)}$  based on (6), (8), (9) and (10) and the approximate  $NAC^{(k,n)}$  under the assumption of  $n \gg 1$ .

As shown in Fig. 6, the  $NAC^{(k,n)}$  dramatically decreases from  $k = 2$  to  $k = 4$  and obviously decreases from  $k = 4$  to  $k = 6$ . When  $k > 6$ , we do not plot any curves corresponding

TABLE I  
THE NORMALIZED AVERAGE CRLB

$k$	$NAC^{(k,n)}$	approximate $NAC^{(k,n)}$
2	$\frac{0.5}{n} + 0.166667$	0.166667
4	$\frac{0.294552}{n} + 0.0333347$	0.0333347
6	$\frac{0.178870}{n} + 0.0119048$	0.0119048
$2n$	$\frac{n+1}{n^2(n+2)}$	0

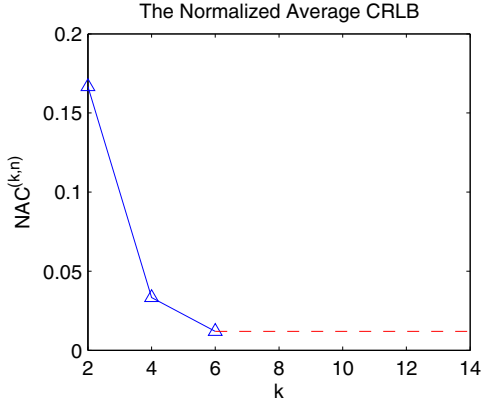


Fig. 6. The  $NAC^{(k,n)}$  in regular WSNs with respect to different  $k$ . The blue curve denotes the true values of  $NAC^{(k,n)}$  when  $k = 2, 4, 6$  and the red dashed line denotes the upper bound of  $NAC^{(k,n)}$  when  $k > 6$ .

to the  $NAC^{(k,n)}$ . From Theorem 2, we know in regular WSNs increasing the node degree  $k$  will not increase the CRLBs, and hence if  $k_1 > k_2$ ,  $NAC^{(k_1,n)} \leq NAC^{(k_2,n)}$ . So,  $NAC^{(k,n)}$  monotonically decreases with  $k$  and specifically when  $6 < k < 2n$ ,  $NAC^{(k,n)}$  monotonically decreases in the small interval  $[0, 0.0119048]$  (or the strip area between the dashed line and the horizontal axis) with  $k$  increasing.

The decrement of  $NAC^{(k,n)}$  in going from  $k = 6$  to  $k = 8$ , or to an arbitrarily large even integer which is smaller than  $2n$ , is less than one half of that going from  $k = 4$  to  $k = 6$  and even less than 8% of that going from  $k = 2$  to  $k = 6$ , which reveals that when  $k > 6$  increasing  $k$  makes little improvement on system localization accuracies. Moreover, in Fig. 7 we plot the average CRLBs derived through simulations in regular WSNs with  $n = 100$  and  $2 \leq k \leq 14$ , which is evidently consistent with  $NAC^{(k,n)}$ .

In the literature of networks, communications, etc., 6 has been recognized as a magic number relating to several topics associated with topologies [15]. For the localization in 1-D WSNs, the node degree 6 turns out to be a turning point of system localization accuracies and thus is also a magic number.

## V. CONCLUSIONS AND FUTURE WORK

In this paper, we investigated the error propagation problem in 1-D regular WSNs through the CRLB. Approximate analytic formulas for the CRLBs in regular WSNs with respect to several different node degrees were obtained. Based on the formulas, properties of error propagation had been observed. For nonregular WSNs, it would be rather difficult to have exact

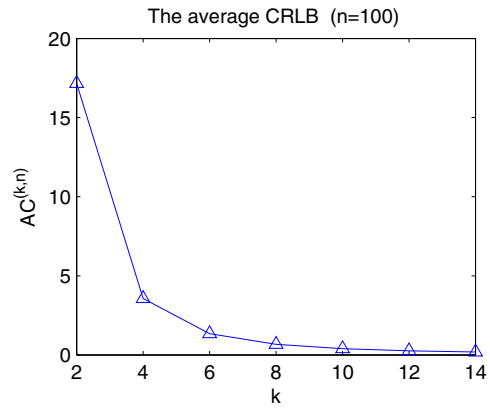


Fig. 7. The average CRLBs in regular WSNs with 101 sensors and  $k$  between 2 and 14. The data are derived through simulations.

analytic formulas describing the CRLBs and then to obtain any properties. But, it is reasonable that if a nonregular WSN is close to a regular WSN, their error propagation characteristics are similar to some extent. In other words, the regular WSNs can be used to analyze error propagation in nonregular WSNs, an aspect we are now studying.

In addition, the node degree 6 appears to be a magic number in regular 1-D WSNs with regard to system localization accuracies. We hope to extend this result to nonregular WSNs, where the node degree varies from sensor to sensor, in the sense that the average node degree 6 may be a magic number.

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## APPENDIX

### A. Proof of Theorem 1

Define  $u_i$  to be the sum of the off-diagonal entries in the  $i$ -th row (or column). In fact  $-u_i$  is the number of range measurements between sensor  $i$  and all other (non-anchor) sensors and  $u_i \leq 0$ . We know

$$d_i \geq -u_i \quad (13)$$

Equation (13) will hold with equality if and only if sensor  $i$  has no range measurements from anchors.

Define  $Y_n = [y_0, y_1, \dots, y_n]^T$  to be a column vector with  $n+1$  real entries where  $[\cdot]^T$  denotes matrix transpose. Then

$$Y_n^T J_n Y_n = \sum_{e_{ij} \in E} (y_i - y_j)^2 + \sum_{i=0}^n (d_i + u_i) y_i^2 \quad (14)$$

Obviously, the FIM  $J_n$  is positive-semidefinite. We need to prove that  $J_n$  is positive-definite, which is equivalent to the statement that  $J_n$  is nonsingular, or equivalently that  $y_0 = y_1 = \dots = y_n = 0$  is the unique solution to

$$\sum_{e_{ij} \in E} (y_i - y_j)^2 + \sum_{i=0}^n (d_i + u_i) y_i^2 = 0 \quad (15)$$

Due to the non-negativity of the two terms, we can obtain

$$\sum_{e_{ij} \in E} (y_i - y_j)^2 = 0 \quad (16)$$

$$\sum_{i=0}^n (d_i + u_i) y_i^2 = 0 \quad (17)$$

Each sensor is associated with an entry in  $Y_n$ . From (16), we know if two sensors are directly connected (i.e. a range measurement exists between them), the associated entries in  $Y_n$  are equal; hence given two sensors, if a path consisting

of only sensors exists between them, the associated entries are also equal; from (17), we know if a sensor is directly connected to an anchor, its associated entry in  $Y_n$  must be zero. Now if  $A_m = \phi$ , because  $\mathcal{N}$  is connected, paths between any two sensors only consist of sensors and all entries in  $Y_n$  are equal but their value is arbitrary, and hence  $J_n$  is positive-semidefinite; on the other hand, if  $A_m \neq \phi$ , because  $\mathcal{N}$  is connected, each sensor is either directly or indirectly connected to at least one anchor through other sensors and thus the associated entry in  $Y_n$  must be zero, and hence  $J_n$  is positive-definite. We conclude that when  $\mathcal{N}$  is connected,  $J_n$  is positive-definite if and only if  $A_m \neq \phi$ .

### B. Proof of Theorem 2

Represent  $\mathcal{R}$  with  $n+1$  sensors and the node degree  $k$  by  $\mathcal{R}_{k,n}$ . It is evident that  $\mathcal{R}_{k_2,n}$  can be transformed to  $\mathcal{R}_{k_1,n}$  by adding necessary anchors and range measurements. If all these operations do not increase the CRLB for each sensor, the theorem is proved.

Given a WSN  $\mathcal{N}$  (Notice that we do not emphasize whether  $\mathcal{N}$  is regular) with  $n+1$  sensors, suppose  $J_n$  is the FIM and  $C_n = J_n^{-1}$ . Assume that adding a measurement between sensor  $i$  and an anchor in  $\mathcal{N}$  results in a new WSN  $\mathcal{N}'$ . Let  $J'_n$  be the FIM in  $\mathcal{N}'$  and  $C'_n = J_n'^{-1}$ . According to the calculations of the FIM, we have

$$J'_n = J_n + e_n^{(i)} [e_n^{(i)}]^T \quad (18)$$

By using the *Sherman-Morrison-Woodbury formula*, see e.g. [16], we can obtain

$$C'_n = C_n - \frac{(C_n e_n^{(i)})(C_n e_n^{(i)})^T}{1 + e_n^{(i)T} C_n e_n^{(i)}} \quad (19)$$

$$C'_{ssn} = C_{ssn} - \frac{(C_{sin})^2}{1 + C_{iin}} \quad (20)$$

Because of  $C_{iin} > 0$ , we can derive  $C'_{ssn} \leq C_{ssn}$ .

Next, assume that adding a measurement between sensors  $i$  and  $j$  ( $i < j$ ) in  $\mathcal{N}$  results in a new WSN  $\mathcal{N}'$ . Let  $J'_n$  be the FIM in  $\mathcal{N}'$  and  $C'_n = J_n'^{-1}$ . Using the method of inverting perturbed matrices, see e.g. [8], we can obtain

$$C'_n = C_n - \frac{(C_{in} - C_{jn})(C_{in} - C_{jn})^T}{1 - 2C_{ijn} + C_{iin} + C_{jjn}} \quad (21)$$

Because  $C_n$  is a covariance matrix, we know  $|(C_n)_{ij}| \leq \sqrt{(C_n)_{ii}(C_n)_{jj}}$  and then can obtain,

$$C'_{ssn} = C_{ssn} - \frac{(C_{isin} - C_{jjsn})^2}{1 - 2C_{ijn} + C_{iin} + C_{jjn}} \quad (22)$$

$$C'_{ssn} \leq C_{ssn} \quad (23)$$

To sum up, all the operations do not increase the CRLBs and hence for each sensor  $s$ ,  $C'_{ssn} \leq C_{ssn}^{(k_2)}$ .