Nonlinear Analysis for Verifying Closed-Loop Stability before Inserting a New Controller

Sung H. Cha, Arvin Dehghani and Brian D. O. Anderson

Abstract— **Suppose a nonlinear plant is stabilized by a nonlinear controller, and further assume that the plant is unknown but the controller is known. Suppose also that the use of a new controller appears attractive. This paper extends our results for linear time-invariant systems and proposes novel nonlinear analysis results, utilizing the kernel representation of nonlinear systems, for ensuring that the introduction of the new nonlinear controller will stabilize the plant.**

I. INTRODUCTION

DURING the operation of a stable closed-loop system—consisting of a plant and a pre-designed stabilizing controller in a standard feedback setting—situations arise where the controller needs to be partially or entirely replaced by a newly designed controller [1]. Such scenarios include the situation where the plant requires the use of a new controller due to aging or damage [2], or where one seeks to improve some aspects of the closed-loop performance, e.g. introducing a nonlinear controller in place of a linear controller connected to a linear plant to secure faster rise-time without increasing the percentage overshoot [3]. In such situations, caution must be exercised to ensure that the replacing controller is in fact not destabilizing.

This assurance in practice, however, may not be easily achievable in advance, or may even be sometimes impossible to achieve [1], [4], [2], [5], [6]. Examples of such cases arise in adaptive control or iterative identification and controller design algorithms [7], [8], [9] where the plant is assumed to be unknown or partially unknown. Clearly, some form of validation tests are needed for verifying the suitability of the new controller before its insertion into the closed-loop.

For the case of linear time-invariant (LTI) plant and controllers, there exist data-based validation tests [10], [11] for verifying that the introduction of a new controller will stabilize an unknown plant, originally in a standard feedback interconnection with a stabilizing controller. However, if any of the plant and/or controllers are nonlinear, the problem of making such validation becomes harder as there exist fewer tools for analyzing nonlinear systems.

This paper lays the foundation and presents novel nonlinear system analysis tools for verifying that the introduction of a new nonlinear controller will stabilize an unknown nonlinear plant, which is originally in a standard and stable feedback interconnection with a nonlinear controller. This is achieved by advancing the existing results in [10] for the LTI case to include analysis results which are general enough to cover nonlinear plant and controllers. In particular, we extend the applicability of the 'kernel representation' of a nonlinear system [12] as a generalization for the existing LTI results.

The structure of the paper is as follows. Section II collects the required definitions and notations, followed by a review of different system representations in Section III. Section IV elaborates on the problem of our interest and bridges between the existing analysis results for the LTI case in [10]. Our novel nonlinear analysis results utilizing kernel representation are presented in Section V. Section VI summarizes our results and remarks on possible future research directions.

II. PRELIMINARIES

We shall outline relevant and necessary definitions and preliminary results from [13], [14], [10] here.

Let $\mathscr{L}_2^m[0,\infty)$ (in short $\mathscr{L}_2^m$ or $\mathscr{L}_2$) denote a vector space of Lebesgue measurable functions $f : [0,\infty) \to \mathbb{R}^m$ such that $\|f\|_2 := (\int_0^\infty f^T f \, dt)^{1/2} < \infty$. If a truncation operator $\mathcal{T}_T$ is defined on the vector space of functions mapping from $\mathbb{R}$ to $\mathbb{R}^m$ by

$$\mathcal{T}_T f(t) := \begin{cases} f(t) & t \le T \\ 0 & t > T, \end{cases} \quad (1)$$

then $\mathscr{L}_{2e}^m[0,\infty)$ (in short $\mathscr{L}_{2e}^m$ or $\mathscr{L}_{2e}$) denotes the extended Lebesgue space of functions $f : \mathbb{R} \to \mathbb{R}^m$ satisfying $\mathcal{T}_T f \in \mathscr{L}_2$, $\forall T > 0$. Alternatively, let $\mathscr{H}_2$ denote a vector space of matrix-valued functions $F(s)$ analytic in the open right-half plane (ie. $Re(s) > 0$) satisfying $\|F\|_2 := \sup_{\sigma>0}(\frac{1}{2\pi}\int_\infty^\infty |F(\sigma + j\omega)|^2 \, d\omega)^{1/2} < \infty$. Also, let $\mathscr{H}_\infty$ denote the space of bounded functions in the open right-half plane such that $\|F\|_\infty := \sup_{\omega\in\mathbb{R}} \bar{\sigma}[F(j\omega)] < \infty$, where $\bar{\sigma}(F)$ denotes the largest singular value of $F(s)$. We shall denote the real rational subspace of $\mathscr{H}_2$ (resp. $\mathscr{H}_\infty$) by $\mathscr{RH}_2$ (resp. $\mathscr{RH}_\infty$). The Parseval's relations provide that the Laplace transform yields an isomorphism between $\mathscr{L}_2$ and $\mathscr{H}_2$. Thus, $f(t) \in \mathscr{L}_2$ and $F(s) \in \mathscr{H}_2$ will be used interchangeably here. Now, consider a general operator $\Sigma^x : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$ with an initial condition $x \in \mathcal{X}_\Sigma \subset \mathbb{R}^n$.

Definition 1: The operator Σ^x is said to be ***causal*** if $\Sigma^x(f) \in \mathscr{L}_{2e}^k$ is uniquely determined for $\forall f \in \mathscr{L}_{2e}^m$ and $\forall x \in \mathcal{X}_\Sigma$, and $\mathcal{T}_T\Sigma^x\mathcal{T}_T = \mathcal{T}_T\Sigma^x$ holds for $\forall T > 0$ and $\forall x \in \mathcal{X}_\Sigma$.

Definition 2: The operator is said to be ***(causally) invertible*** if it is causal, $m \equiv k$ holds and there exists a causal operator $(\Sigma^x)^{-1} : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^m$ (also denoted as $(\Sigma^{-1})^x$ or Σ^{-1}) such that $\Sigma^x(\Sigma^x)^{-1} = (\Sigma^x)^{-1}\Sigma^x = I \; \forall x \in \mathcal{X}_\Sigma$.

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Definition 3: The operator is said to be ***bounded*** if it is causal and there exists a finite constant γ and a scalar function ϕ with $\phi(0) = 0$ such that $\|\Sigma^x(u)\|_2 \leq \gamma\|u\|_2 + \phi(x)$ for $\forall u \in \mathscr{L}_2^m$ and $\forall x \in \mathcal{X}_\Sigma$.

Definition 4: The operator is said to be ***weakly Lipschitz*** (or weakly Lipschitz continuous) if it is causal and its Lipschitz semi-norm

$$\|\mathcal{T}_T\Sigma^x\|_L := \sup_{\substack{u,\nu\in\mathscr{L}_{2e}^m \\ \mathcal{T}_T u\neq\mathcal{T}_T\nu}} \frac{\|\mathcal{T}_T\Sigma^x u - \mathcal{T}_T\Sigma^x \nu\|_2}{\|\mathcal{T}_T u - \mathcal{T}_T\nu\|_2} \qquad (2)$$

is finite for every $T > 0$ and $x \in \mathcal{X}_\Sigma$.

Remark 5: [15] The sum (or cascade) of two weakly Lipschitz operators is also weakly Lipschitz.

Definition 6: The operator $\Sigma_w^x : \mathscr{L}_{2e}^m \Rightarrow \mathscr{L}_{2e}^k$ is said to be ***parameterized*** with $w \in \mathscr{L}_{2e}^l$ if there exists an associated operator $\Sigma^x : \mathscr{L}_{2e}^{(l+m)} \Rightarrow \mathscr{L}_{2e}^k$ such that $\Sigma_w^x(u) = \Sigma^x(w, u)$ $\forall u \in \mathscr{L}_{2e}^m$, $\forall w \in \mathscr{L}_{2e}^l$ and $\forall x \in \mathcal{X}_{\Sigma_w}$. Furthermore, a parameterized operator Σ_w^x is said to be ***parametrically linearly bounded*** if there exists a finite constant γ and a scalar function ϕ with $\phi(0) = 0$ such that $\|\Sigma_w^x(u)\|_2 \leq \gamma\|(w, u)\|_2 + \phi(x)$ $\forall u \in \mathscr{L}_{2e}^m$, $\forall w \in \mathscr{L}_{2e}^l$ and $\forall x \in \mathcal{X}_\Sigma$.

Remark 7: From Definition 3 and 6, one can show that a parameterized operator is parametrically linearly bounded *if and only if* its associated operator is bounded.

III. System representations

We shall outline some conventional system representations, followed by a more general representation, known as the *kernel representation*; see [12] for more details.

A. Conventional representations

Let $P^{x_P} : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$ (resp. $C^{x_C} : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^m$) denote the *input-output representation* of a plant (resp. controller) with an initial condition $x_P \in \mathcal{X}_P$ (resp. $x_C \in \mathcal{X}_C$).

Definition 8: For a causal operator $P^{x_P} : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$, a pair of operators (M, N) is a ***right fractional representation*** (resp. left fractional representation) if $\forall x_P \in \mathcal{X}_P$ it can be factorized into $P^{x_P} = MN^{-1}$ (resp. $P^{x_P} = N^{-1}M$), where M and N are bounded and N is (causally) invertible.

For an LTI system, its input-output representation (in time-domain) has an associated analogue operator in the s-domain, the ***transfer function***. Let $\mathbf{P}(s) : U(s) \mapsto Y(s)$ denote a transfer function of $P : u \in \mathscr{L}_{2e}^m \mapsto y \in \mathscr{L}_{2e}^k$, where $U(s)$ and $Y(s)$ are the Laplace transforms of $u(t)$ and $y(t)$.

Definition 9: [16] A pair $(\mathbf{M}(s), \mathbf{N}(s)) \in \mathscr{RH}_\infty$ is ***right coprime*** (resp. left coprime) over $\mathscr{RH}_\infty$ if they have the same numbers of rows and $\exists\mathbf{X}(s), \mathbf{Y}(s) \in \mathscr{RH}_\infty$ that satisfy $\mathbf{X}(s)\mathbf{M}(s)+\mathbf{Y}(s)\mathbf{N}(s) = I$ (resp. $\mathbf{M}(s)\mathbf{X}(s)+\mathbf{N}(s)\mathbf{Y}(s) = I$).

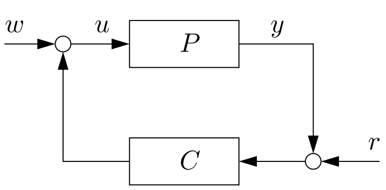

Fig. 1. Standard Feedback Configuration.

Definition 10: [14] The interconnection $[P, C]$ with $P : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$ and $C : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^m$ in the feedback setting of Fig. 1 is said to be ***well-posed*** if the mapping $H_{[P,C]} : \binom{r}{w} \mapsto \binom{y}{u}$ exists and is weakly Lipschitz. Furthermore, this interconnection is said to be ***internally stable*** if it is well-posed and $H_{[P,C]}$ is bounded.

Remark 11: [16] For the LTI and finite dimensional case where both $\mathbf{P}(s)$ and $\mathbf{C}(s)$ are in $\mathscr{RH}_\infty$, well-posedness is equivalent to the condition that $\begin{bmatrix} I & -\mathbf{C}(s) \\ -\mathbf{P}(s) & I \end{bmatrix}^{-1}$ exists and is proper. Also, internal stability is equivalent to the condition $\begin{bmatrix} I & -\mathbf{C}(s) \\ -\mathbf{P}(s) & I \end{bmatrix}^{-1} \in \mathscr{RH}_\infty$.

B. Kernel Representation

Due to the absence of a left fractional representation for nonlinear systems in most cases, we shall utilize the *kernel representation*.

Definition 12: Consider a causal operator $P : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$ with an initial condition space $\mathcal{X}_P$. Then a causal operator $R_P^{x_P} : \mathscr{L}_{2e}^m \times \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$, $\forall x_P \in \mathcal{X}_P$ is called a ***kernel representation*** of P if $\forall x_P \in \mathcal{X}_P$ and $\forall u \in \mathscr{L}_{2e}^m$ $y = P^{x_P}u \Leftrightarrow R_P^{x_P}(u, y) = 0$ holds with $y \in \mathscr{L}_{2e}^k$.

Definition 13: A kernel operator $R_P^{x_P}$ is ***well-defined*** if there exists the causal operator $(R_P^{x_P})^\# : \mathscr{L}_{2e}^m \times \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$ such that $y = (R_P^{x_P})^\#(u, z) \Leftrightarrow R_P^{x_P}(u, y) = z, \forall x_P \in \mathcal{X}_P$, $\forall u \in \mathscr{L}_{2e}^m$ and $y, z \in \mathscr{L}_{2e}^k$.

We assume all kernel representations used are well-defined.

Definition 14: A bounded operator $R_\Sigma^{x_P} : \mathscr{L}_{2e}^{(m+k)} \to \mathscr{L}_{2e}^k$ is ***coprime*** if there exists a bounded operator $M^{x_P} : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^{(m+k)}$ such that $R_\Sigma^{x_P} M^{x_P} = I$ $\forall x_P \in \mathcal{X}_\Sigma$.

The analogous feedback configuration of Fig. 1 in the kernel representation is shown in Fig. 2.

Definition 15: Consider $P : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$ with $\mathcal{X}_P$ and $C : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^m$ with $\mathcal{X}_C$. Suppose that kernel representations of P, $R_P : \mathscr{L}_{2e}^{(m+k)} \to \mathscr{L}_{2e}^k$, and C, $R_C : \mathscr{L}_{2e}^{(k+m)} \to \mathscr{L}_{2e}^m$, exist and consider the feedback interconnection in Fig. 2. Then a ***closed-loop kernel representation*** $R_{[P,C]} : \mathscr{L}_{2e}^{(m+k)} \to \mathscr{L}_{2e}^{(k+m)}$, $R_{[P,C]} = (z_P, z_C)$, is defined as

$$R_{[P,C]}^{(x_p x_c)}(u, y) := \begin{pmatrix} R_P(u, y) \\ R_C(y, u) \end{pmatrix} \quad \forall(x_P, x_C) \in \mathcal{X}_{PC} \qquad (3)$$

where $\mathcal{X}_{PC} := \mathcal{X}_P \times \mathcal{X}_C$.

Remark 16: If R_P and R_C are causal, $R_{[P,C]}$ is also causal from definition 15.

Definition 17: [14] $[P, C]$ with a weakly Lipschitz kernel representation $R_{[P,C]} : (u, y) \in \mathscr{L}_{2e}^{(m+k)} \mapsto (z_P, z_C) \in$

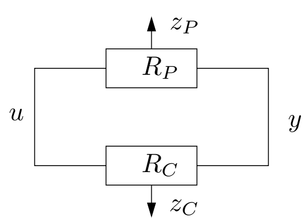

Fig. 2. Kernel Representation of $[P, C]$ without any disturbance.

$\mathscr{L}_{2e}^{(k+m)}$ of Fig. 2 is ***null well-posed*** if $\forall(x_P, x_C) \in \mathcal{X}_{PC}$, $R_{[P,C]}^{-1} : (z_P, z_C) \mapsto (u, y)$ exists and $R_{[P,C]}^{-1}$ is weakly Lipschitz. Also, $[P, C]$ is ***null internally stable*** if it is null well-posed and $R_{[P,C]}^{-1}$ is bounded.

IV. Problem Set-up and Existing Results

Our problem of interest can be described as follows: Given a plant P, which is stabilized by a known controller C_0 in a feedback setting, how can one verify if the introduction of the new controller C_1 will stabilize the P?

The existing key results for the LTI case from [10] are captured in the following theorem.

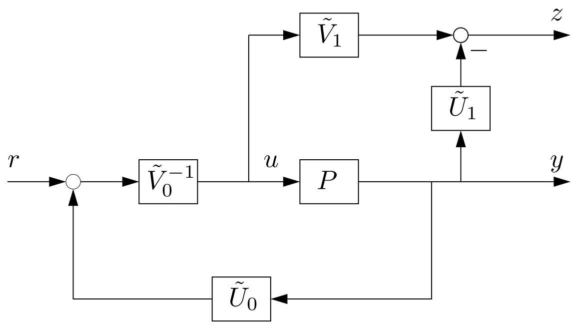

Fig. 3. Experiment setting for the LTI case: $C_0 = \tilde{V}_0^{-1}\tilde{U}_0$, $C_1 = \tilde{V}_1^{-1}\tilde{U}_1$.

Theorem 18: Let $[P, C_0]$ be internally stable. Let $C_0 = \tilde{V}_0^{-1}\tilde{U}_0$ and $C_1 = \tilde{V}_1^{-1}\tilde{U}_1$ be left coprime factorizations over $\mathscr{RH}_\infty$. Consider the configuration in Fig. 3 and define the mapping $T : r \mapsto z$ to be

$$T = [-\tilde{U}_1 \;\; \tilde{V}_1] \begin{bmatrix} P(I - C_0P)^{-1} \\ (I - C_0P)^{-1} \end{bmatrix} \tilde{V}_0^{-1}. \quad (4)$$

Let wno denote the winding number[1] of the Nyquist diagram of T and unwarg denote the unwrapped phase[2]. Then the following are equivalent:

(a) $[P, C_1]$ is internally stable;
(b) $T^{-1} \in \mathscr{RH}_\infty$;
(c) det $T(j\omega) \neq 0 \; \forall\omega$ and wno det $T = 0$;
(d) det $T(j\omega) \neq 0 \; \forall\omega$ and unwarg det $T(j\infty)$ = unwarg det $T(j0)$.

Proof: See [10]. ∎

In this paper we focus on extending the results in Theorem 18 to cover situations where P and/or C_0 and C_1 are nonlinear. In particular, we shall develop methods via the use of kernel representations (see Definitions 12, 14 and 15) for offline-based analysis of nonlinear systems of interest. However, the results can be principally utilized in developing online validation tests analogous to those presented in [10]. These tests use the available knowledge of the controllers and limited measurements on the closed-loop to infer the closed-loop stability with the introduction of the new controller C_1.

[1]The winding number wno$X$ is the number of integral multiples of 2π by which the argument of $X(j\omega)$ changes as ω moves from $-\infty$ to ∞, under the assumption that X is nonzero for all ω.

[2]The unwrapped phase is a continuous function of ω derived from the original phase function by removing the discontinuities of 2π. The process of removing the discontinuities is called unwrapping the phase [17].

V. Proposed nonlinear analysis using kernel representation

Prior to presenting our main analysis results, we shall consider the following two lemmas that provide causal and bounded relationships between two individual kernel operators R_P, R_C and their closed-loop form $R_{[P,C]}$.

Lemma 19: Consider a closed-loop kernel operator $R_{[P,C]}$ as in (3) with $R_P : \mathscr{L}_{2e}^{(m+k)} \to \mathscr{L}_{2e}^{k}$ and $R_C : \mathscr{L}_{2e}^{(k+m)} \to \mathscr{L}_{2e}^{m}$ as shown in Fig. 2. Then $R_{[P,C]}$ is weakly Lipschitz ***iff*** R_P and R_C are both weakly Lipschitz.

Proof: ($\Leftarrow$) Suppose $R_P : (u, y) \in \mathscr{L}_{2e}^{(m+k)} \mapsto z_P \in \mathscr{L}_{2e}^{k}$ and $R_C : (y, u) \in \mathscr{L}_{2e}^{(k+m)} \mapsto z_C \in \mathscr{L}_{2e}^{m}$ are weakly Lipschitz. Here R_P and R_C are causal by Definition 4. Since $R_{[P,C]}$ is simply an interconnection of two operators that maps $(u, y) \in \mathscr{L}_{2e}^{(m+k)}$ to $(z_P, z_C) \in \mathscr{L}_{2e}^{(k+m)}$, R_P and R_C being causal implies $R_{[P,C]}$ is causal. Now we need to show that $R_{[P,C]}$ has a finite Lipschitz semi-norm (i.e. $\|\mathcal{T}_T R_{[P,C]}^{x}\|_L < \infty$ for every $T > 0$ and $x_{PC} := (x_P, x_C) \in \mathcal{X}_{PC}$). Since we have R_P and R_C are weakly Lipschitz, $\|\mathcal{T}_T R_P^{x_P}\|_L < \infty$ and $\|\mathcal{T}_T R_C^{x_C}\|_L < \infty$. Then

$$\|\mathcal{T}_T R_{[P,C]}^{x_{PC}}\|_L \leq (\|\mathcal{T}_T R_P^{x_P}\|_L^2 + \|\mathcal{T}_T R_C^{x_C}\|_L^2)^{1/2} < \infty,$$

where the inequality comes from the property of 2-norm with the definition of Lipschitz semi-norm given in (2). Thus $R_{[P,C]}$ is weakly Lipschitz.

($\Rightarrow$) $R_{[P,C]}$ being causal implies each interconnected operator (R_P or R_C) is causal. Also, $\|\mathcal{T}_T R_{[P,C]}^{x_{PC}}\|_L < \infty$ implies $\|\mathcal{T}_T R_P^{x_P}\|_L < \infty$ and $\|\mathcal{T}_T R_C^{x_C}\|_L < \infty$ since $\|\mathcal{T}_T R_P^{x_P}\|_L \leq \|\mathcal{T}_T R_{[P,C]}^{x_{PC}}\|_L$ and $\|\mathcal{T}_T R_C^{x_C}\|_L \leq \|\mathcal{T}_T R_{[P,C]}^{x_{PC}}\|_L$ by the definition of $\mathscr{L}_2$-norm and Definition 15. ∎

Lemma 20: Suppose the hypothesis of Lemma 19 hold and consider the setting in Fig. 2. Then $R_{[P,C]}$ is bounded ***iff*** R_P and R_C are both bounded.

Proof: ($\Leftarrow$) Suppose $R_P : (u, y) \in \mathscr{L}_{2e}^{(m+k)} \mapsto z_P \in \mathscr{L}_{2e}^{k}$ and $R_C : (y, u) \in \mathscr{L}_{2e}^{(k+m)} \mapsto z_C \in \mathscr{L}_{2e}^{m}$ are bounded. Here R_P and R_C are causal by Definition 3. Since $R_{[P,C]}$ is simply an interconnection of two operators that maps $(u, y) \in \mathscr{L}_{2e}^{(m+k)}$ to $(z_P, z_C) \in \mathscr{L}_{2e}^{(k+m)}$, R_P and R_C being causal implies $R_{[P,C]}$ being causal. Also, by R_P and R_C being bounded we have $\exists\gamma_P, \gamma_C > 0$ and $\exists\phi_P, \phi_C$ such that

$$\|R_P^{x_P}(u, y)\|_2 \leq \gamma_P\|(u, y)\|_2 + \phi_P(x_P)$$
$$\|R_C^{x_C}(y, u)\|_2 \leq \gamma_C\|(y, u)\|_2 + \phi_C(x_C),$$

$\forall(u, y) \in \mathscr{L}_{2e}^{(m+k)}$, $\forall x_P \in \mathcal{X}_P$ and $\forall x_C \in \mathcal{X}_C$. Hence, by (3)

$$\begin{aligned}\|R_{[P,C](u,y)}\|_2 &\leq \|R_P(u, y)\|_2 + \|R_C(y, u)\|_2 \\ &\leq \gamma_{[P,C]}\|(u, y)\|_2 + \phi_{[P,C]}(x_P, x_C), \quad (5)\end{aligned}$$

where $\gamma_{[P,C]} := \gamma_P + \gamma_C$ and $\phi_{[P,C]}(x_P, x_C) := \phi_P(x_P) + \phi_C(x_C)$.

($\Rightarrow$) $R_{[P,C]}$ being causal implies each interconnected operator (R_P or R_C) is causal. Since $\|R_P(u, y)\|_2 \leq \|R_{[P,C]}(u, y)\|_2$ and $\|R_C(y, u)\|_2 \leq \|R_{[P,C]}(u, y)\|_2$ (by Definition 15), R_P and R_C are bounded. ∎

We construct a specific mapping from $R_{[P,C_I]}$ and $R_{[P,C_J]}$.

Lemma 21: Let R_P, R_{C_I} and R_{C_J} be bounded and weakly Lipschitz kernel representations for the operators P, C_I and C_J, respectively. Suppose $R_{[P,C_I]}$ and $R_{[P,C_J]}$ are kernel representations of $[P, C_I]$ and $[P, C_J]$, and $[P, C_I]$ is null internally stable. One can define

$$Q_{C_I}^{C_J} : \mathcal{Z}_{PC_I} \to \mathcal{Z}_{PC_J} := R_{[P,C_J]}^{x_{PC_J}} \circ [R_{[P,C_I]}^{x_{PC_I}}]^{-1}, \quad (6)$$

where

$$Q_{C_I}^{C_J}(z_P, z_{C_I}) = ([Q_{C_I}^{C_J}]_1(z_P, z_{C_I}), [Q_{C_I}^{C_J}]_2(z_P, z_{C_I})),$$

and there holds $[Q_{C_I}^{C_J}]_1(z_P, z_{C_I}) = z_P \in \mathcal{Z}_P$.

Proof: For an external input $(z_P, z_{C_I}) \in \mathcal{Z}_{PC_I}$, suppose that we have $[P, C_I]$ with the plant input u and output y related by

$$R_{[P,C_I]}^{x_{PC_I}}(u, y) = (z_P, z_{C_I}), \quad (7)$$

which includes $R_P^{x_P}(u, y) = z_P$. Since $[P, C_I]$ is null internally stable (ie. $[R_{[P,C_I]}^{x_{PC_I}}]^{-1}$ exists, is weakly Lipschitz and bounded), we have the inverse relationship of (7), $[R_{[P,C_I]}^{x_{PC_I}}]^{-1}(z_P, z_{C_I}) = (u, y)$, such that

$$R_{[P,C_I]}^{x_{PC_I}} \circ [R_{[P,C_I]}^{x_{PC_I}}]^{-1} = [R_{[P,C_I]}^{x_{PC_I}}]^{-1} \circ R_{[P,C_I]}^{x_{PC_I}} = I. \quad (8)$$

Note that when $R_P^{x_P}(u, y) = z_P$ and $R_{C_J}^{x_{C_J}}(y, u) = z_{C_J}$ are inter-connected, we have

$$R_{[P,C_J]}^{x_{PC_J}}(u, y) = (z_P, z_{C_J}). \quad (9)$$

We can now define $Q_{C_I}^{C_J}$ as in (6)

$$\begin{aligned} Q_{C_I}^{C_J}(z_P, z_{C_I}) &= R_{[P,C_J]}^{x_{PC_J}} \circ [R_{[P,C_I]}^{x_{PC_I}}]^{-1}(z_P, z_{C_I}) \\ &= R_{[P,C_J]}^{x_{PC_J}}(u, y) \quad \text{via (7)} \\ &= (z_P, z_{C_J}) \quad \text{via (9)} \end{aligned} \quad (10)$$

and thus $[Q_{C_I}^{C_J}]_1(z_P, z_{C_I}) = z_P$. ■

Lemma 22: The projection, Proj : $\mathscr{L}_{2e}^{(m+k)} \to \mathscr{L}_{2e}^{m}$, defined as Proj$(a, b) = a$, is weakly Lipschitz and bounded.

Proof: The proof is trivial. ■

A. Proposed Stability Verification Results

We shall establish an experimental setting analogous to Fig. 3. First, we assume that $[P, C_0]$ is internally stable (as in Theorem 18).

Let R_P be the kernel representation of P, $R_P(u, y) = w$. Via Definition 12, we have

$$[-\tilde{U}_0 \ \tilde{V}_0]\begin{pmatrix} y \\ u \end{pmatrix} = r \ \Leftrightarrow \ R_{C_0}(y, u) = r$$

and

$$[-\tilde{U}_1 \ \tilde{V}_1]\begin{pmatrix} y \\ u \end{pmatrix} = z \ \Leftrightarrow \ R_{C_1}(y, u) = z.$$

Since $[P, C_0]$ is internally stable, we have a bounded operator that maps (w, r) into (u, y). This is analogous to $R_{[P,C_0]}^{-1}$ with (w, r) as its input in the kernel representation form. The setting in Fig. 4 is formed by adding $R_{C_1} : (y, u) \mapsto z$ to $R_{[P,C_0]}^{-1}$. The nonlinear version of Theorem 18 using the kernel representation is proposed next.

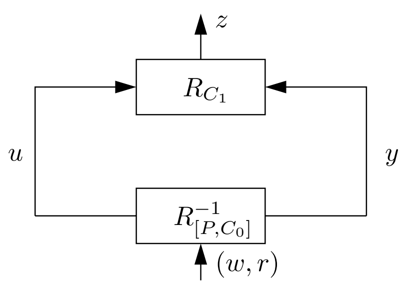

Fig. 4. Experiment setting for Kernel Representation.

Theorem 23: Let R_P, R_{C_0} and R_{C_1} be bounded and weakly Lipschitz kernel representations for the operators P, C_0 and C_1, respectively. Further suppose that $R_{[P,C_0]}$ and $R_{[P,C_1]}$ are kernel representations of $[P, C_0]$ and $[P, C_1]$ and assume $[P, C_0]$ is null internally stable (ie. $[R_{[P,C_0]}^{x_{PC_0}}]^{-1}$ exists, is weakly Lipschitz and bounded). Then one can define a family of mappings $T_w : r \in \mathcal{Z}_{C_0} \mapsto z \in \mathcal{Z}_{C_1}$ parameterized by w

$$T_w(r) := \ R_{C_1}^{x_{C_1}} \circ [R_{[P,C_0]}^{x_{PC_0}}]^{-1}(w, r) \quad (11)$$

as shown in Fig. 4. In this setting the following are equivalent:

(a) $[P, C_1]$ is null internally stable;
(b) $T_w^{-1} : z \in \mathcal{Z}_{C_1} \mapsto r \in \mathcal{Z}_{C_0}$ exists, is weakly Lipschitz and parametrically linearly bounded.

Proof: Since $[P, C_0]$ is assumed to be null internally stable, Lemma 21 gives

$$Q_{C_0}^{C_1} : \mathcal{Z}_{PC_0} \to \mathcal{Z}_{PC_1} := R_{[P,C_1]}^{x_{PC_1}} \circ [R_{[P,C_0]}^{x_{PC_0}}]^{-1}, \quad (12)$$

where

$$[Q_{C_0}^{C_1}]_1(w, r) = w. \quad (13)$$

Note that

$$T_w(r) \equiv [Q_{C_0}^{C_1}]_2(w, r). \quad (14)$$

(a $\Rightarrow$ b) Since $[P, C_1]$ is null internally stable (ie. by Definition 17, $[R_{[P,C_1]}^{x_{PC_1}}]^{-1}$ exists, is weakly Lipschitz and bounded), Lemma 21 provides that

$$Q_{C_1}^{C_0} : \mathcal{Z}_{PC_1} \to \mathcal{Z}_{PC_0} := R_{[P,C_0]}^{x_{PC_0}} \circ [R_{[P,C_1]}^{x_{PC_1}}]^{-1}, \quad (15)$$

where

$$[Q_{C_1}^{C_0}]_1(w, z) = w. \quad (16)$$

Note that it is straightforward to verify that $Q_{C_1}^{C_0}$ is, in fact, the inverse of $Q_{C_0}^{C_1}$.

Let us define

$$S_w(z) := [Q_{C_1}^{C_0}]_2(w, z) \quad (17)$$

and show that it is the inverse of T_w for arbitrary but fixed w. First, observe that

$$\begin{aligned} T_w \circ S_w(z) &= T_w \circ [Q_{C_1}^{C_0}]_2(w, z) \ \text{via (17)} \\ &= [Q_{C_0}^{C_1}]_2 \left(w, [Q_{C_1}^{C_0}]_2(w, z)\right) \ \text{via (14)} \\ &= [Q_{C_0}^{C_1}]_2 \left([Q_{C_1}^{C_0}]_1(w, z), [Q_{C_1}^{C_0}]_2(w, z)\right) \ \text{via (16)} \\ &= [Q_{C_0}^{C_1}]_2 \circ Q_{C_1}^{C_0}(w, z) \\ &= R_{C_1}^{x_{C_1}} \circ [R_{[P,C_1]}^{x_{PC_1}}]^{-1}(w, z) = z. \end{aligned} \quad (18)$$

Second, we have

$$\begin{aligned} S_w \circ T_w(r) &= S_w \circ [Q_{C_0}^{C_1}]_2(w,r) \text{ via (14)} \\ &= [Q_{C_1}^{C_0}]_2 \left(w, [Q_{C_0}^{C_1}]_2(w,r)\right) \text{ via (17)} \\ &= [Q_{C_1}^{C_0}]_2 \left([Q_{C_0}^{C_1}]_1(w,r), [Q_{C_0}^{C_1}]_2(w,r)\right) \text{ via (13)} \\ &= [Q_{C_1}^{C_0}]_2 \circ Q_{C_0}^{C_1}(w,r) \\ &= R_{C_0}^{x_{C_0}} \circ [R_{[P,C_0]}^{x_{PC_0}}]^{-1}(w,r) = r. \end{aligned} \tag{19}$$

Hence for arbitrary but fixed w we can define

$$\begin{aligned} T_w^{-1}(z) &:= S_w(z) = [Q_{C_1}^{C_0}]_2(w,z) \\ &= R_{C_0}^{x_{C_0}} \circ [R_{[P,C_1]}^{x_{PC_1}}]^{-1}(w,z), \end{aligned} \tag{20}$$

where $T_w T_w^{-1} = T_w^{-1} T_w = I$. Here, T_w^{-1} is weakly Lipschitz since $R_{C_0}^{x_{C_0}}$ and $[R_{[P,C_1]}^{x_{PC_1}}]^{-1}$ are weakly Lipschitz, and the cascade is also weakly Lipschitz (See Remark 5). Furthermore, Remark 7 provides that T_w^{-1} is parametrically linearly bounded since $R_{C_0}^{x_{C_0}} \circ [R_{[P,C_1]}^{x_{PC_1}}]^{-1}$ is bounded as the cascade of two bounded operators is bounded.

(a $\Leftarrow$ b) Suppose that for fixed w, there exists $T_w^{-1} : z \in \mathscr{Z}_{C_1} \mapsto r \in \mathscr{Z}_{C_0}$ such that

$$T_w \circ T_w^{-1} = T_w^{-1} \circ T_w = I, \tag{21}$$

with T_w^{-1} weakly Lipschitz and parametrically linearly bounded.

If we define

$$\begin{aligned} W(w,z) &:= (W_1(w,z), W_2(w,z)) \\ &= (w, T_w^{-1}(z)) = (w,r), \end{aligned} \tag{22}$$

then we have

$$\begin{aligned} W \circ Q_{C_0}^{C_1}(w,r) &= W\left([Q_{C_0}^{C_1}]_1(w,r), [Q_{C_0}^{C_1}]_2(w,r)\right) \text{ via (12)} \\ &= \left([Q_{C_0}^{C_1}]_1(w,r), T_w^{-1} \circ [Q_{C_0}^{C_1}]_2(w,r)\right) \text{ via (22)} \\ &= \left(w, T_w^{-1} \circ [Q_{C_0}^{C_1}]_2(w,r)\right) \text{ via (13)} \\ &= \left(w, T_w^{-1} \circ T_w(r)\right) \text{ via (14)} \\ &= (w,r) \quad \text{via (21).} \end{aligned}$$

Similarly, we also have

$$\begin{aligned} Q_{C_0}^{C_1} \circ W(w,z) &= Q_{C_0}^{C_1}(w,r) \text{ via (22)} \\ &= \left([Q_{C_0}^{C_1}]_1(w,r), [Q_{C_0}^{C_1}]_2(w,r)\right) \text{ via (12)} \\ &= \left(w, [Q_{C_0}^{C_1}]_2(w,r)\right) \text{ via (13)} \\ &= (w, T_w(r)) \text{ via (14)} \\ &= (w,z). \end{aligned}$$

Hence, W is the inverse of $Q_{C_0}^{C_1}$; ie. $W(w,z) = Q_{C_1}^{C_0}(w,z)$. This implies that $Q_{C_1}^{C_0}$ exists for all input (w,z) and since

$$Q_{C_1}^{C_0} : \mathscr{Z}_{PC_1} \to \mathscr{Z}_{PC_0} := R_{[P,C_0]}^{x_{PC_0}} \circ [R_{[P,C_1]}^{x_{PC_1}}]^{-1}, \tag{23}$$

one can readily conclude that $[R_{[P,C_1]}^{x_{PC_1}}]^{-1}$ exists and $T_w^{-1}(z) = R_{C_0}^{x_{C_0}} \circ [R_{[P,C_1]}^{x_{PC_1}}]^{-1}(w,z)$.

To complete the proof, we need to show that $[R_{[P,C_1]}^{x_{PC_1}}]^{-1}$ is weakly Lipschitz and bounded. Observe first that $W_1(w,z) = w$ is a projection operator and by Lemma 22, W_1 is weakly Lipschitz and bounded. Since we have $T_w^{-1}(z) \equiv W_2(w,z)$, T_w^{-1} is weakly Lipschitz and parametrically linearly bounded, $Q_{C_1}^{C_0} = W$ is also weakly Lipschitz and bounded via Remark 7 (as each component of $W(w,z) = (W_1(w,z), W_2(w,z))$ is weakly Lipschitz and bounded).

Since $[P, C_0]$ is assumed be null internally stable (ie. $[R_{[P,C_0]}^{x_{PC_0}}]^{-1}$ exists, is weakly Lipschitz and bounded), one can conclude that

$$[R_{[P,C_1]}^{x_{PC_1}}]^{-1} = [R_{[P,C_0]}^{x_{PC_0}}]^{-1} \circ Q_{C_1}^{C_0} \tag{24}$$

is weakly Lipschitz (Remark 5) and bounded (as the cascade of two bounded operators is also bounded). Given Definition 17, $[P, C_1]$ is null internally stable. ■

Remark 24: For the linear case, let $P = NM^{-1}$ be a right coprime factorization over $\mathscr{RH}_\infty$, $C_0 = \tilde{V}_0^{-1}\tilde{U}_0$ and $C_1 = \tilde{V}_1^{-1}\tilde{U}_1$ be left coprime factorizations over $\mathscr{RH}_\infty$. Then, one can express P and $C_i,\ i \in \{1,2\}$ as

$$\begin{aligned} y = Pu \quad &\Leftrightarrow \quad R_P^{x_P}(u,y) = 0 \\ u = C_i y \quad &\Leftrightarrow \quad R_{C_i}^{x_{C_i}}(y,u) := [-\tilde{U}_i \ \tilde{V}_i]\tbinom{y}{u} = 0 \end{aligned}$$

This implies that the feedback interconnection of $[P, C_0]$ can be expressed with $(0,r) \in \mathscr{Z}_{PC_0}$ as

$$(u,y) = [R_{[P,C_0]}^{x_{PC_0}}]^{-1}(0,r) \quad \Leftrightarrow \quad \begin{pmatrix} y \\ u \end{pmatrix} = G(\tilde{K}_0 G)^{-1} r, \tag{25}$$

where $G := \binom{N}{M}, \tilde{K}_i := [-\tilde{U}_i \quad \tilde{V}_i]$. Hence,

$$T_0 = R_{C_1}^{x_{C_1}} \circ [R_{[P,C_0]}^{x_P, x_{C_0}}]^{-1} = (\tilde{K}_1 G)(\tilde{K}_0 G)^{-1}. \tag{26}$$

In fact, (26) is equivalent to (4), and [10] provides the data-based stability tests with this T_0.

Theorem 23 puts forward a mechanism, at least in principle, to verify the stability of the closed-loop system $[P, C_1]$ by checking the invertibility of T_w, together with the weakly Lipschitz condition and boundedness of T_w^{-1}. One should note that as (11) indicates, R_P (or at least a good approximation or model of R_P) is required in order to be able to compute T_w or to check whether T_w^{-1} exists.

Now we conclude our main results with the following lemma that connects results of Theorem 18 with Theorem 23 with all P, C_0 and C_1 being LTI.

Lemma 25: Suppose the hypotheses of Theorem 23 hold and consider the setting in Fig. 4. If we assume P, C_0 and C_1 are all LTI and define $z := T_w(r)$ and $z_1 = T_0(r)$, then T_0^{-1} exists *iff* T_w^{-1} exists.

Proof: Since P, C_0 and C_1 are all LTI, one can separate $T_w(r)$ as

$$\begin{aligned} T_w(r) &= R_{C_1}^{x_{C_1}} \circ [R_{[P,C_0]}^{x_{PC_0}}]^{-1}(w,r) \\ &= R_{C_1}^{x_{C_1}} \circ [R_{[P,C_0]}^{x_{PC_0}}]^{-1}(0,r) + R_{C_1}^{x_{C_1}} \circ [R_{[P,C_0]}^{x_{PC_0}}]^{-1}(w,0) \\ &= T_0(r) + T_w(0). \end{aligned} \tag{27}$$

One should notice that for a fixed w, $T_w(0)$ can be regarded as a constant for all r.

($\Rightarrow$) Suppose T_0^{-1} exists, ie. $T_0(r) = z_1 \;\Leftrightarrow\; T_0^{-1}(z_1) = r$. Since $z = z_1 + T_w(0)$ (or $z_1 = z - T_w(0)$) , we have

$$T_0(r) = z - T_w(0) \quad \Leftrightarrow \quad r = T_0^{-1}(z - T_w(0)) \qquad (28)$$

If we define $S_w(z) := T_0^{-1}(z - T_w(0))$, then we have

$$T_w(S_w(z)) = T_w(T_0^{-1}(z - T_w(0))) = T_w(r) = z$$

and

$$S_w(T_w(r)) = S_w(z) = T_0^{-1}(z - T_w(0)) = T_0^{-1}(z_1) = r.$$

Hence S_w is in fact the inverse of T_w (ie. $T_w^{-1} := T_0^{-1}(z - T_w(0))$).

($\Leftarrow$) If T_w^{-1} exists for all w, then T_0^{-1} also exists as it is a special case with $w = 0$. ∎

As shown in Remark 24, $T_0(r)$ is equivalent to (26) in the LTI case. Hence, Lemma 25 shows that for the LTI case (ie. P, C_0 and C_1 are all LTI), Theorem 18 and Theorem 23 are equivalent.

VI. Conclusions

In this paper, we have proposed a generalized nonlinear version of the LTI results proposed in [10]. Given an existing stable feedback $[P, C_0]$, our novel nonlinear results provide a mechanism for ensuring the stability of the closed-loop with the introduction of a new controller C_1. In particular, the kernel representation was utilized as a generalization of the left fractional representations for nonlinear plant and controllers. We have also shown that our proposed nonlinear analysis results are in fact analogous to their LTI counterparts and provide an equivalent condition ensuring $[P, C_1]$ to be (null) internally stable.

Our current research attempts focus on developing data-based stability results to ensure stability of the closed-loop with the introduction of C_1. Furthermore, we are working on developing tools for verifying the closed-loop performance with the new controller in advance.

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