Control of Minimally Persistent Leader-Remote-Follower Formations in the Plane

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***Abstract*—This paper addresses the n-vehicle formation shape maintenance problem in the plane. The objective is to design decentralized motion control laws for each vehicle to restore formation shape in the presence of small perturbations from the desired shape. Formation shape is restored by actively controlling a certain set of interagent distances, and we assign the task of controlling a particular interagent distance to only one of the involved agents. We restrict our attention to a class of directed information architectures called *minimally persistent leader-remote-follower*. The nonlinear closed-loop system has a manifold of equilibria, which implies that the linearized system is nonhyperbolic. We apply center manifold theory to show local exponential stability of the desired formation shape. Choosing stabilizing gains is possible if a certain submatrix of the rigidity matrix has all leading principal minors nonzero, and we show that this condition holds for all leader-remote-follower formations with generic agent positions. Simulations are provided.**

I. INTRODUCTION

There has been much interest in cooperative control of autonomous vehicle formations and mobile sensor networks. Advances in computation, communication, sensing, and control technologies have made possible systems in which multiagent cooperation allows not only improved capabilities, but also entirely new capabilities over what can be achieved with a single agent. The motivation for studying such systems comes from both the potential for scientific and engineering applications and the unique technical challenges that such systems present. Applications include teams of UAVs performing military reconnaissance and surveillance missions in hostile environments [1], satellite formations for high-resolution Earth and deep space imaging [2], and submarine swarms for oceanic exploration and mapping [3]. For large formations, an overarching requirement is that the formation operates in a decentralized fashion, where each agent operates using only local information (apart from information such as waypoints which may be provided to the leader(s) in a formation).

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A basic task for autonomous vehicle formations is formation shape control. Precisely controlled formations can maintain mobile sensing agents in optimal sensing configurations. We study the n-vehicle formation shape maintenance problem for designing decentralized control laws for each vehicle to restore a desired formation shape in the presence of small perturbations from the nominal shape. Formation shape is restored by actively controlling a certain set of interagent distances.

The information architecture is modeled as a graph $G(V, E)$ where V is a set of vertices representing agents and E is a set of edges representing information flow amongst the agents. For the formation shape control task, the edge set represents the set of interagent distances to be actively held constant via control of individual agent motion. If a suitably large and well-chosen set of interagent distances is held constant, then all remaining interagent distances will be constant as a consequence, thus maintaining formation shape. Just which interagent distances should be held constant is the topic of *rigidity theory* [4], [5].

We assign the task of controlling a particular interagent distance to only one of the involved agents, resulting in a directed information architecture (as opposed to assigning it to both agents which results in an undirected information architecture). In this case, G is a directed graph where a direction is assigned to every edge in E with an outward arrow from the agent responsible for controlling the interagent distance. In order to maintain formation shape, G is required to be *persistent*. The persistence concept includes rigidity, but also requires a further condition called *constraint consistence* that precludes certain directed information flow patterns [5].

In [6], Yu et al consider *minimally persistent leader-first-follower (LFF)* formations *with cycles*. Note that LFF is not an acyclic formation in general like the type considered in [7]. Yu et al present decentralized nonlinear control laws to restore formation shape in the presence of small distortions from the desired shape. They show that choosing stabilizing control gains is possible if a certain submatrix of the *rigidity matrix* has all leading principal minors nonzero and prove that all minimally persistent LFF formations generically obey this principal minor condition. In [8], Krick et al present decentralized gradient-based control laws for a minimally rigid formation (with undirected information architecture) to restore formation shape in the presence of small distortions from the desired shape. Since the linearized system is nonhyperbolic, they utilize center manifold theory to prove local exponential stability.

We consider formations in the plane with *minimally per-*

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sistent leader-remote-follower (LRF) structure. After an initial perturbation from the desired shape, the leader remains stationary and the remaining agents move to restore the distances they must meet. We present decentralized nonlinear control laws analogous to [6]. The nonlinear closed-loop system has a manifold of equilibria, i.e. that the linearized system is nonhyperbolic. For LFF formations, [6] obtains a hyperbolic system via the choice of a particular global coordinate system and prove local stability through eigenvalue analysis. For LRF formations, this choice is not possible and so we apply a new result based on center manifold theory to show local exponential stability of the desired formation shape. Again, it is possible to choose stabilizing control gains whenever a certain submatrix of the rigidity matrix has all leading principal minors nonzero, and we show that this condition holds for leader-remote-follower formations as well.

Section II describes minimally persistent formations and center manifold theory. Section III describes the nonlinear equations of motion and shows how center manifold theory can be applied to prove that the desired formation shape is locally exponentially stable. Section IV shows that the principal minor condition holds for LRF formations. Section V presents a numerical simulation to demonstrate the performance of our algorithm. Section VI gives concluding remarks and future research directions.

II. BACKGROUND

In this section, we review (a) the structure of information architectures for minimally persistent formations, and (b) center manifold theory, which offers tools for analyzing stability of dynamical systems near nonhyperbolic equilibrium points.

A. Minimally Persistent Formations

Let $F(G,p)$ denote a formation of n agents in the plane. Suppose $G(V,E)$ is a directed graph that represents the information architecture where the vertex set V represents the agents and the edge set E represents the set of interagent distances to be controlled to maintain formation shape. $p : V \to \Re^{2n}$ is a position function mapping each vertex to a position in the plane. A formation is called *minimally persistent* if the information architecture G is minimally persistent, and G is minimally persistent if it is *minimally rigid* and *constraint consistent*. A minimally rigid graph on n vertices has $2n-3$ edges which are well-distributed according to Laman's Theorem [9]. A constraint consistent graph precludes certain directed information flow patterns that make it impossible to control formation shape [5].

We consider a particular type of minimally persistent leader-follower formation called *leader-remote-follower (LRF)*. In minimally persistent leader-follower formations, one agent (the leader) has zero distances to maintain and thus two degrees of freedom (DOF), one agent (the first follower or remote follower) has only one distance to maintain and thus one DOF, and all other agents (the ordinary followers) have two distances to maintain and thus zero DOF. If the agent with one DOF is connected to the leader by an edge in G, then we call the formation *leader-first-follower (LFF)*. Otherwise, we call the formation *leader-remote-follower (LRF)*. Figure 1 illustrates examples LFF and LRF formations.

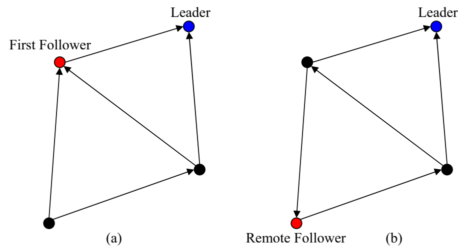

Fig. 1. Examples of LFF and LRF formations with four agents: (a) in LFF formations the one-DOF agent is connected to the leader, and (b) in LRF formations the one-DOF agent is not connected to the leader.

The distinction between LFF and LRF formations is important in the stability analysis for the formation shape maintenance control laws. For LFF formations one can define a global coordinate basis to obtain a hyperbolic reduced-order system in which local stability requires only eigenvalue analysis of the linearized system [6]. This is so because in the framework of both [6] and this paper, after its "small" initial move, the leader stops moving. Thereafter, [6] forces the first follower to move in the direction of the leader. Thus the direction of movement of the first follower in the LFF framework is fixed. This direction defines the stated coordinate basis in [6]. In contrast, for LRF formations the direction associated with the remote follower's DOF is not fixed in space since it is following an agent other than the leader to satisfy its distance constraint. Thus, the device used in [6] to obtain a global coordinate system that provides a hyperbolic reduced-order system no longer applies. Consequently, one cannot draw conclusions about the local stability of the nonlinear system near the desired formation shape by analyzing the linearized system alone; more sophisticated techniques are needed. Center manifold theory provides tools for determining stability near nonhyperbolic equilibrium points.

B. Center Manifold Theory

Standard treatments of center manifold theory can be found in e.g [10], [11], or [12]. These treatments concentrate on isolated equilibria. In the formation shape maintenance problem, the dynamic system has a manifold of non-isolated equilibrium points corresponding to the desired formation shape. In [13], Malkin proves a local stability result where trajectories converge to a point on an equilibrium manifold. More general results for equilibrium manifolds are presented in [14]. In [8], the importance of compactness for proving stability is emphasized. Here, we state a new result for stability of equilibrium manifolds that does not explicitly need compactness.

Consider the nonlinear autonomous dynamic system

$$\dot{x} = f(x), \quad x \in \Re^n \tag{1}$$

where the function f is $\mathbf{C}^r$, $r \geq 2$ almost everywhere including a neighbourhood of the origin. Suppose the origin is an equilibrium point and that the Jacobian of f (we will use the notation $J_f(x)$) at the origin has m eigenvalues with zero real part and $n-m$ eigenvalues with negative real part. Then (1) can be transformed into the following form

$$\begin{aligned} \dot{\theta} &= A_c\theta + g_1(\theta,\rho) \\ \dot{\rho} &= A_s\rho + g_2(\theta,\rho), \quad (\theta,\rho) \in \Re^m \times \Re^{n-m} \end{aligned} \quad (2)$$

where A_c is a matrix having eigenvalues with zero real parts, A_s is a matrix having eigenvalues with negative real parts, and the functions g_1 and g_2 satisfy

$$\begin{aligned} g_1(0,0) = 0, \quad J_{g_1}(0,0) = 0 \\ g_2(0,0) = 0, \quad J_{g_2}(0,0) = 0. \end{aligned} \quad (3)$$

Definition 1. *An invariant manifold is called a* center manifold *for (2) if it can be locally represented as follows*

$$W^c(0) = \{(\theta,\rho) \in U \subset \Re^m \times \Re^{n-m} | \rho = h(\theta)\} \quad (4)$$

for some sufficiently small neighbourhood of the origin U where the function h satisfies $h(0) = 0$ and $J_h(0) = 0$.

We have the following standard result.

Theorem 1 ([12])**.** *Suppose there exists a $\mathbf{C}^r$ center manifold for (2) with dynamics restricted to the center manifold given by the following m-dimensional nonlinear system for sufficiently small ξ*

$$\dot{\xi} = A_c\xi + g_1(\xi, h(\xi)), \quad \xi \in \Re^m. \quad (5)$$

If the origin of (5) is stable (asymptotically stable) (unstable), then the origin of (2) is stable (asymptotically stable) (unstable). Suppose the origin of (5) is stable. Then if $(\theta(t), \rho(t))$ is a solution of (2) for sufficiently small $(\theta(0), \rho(0))$, there is a solution $\xi(t)$ of (5) such that as $t \to \infty$

$$\begin{aligned} \theta(t) = \xi(t) + O(e^{-\gamma t}) \\ \rho(t) = h(\xi(t)) + O(e^{-\gamma t}) \end{aligned} \quad (6)$$

where γ is a positive constant.

Thus, if a center manifold exists then to determine stability near the nonhyperbolic equilibrium point of (1), one can analyze a reduced-order system, viz. (5). If the origin of (5) is stable, then the solutions of the original system converge exponentially to a trajectory on the center manifold.

We have the following result when there is a manifold of equilibria. Observe that although the theorem postulates the existence of a center manifold, it makes no explicit compactness assumptions, in contrast to [8].

Theorem 2. *Suppose there is an m-dimensional ($m > 0$) manifold of equilibrium points $M_1 = \{x \in \Re^n | f(x) = 0\}$ for (1) that contains the origin. Suppose at the origin the Jacobian of f has m eigenvalues with zero real part and $n-m$ eigenvalues with negative real part. Then M_1 is a center manifold for (1), i.e. there exists a function $h_1 : \Re^m \to \Re^{n-m}$ such that $h_1(0) = 0, J_{h_1}(0) = 0$ and in a suitably small neighborhood U of the origin, the equilibrium set can be represented as $\rho = h_1(\theta)$. Further, there are neighborhoods Ω_1 and Ω_2 of the origin such that $M_2 = \Omega_2 \cap M_1$ is locally exponentially stable[1] and for each $x(0) \in \Omega_1$ there is a point $q \in M_2$ such that $lim_{t\to\infty} x(t) = q$.*

In our problem, the manifold of equilibria will correspond to formation positions with the desired shape. In the plane, the manifold is three-dimensional due to the three possible Euclidean motions of the formation in the plane (two translational and one rotational). In the following section, we develop equations of motion and apply the results in this section to show local exponential stability of the desired shape.

III. EQUATIONS OF MOTION

In this section, we present equations of motion for the formation shape maintenance problem and study the local stability properties of the desired formation shape. Suppose the formation is initially in the desired shape. Then the position of each agent is perturbed by a small amount. The leader then remains stationary, and the remaining agents move under distance control laws to meet their distance specifications in order to restore the desired formation shape. This shape is realized by every point on a three-dimensional equilibrium manifold. We consider a reduced-order system and show that Theorem 2 can be directly applied to prove local exponential convergence to the invariant manifold.

A. Nonlinear Equations of Motion

Consider a minimally persistent formation $F(G,p)$ of n agents in the plane where the leader and remote follower are agents n and $n-1$, respectively. We define the rigidity function

$$r(p) = [..., ||p_j - p_k||^2, ...]^T \quad (7)$$

where the ith entry of r, viz. $||p_j - p_k||^2$, corresponds to an edge $e_i \in E$ connecting two vertices j and k. Let $d = [..., d_{jk}^{*2}, ...]$ represent a vector of the squares of the desired distances corresponding to each edge. We assume that there exist agent positions p such that $p = r^{-1}(d)$, i.e. the set of desired interagent distances corresponds to a realizable formation. Formation shape is controlled by controlling the interagent distance corresponding to each edge.

Following [6] and [8], we adopt a single integrator model for each agent:

$$\dot{p}_i = u_i. \quad (8)$$

Consider an ordinary follower agent denoted by i that is required to maintain constant distances d_{ij}^* and d_{ik}^* from agents j and k, respectively, and can measure the instantaneous relative positions of these agents. We use the same law as in [6] for ordinary followers:

$$u_i = K_i(p_i^* - p_i) = K_i f_i(p_j - p_i, p_k - p_i, d_{ij}^*, d_{ik}^*) \quad (9)$$

[1] Saying that M_2 is locally exponentially stable means that there is a single exponent γ such that all trajectories converge to M_2 from the neighbourhood Ω_1 at least as fast as $e^{-\gamma t}$. One could envisage a non-compact M_2 where any single trajectory approaches M_1 exponentially fast but no single γ could be found applicable to all trajectories.

where K_i is a gain matrix and p_i^* is the instantaneous target position for agent i in which the distances from agents j and k are correct. Since the perturbations from the desired shape are small, the instantaneous target positions are well-defined and unique. For the remote follower, we have

$$\begin{aligned} u_{n-1} &= K_{n-1}(p_{n-1}^* - p_{n-1}) \\ &= K_{n-1}\frac{||p_l - p_{n-1}|| - d_{n-1,l}^*}{||p_l - p_{n-1}||}(p_l - p_{n-1}) \end{aligned} \quad (10)$$

where K_{n-1} is a gain matrix and agent l is the agent from which the remote follower is maintaining the constant distance $d_{n-1,l}^*$. For the leader, we have

$$\dot{p}_n = 0. \quad (11)$$

Equations (9)-(11) represent the dynamics of the autonomous closed-loop system, which may be written in the form

$$\dot{p} = \begin{bmatrix} f^T(p) & 0 \end{bmatrix}^T \quad (12)$$

where $f : \Re^{2n} \rightarrow \Re^{2n-2}$ is smooth almost everywhere including a neighbourhood of the desired formation.

There is a *manifold* of equilibria for (12) given by

$$\Psi = \{p \in \Re^{2n} | p = r^{-1}(d)\} \quad (13)$$

corresponding to formations where all distance constraints are satisfied. The manifold Ψ is a three-dimensional manifold because a formation with correct distances has three degrees of freedom associated with the planar Euclidean motions (two for translation and one for rotation). Given these degrees of freedom, it is evident that Ψ is not compact. In the following, we define a reduced-order system by fixing the position of the leader and obtain a compact equilibrium manifold.

B. Linearized Equations

We represent the position of the formation as $p(t) = \delta p(t) + \bar{p}$, where $\bar{p}$ is any equilibrium position with desired shape close to the perturbed formation, and the displacements $\delta p(t)$ are assumed to be small. In particular, for agent i we have $p_i(t) = \delta p_i(t) + \bar{p}_i$ where $\bar{p}_i$ corresponds to positions of agent i that meet its distance constraints. Let $p_i(t) = [x_i(t), y_i(t)]^T$, $\bar{p}_i = [\bar{x}_i, \bar{y}_i]^T$, and $\delta p_i(t) = [\delta x_i(t), \delta y_i(t)]^T$ in a global coordinate system to be defined later.

From [6], the linear part for the ordinary followers ($i = 1, ..., n-2$) is given by

$$\begin{bmatrix} \dot{\delta x}_i \\ \dot{\delta y}_i \end{bmatrix} = K_i R_{ei}^{-1} R_{ij,ik} \begin{bmatrix} \delta x_i \\ \delta y_i \\ \delta x_j \\ \delta y_j \\ \delta x_k \\ \delta y_k \end{bmatrix} \quad (14)$$

where

$$R_{ei} = \begin{bmatrix} (\bar{p}_j - \bar{p}_i)^T \\ (\bar{p}_k - \bar{p}_i)^T \end{bmatrix}$$

$$R_{ij,ik} = \begin{bmatrix} (\bar{p}_i - \bar{p}_j)^T & (\bar{p}_j - \bar{p}_i)^T & 0 \\ (\bar{p}_i - \bar{p}_k)^T & 0 & (\bar{p}_k - \bar{p}_i)^T \end{bmatrix}.$$

Similarly, the linear part for the remote follower is given by

$$\begin{bmatrix} \dot{\delta x}_{n-1} \\ \dot{\delta y}_{n-1} \end{bmatrix} = K_{n-1} R_{e,n-1}^{-1} R_{(n-1)l,00} \begin{bmatrix} \delta x_i \\ \delta y_i \\ \delta x_l \\ \delta y_l \end{bmatrix} \quad (15)$$

where

$$R_{e,n-1} = \begin{bmatrix} \bar{x}_l - \bar{x}_{n-1} & \bar{y}_l - \bar{y}_{n-1} \\ \bar{y}_{n-1} - \bar{y}_l & \bar{x}_l - \bar{x}_{n-1} \end{bmatrix}$$

$$R_{(n-1)l,00} = \begin{bmatrix} (\bar{p}_{n-1} - \bar{p}_l)^T & (\bar{p}_l - \bar{p}_{n-1})^T \\ 0 & 0 \end{bmatrix}.$$

The leader equations are of course

$$\begin{bmatrix} \dot{\delta x}_n & \dot{\delta y}_n \end{bmatrix}^T = 0. \quad (16)$$

Putting the equations together, we have

$$\dot{\delta p} = K R_e^{-1} \begin{bmatrix} R^T & 0 \end{bmatrix}^T \delta p \quad (17)$$

where $K = diag[K_1, ..., K_{n-1}, 0]$ with 2×2 K_i to be specified, $R_e = diag[R_{e1}, ..., R_{e,n-1}, I_2]$ with each block being a 2×2 submatrix of the rigidity matrix $R \in \Re^{2n-3 \times 2n}$.

C. A Reduced-Order System

We define a reduced-order system by neglecting the stationary leader dynamics since $\dot{p}_n(t) = 0$. Let the global coordinate basis have the leader at the origin and let the x-axis be an arbitrary direction. Let $z = [p_1, ..., p_{n-1}]^T \in \Re^{2n-2}$, $\bar{z} = [\bar{p}_1, ..., \bar{p}_{n-1}]$, and $z = \delta z + \bar{z}$ where δz is assumed to be small. The reduced-order nonlinear system may then be written in the form

$$\dot{z} = \bar{f}(z). \quad (18)$$

The rigidity function associated with (18) is $r_z(z) = [..., ||z_j - z_k||^2, ...]^T$ where the ith entry of r_z corresponds to an edge $e_i \in E$ connecting two vertices j and k. If a vertex l is connected to the leader, then the corresponding entry in r_z is $||z_l||^2$. The equilibrium manifold associated with (18) is

$$\Psi_z = \{z \in \Re^{2n-2} | z = r_z^{-1}(d)\}. \quad (19)$$

Ψ_z is a one-dimensional manifold that can be characterized by a rotation around the leader since the the position of the leader is fixed. Therefore, since Ψ_z is a closed and bounded subset of Euclidean space, it is compact.

Expanding in a Taylor series about the equilibrium position, (18) becomes

$$\dot{\delta z} = J_{\bar{f}}(\bar{z})\delta z + g(\delta z) \quad (20)$$

where the first term represents the reduced-order linear system and the second term represents the nonlinear part of order two or higher. The reduced-order linear system may be written in the form

$$\dot{\delta z} = \tilde{K}\tilde{R}_e^{-1} \begin{bmatrix} \tilde{R}^T & 0 \end{bmatrix}^T \delta z \quad (21)$$

where $\tilde{K} = diag[K_1, ..., K_{n-1}]$, $\tilde{R}_e = diag[R_{e1}, ..., R_{e,n-1}]$, and $\tilde{R}$ is the submatrix of the rigidity matrix with the last two columns associated with the leader removed.

Observe that the Jacobian $J_{\bar{f}}(\bar{z})$ is rank deficient by one because of the row of zeros below the rigidity matrix. Thus one of its eigenvalues is zero. Thus, the equilibrium position is nonhyperbolic, and we can apply center manifold theory as developed in Section II to determine local stability of the equilibrium position. Since $J_{\bar{f}}(\bar{z})$ has one zero eigenvalue, there exists an invertible matrix Q such that

$$QJ_{\bar{f}}(\bar{z})Q^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & A_s \end{bmatrix}. \tag{22}$$

where $A_s \in \Re^{2n-3\times 2n-3}$ is a nonsingular matrix. Let $[\theta, \rho]^T = Q\delta z$ where $\theta \in \Re$ and $\rho \in \Re^{2n-3}$. Then (20) can be written in the form

$$\begin{aligned} \dot{\theta} &= g_1(\theta,\rho) \\ \dot{\rho} &= A_s\rho + g_2(\theta,\rho) \end{aligned} \tag{23}$$

where g_1 is the first entry of $Qg(Q^{-1}[\theta,\rho]^T)$ and satisfies $g_1(0,0) = 0$ and $J_{g1}(0,0) = 0$, and g_2 is the last $2n-3$ entries of $Qg(Q^{-1}[\theta,\rho]^T)$ and satisfies $g_2(0,0) = 0$ and $J_{g2}(0,0) = 0$. This is in the normal form for center manifold theory.

To apply Theorem 2 we simply need the matrix A_s to be Hurwitz. Here, A_s must be *made* Hurwitz by a suitable choice of the gain matrices $K_1,...,K_{n-1}$. Showing that such a choice of gains is indeed possible is the topic of the next section.

IV. CHOOSING GAINS AND THE PRINCIPAL MINOR CONDITION

In this section we show that it is possible to choose the gain matrices for each vehicle such that all nonzero eigenvalues of the linearized system have negative real parts. This is the case if a certain submatrix of the rigidity matrix has all leading principal minors nonzero. That this condition is satisfied by all LRF formations is shown in the following.

Let the gain matrices $K_1,...,K_{n-1}$ be chosen as follows:

$$\begin{aligned} K_i &= \Lambda_i R_{e,i}, \quad i = 1,...,n-2 \\ K_{n-1} &= \left(\frac{(\bar{x}_l - \bar{x}_{n-1})^2 + (\bar{y}_l - \bar{y}_{n-1})^2}{\bar{x}_l - \bar{x}_{n-1}}\right)\Lambda_{n-1} \end{aligned} \tag{24}$$

where Λ_i is a diagonal matrix. Then we have

$$J_{\bar{f}}(\bar{z}) = \Lambda \begin{bmatrix} \tilde{R}^T & 0 \end{bmatrix}^T \tag{25}$$

where $\Lambda \in \Re^{2n-2\times 2n-2}$ is a diagonal matrix and r is a scalar multiple of the last row of $\tilde{R}$. The following result gives the conditions on the singular matrix $[\tilde{R}^T, r]^T$ so Λ can be chosen such that all nonzero eigenvalues of $\Lambda[\tilde{R}^T, r]^T$ have negative real parts.

Theorem 3. *Partition the singular matrix $[\tilde{R}^T, r]^T$ as follows*

$$\begin{bmatrix} \tilde{R} \\ r^T \end{bmatrix} = \begin{bmatrix} \hat{R} & r_{12} \\ r_{21}^T & r_{22} \end{bmatrix} \tag{26}$$

where $\hat{R}$ is $\tilde{R}$ with the last column removed (r_{12} is the last column). Suppose $\hat{R}$ is a nonsingular matrix with every leading principal minor nonzero. Then, there exists a diagonal matrix Λ such that the real parts of all nonzero eigenvalues of $\Lambda[\tilde{R}^T, r]^T$ are negative.

The matrix $\hat{R}$ in Theorem 3 is the rigidity matrix obtained by removing the two columns corresponding to the leader and one column corresponding to the remote follower (the same matrix in question in [6]). We now have the following result.

Theorem 4. *Consider any minimally persistent LRF formation $F(G, p)$ with agents at generic positions. There exists an ordering of the vertices of F and an ordering of the pair of outgoing edges for each vertex such that all leading principal minors of the associated $\hat{R}$ are generically nonzero.*

This result shows that for any LRF formation with generic agent positions, one can choose the diagonal matrix Λ such that the real parts of all nonzero eigenvalues of the reduced-order linearized system (21) are negative (and accordingly the matrix A_s in (23) is Hurwitz). The stabilizing gains are designed for a particular equilibrium point in Ψ_z. It is important to note here that the control gains proposed in (24) may not be stabilizing for all other points in Ψ_z. Theorem 2 can be directly applied to show that for each $\bar{z} \in \Psi_z$, there is a neighbourhood $\Omega(\bar{z})$ of $\bar{z}$ such that for any initial formation position $z(0) \in \Omega(\bar{z})$ there is a point $z^* \in \Psi$ such that $lim_{t\to\infty} z(t) = z^*$ at an exponential rate, i.e. the formation converges locally exponentially to the desired shape.

Remark. There is an important distinction to be made between decentralized *design* and decentralized *implementation*. The control laws in this paper are based on minimally persistent information architectures, and selecting stabilizing gains requires a suitable ordering of the vertices and edges. Therefore, the *design* of our control laws is inherently centralized. However, persistent information architectures provide a basis from which we can design control laws with decentralized *implementation*. Once the design is established, our control laws require only local information.

V. SIMULATION

In this section, we demonstrate the performance of our algorithm via simulation. Figure 2 shows a LRF formation in the plane where agents 1 and 2 are ordinary followers, agent 3 is the remote follower, and agent 4 is the leader. Suppose the agents are in the desired formation shape in the position $\bar{p} = [0.3123, -0.1574, 0.7359, 0.5710, -0.0609, 0.6901, 0, 0]$. We note that if the gain matrices are all chosen to be identity, the nonzero eigenvalues of the linearized system are $\{-1.5363 \pm 0.9289i, 0.0726, -1, -1\}$, which implies instability. Suppose the gain matrices are chosen with the structure given by (24) where the diagonal multipliers are

$$\Lambda_1 = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix},\ \Lambda_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},\ \Lambda_3 = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}. \tag{27}$$

Then the nonzero eigenvalues of the linearized system are given by $\{-1.9521 \pm 0.3196i, -0.2521 \pm 0.3886i, -0.7532\}$, and the desired formation shape is stable via the analysis in the previous sections. Figure 3 shows the agent trajectories in the plane under the formation shape maintenance control laws. The desired formation shape is restored, though not

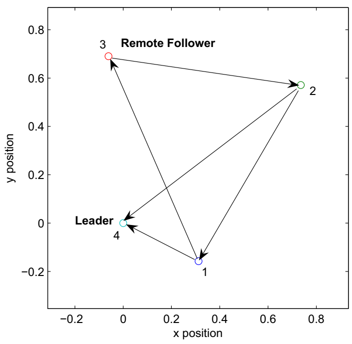

Fig. 2. LRF formation in unstable agent positions for identity gain.

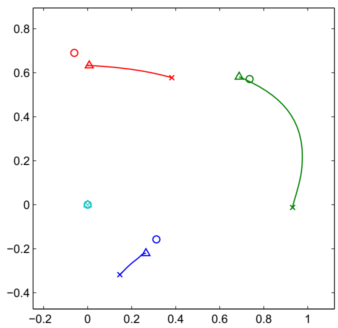

Fig. 3. Agent trajectories in the plane: the circles represent the initial desired formation shape, the triangles represent the perturbed agent positions, and the X's represent the final agents positions under the formation shape maintenance control laws. The desired shape has been restored. The leader does not move.

to the initial unperturbed formation. Figure 4 shows that the interagent distance errors all converge to zero.

VI. CONCLUDING REMARKS

In this paper, we have addressed the n-vehicle formation shape maintenance problem for leader-remote-follower formations. We presented decentralized nonlinear control laws that restore desired formation shape in the presence of small perturbations from the nominal shape. The nonlinear system has a manifold of equilibria, which implies that the linearized system is nonhyperbolic. We applied center manifold theory to show local exponential stability of the equilibrium formation with desired shape. We have also shown that a principal minor condition holds for LRF formations, which allows a choice of stabilizing gain matrices. Finally, we demonstrated our results through numerical simulation.

There are many directions for future research. First, the stability results here are local, and an immediate task would be to determine the size of the region of attraction. Second, also of interest is the formation shape maintenance problem for a separate class of minimally persistent formations called Coleader in which three agents have one DOF and all remaining agents have zero DOF (see [15]). Finally, non-minimally

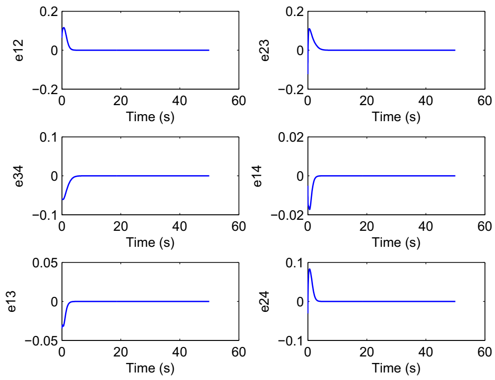

Fig. 4. The interagent distance errors, defined as $e_{ij} = ||p_i - p_j|| - d_{ij}$, all converge to zero, thus recovering the desired formation shape.

persistent formations will eventually be of interest because it may be desirable to control more than the minimum number of distances for formation shape maintenance in order to obtain a level of robustness.

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