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Implementing nonlinear controllers using observer-form via kernel representations

Sung H. Cha, Arvin Dehghani and Brian D. O. Anderson

Abstract— **We propose a connection between the state-space realization and 'observer-like' property of a nonlinear feedback system, consisting of an linear time-invariant (LTI) plant and a nonlinear controller by utilizing 'kernel representation' of dynamical systems. This connection is advanced by noticing that the kernel representation shares many properties with and is indeed a generalization of left fractional representations for nonlinear systems.**

I. INTRODUCTION

AMONG many different representations of dynamical systems, the fractional representation has been one of the favorite approaches for linear time-invariant (LTI) system analysis and synthesis over decades [1], [2]. In particular, the coprime factorizations of the LTI plants and controllers can be expressed as 'well-charaterized' transfer functions, in which one can utilize the fractional representation theory as well as the classical transfer function approach. In addition, the coprime factorization approach is known to have a built-in 'observer-like' property, which naturally provides an estimate of the internal states for LTI systems, via a connection between coprime factors and state-space realization [3].

For nonlinear systems, the fractional representations are expressed as input-output operators, and a nonlinear left fractional representation of such systems is generally not available. However, the right fractional representation for nonlinear systems is available and can be naturally extended from their counterparts for the linear systems [4]. Thus, establishing explicit connections between classical control theory, fractional representation theory and state-space realizations for the nonlinear systems is not straightforward.

In this paper, we shall develop a general framework for a connection between the state-space realization and 'observer-like' property of a nonlinear feedback system. First, we shall establish a more complete analysis of observer-form implementation for general LTI plants and controllers. In particular, we will review existing connections between coprime factorizations and their state-space realizations, see eg. [4], [5], [3], [6] and [7]. Second, we focus on a nonlinear loop which consists of a linear plant and a nonlinear controller. The proposed results relate nonlinear observer-form to a standard state feedback control law, which can be implemented using built-in state estimators. Our results can be extended to include nonlinear plants and provide a framework for implementing nonlinear output feedback and/or state observer (see eg. [8], [9] and [10]). Our proposed results can be utilized in the area of feedback system identification (see [11], [12], [13]), controller verification (see [7] and [14]) and robust and/or adaptive control (see [15], [1] and [6]).

Section II collects the required definitions and different system representations. In Section III, we will review the observer-form implementation of controllers for the LTI case in a standard feedback setting. This is followed by a discussion of the existing relationship between coprime factorizations and state-space realizations of a linear system. This leads to the presentation of a connection between the state-space realization and nonlinear observer-form representation via the kernel representation. The numerical example of Section IV illustrates the effectiveness of the proposed results. Section V summarizes the main results and remarks on possible future research directions.

II. PRELIMINARIES

We shall outline relevant and necessary definitions from [12], [13], [14] here. The vector space of Lebesgue measurable functions $f : [0,\infty) \to \mathbb{R}^m$ such that $\|f\|_2 := (\int_0^\infty f^T f \; dt)^{1/2} < \infty$ is denoted by $\mathscr{L}_2^m[0,\infty)$ (in short $\mathscr{L}_2^m$ or $\mathscr{L}_2$). If a truncation operator $\mathcal{T}_T$ is given on the vector space of functions $f : \mathbb{R} \to \mathbb{R}^m$ as

$$\mathcal{T}_T f(t) := \begin{cases} f(t) & t \le T \\ 0 & t > T, \end{cases}$$

then $\mathscr{L}_{2e}^m[0,\infty)$ (in short $\mathscr{L}_{2e}^m$ or $\mathscr{L}_{2e}$) denotes the extended Lebesgue space of functions $f : \mathbb{R} \to \mathbb{R}^m$ satisfying $\mathcal{T}_T f \in \mathscr{L}_2^m$, $\forall T > 0$. In contrast, let $\mathscr{H}_2$ denote a vector space of matrix-valued functions $F(s)$ analytic in the open right-half plane (RHP) (ie. $Re(s) > 0$) satisfying $\|F\|_2 := \sup_{\sigma>0}(\frac{1}{2\pi}\int_\infty^\infty |F(\sigma + j\omega)|^2 \; d\omega)^{1/2} < \infty$. Also, $\mathscr{H}_\infty$ denotes the space of bounded functions in the open RHP such that $\|F\|_\infty := \sup_{\omega\in\mathbb{R}} \bar{\sigma}[F(j\omega)] < \infty$, where $\bar{\sigma}(F)$ denotes the largest singular value of $F(s)$. We denote the real rational subspace of $\mathscr{H}_2$ (resp. $\mathscr{H}_\infty$) by $\mathscr{RH}_2$ (resp. $\mathscr{RH}_\infty$). The Parseval's relations provide that the Laplace transform yields an isomorphism between $\mathscr{L}_2$ and $\mathscr{H}_2$. Thus, $f(t) \in \mathscr{L}_2$ and $F(s) \in \mathscr{H}_2$ will be used interchangeably here. Now, consider a general operator $\Sigma^x : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$ with an initial condition $x \in \mathcal{X}_\Sigma \subset \mathbb{R}^n$.

Definition 1: The operator Σ^x is said to be ***causal*** if $\Sigma^x(f) \in \mathscr{L}_{2e}^k$ is uniquely determined $\forall f \in \mathscr{L}_{2e}^m$ $\forall x \in \mathcal{X}_\Sigma$, and $\mathcal{T}_T\Sigma^x\mathcal{T}_T = \mathcal{T}_T\Sigma^x$ holds $\forall T > 0$ and $\forall x \in \mathcal{X}_\Sigma$.

The authors are all with the Department of Information Engineering, Research School of Information Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia.

Brian D. O. Anderson is also with National ICT Australia Ltd., Locked Bag 8001, Canberra, ACT 2601, Australia.

Corresponding Author: Sung H. Cha.

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A. Different System Representations

One of the most natural ways to represent a dynamic system is via a mapping from an input signal to an output signal. Let $P^{x_P} : \mathscr{L}^m_{2e} \to \mathscr{L}^k_{2e}$ (resp. $C^{x_C} : \mathscr{L}^k_{2e} \to \mathscr{L}^m_{2e}$) denote the *input-output representation* of a plant (resp. controller) with an initial condition $x_P \in \mathcal{X}_P$ (resp. $x_C \in \mathcal{X}_C$).

Definition 2: For a causal operator $P^{x_P} : \mathscr{L}^m_{2e} \to \mathscr{L}^k_{2e}$, a pair of operators (M, N) is a ***right fractional representation*** (resp. left fractional representation) if $\forall x_P \in \mathcal{X}_P$ it can be factorized into $P^{x_P} = MN^{-1}$ (resp. $P^{x_P} = N^{-1}M$), where M and N are bounded and N is (causally) invertible.

For an LTI system, its input-output representation in the time-domain has an associated analogue operator in the s-domain, the ***transfer function***. Let $\mathbf{P}(s) : U(s) \mapsto Y(s)$ denote a transfer function of $P : u \in \mathscr{L}^m_{2e} \mapsto y \in \mathscr{L}^k_{2e}$, where $U(s)$ and $Y(s)$ are the Laplace transforms of $u(t)$ and $y(t)$.

Definition 3: [6] A pair $(\mathbf{M}(s), \mathbf{N}(s)) \in \mathscr{RH}_\infty$ is ***right coprime*** (resp. left coprime) over $\mathscr{RH}_\infty$ if they have the same numbers of rows and $\exists \mathbf{X}(s), \mathbf{Y}(s) \in \mathscr{RH}_\infty$ that satisfy $\mathbf{X}(s)\mathbf{M}(s) + \mathbf{Y}(s)\mathbf{N}(s) = I$ (resp. $\mathbf{M}(s)\mathbf{X}(s) + \mathbf{N}(s)\mathbf{Y}(s) = I$).

Here, $[P, C]$ denotes the closed-loop interconnection in Fig. 1 with external inputs (disturbances or reference signals) w and r, and internal signals u and y.

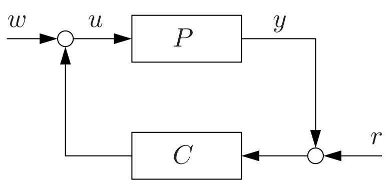

Fig. 1. Standard Feedback Configuration.

Definition 4: [13] The interconnection $[P, C]$, with $P : \mathscr{L}^m_{2e} \to \mathscr{L}^k_{2e}$ and $C : \mathscr{L}^k_{2e} \to \mathscr{L}^m_{2e}$ in Fig. 1, is said to be ***well-posed*** if the mapping $H_{[P,C]} : \binom{r}{w} \mapsto \binom{y}{u}$ exists and is weakly Lipschitz. Furthermore, $[P, C]$ is said to be ***internally stable*** if it is well-posed and $H_{[P,C]}$ is bounded.

Remark 5: [6] For the LTI and finite dimensional case where both $\mathbf{P}(s)$ and $\mathbf{C}(s)$ are in $\mathscr{RH}_\infty$, well-posedness is equivalent to the condition that $\begin{bmatrix} I & -\mathbf{C}(s) \\ -\mathbf{P}(s) & I \end{bmatrix}^{-1}$ exists and is proper. Also, internal stability is equivalent to the condition $\begin{bmatrix} I & -\mathbf{C}(s) \\ -\mathbf{P}(s) & I \end{bmatrix}^{-1} \in \mathscr{RH}_\infty$.

Due to the absence of a left fractional representation for nonlinear systems in most cases, we shall utilize a more general representation of systems, known as the *kernel representation*; see [16] for more details.

Definition 6: Consider a causal operator $P : \mathscr{L}^m_{2e} \to \mathscr{L}^k_{2e}$ with an initial condition space $\mathcal{X}_P$. Then a causal operator $R^{x_P}_P : \mathscr{L}^m_{2e} \times \mathscr{L}^k_{2e} \to \mathscr{L}^k_{2e}$ is called a ***kernel representation*** of P if $\forall x_P \in \mathcal{X}_P \; \forall u \in \mathscr{L}^m_{2e}, y = P^{x_P} u \Leftrightarrow R^{x_P}_P(u, y) = 0$ holds with $y \in \mathscr{L}^k_{2e}$.

Definition 7: A kernel operator $R^{x_P}_P$ is ***well-defined*** if there exists the causal operator $(R^{x_P}_P)^\# : \mathscr{L}^m_{2e} \times \mathscr{L}^k_{2e} \to \mathscr{L}^k_{2e}$ such that $y = (R^{x_P}_P)^\#(u, z) \Leftrightarrow R^{x_P}_P(u, y) = z, \forall x_P \in \mathcal{X}_P$, $\forall u \in \mathscr{L}^m_{2e}$ and $y, z \in \mathscr{L}^k_{2e}$.

Definition 8: The operator is said to be ***weakly Lipschitz*** (or weakly Lipschitz continuous) if it is causal and its Lipschitz semi-norm

$$\|\mathcal{T}_T \Sigma^x\|_L := \sup_{u,\nu \in \mathscr{L}^m_{2e}, \;\; \mathcal{T}_T u \neq \mathcal{T}_T \nu} \frac{\|\mathcal{T}_T \Sigma^x u - \mathcal{T}_T \Sigma^x \nu\|_p}{\|\mathcal{T}_T u - \mathcal{T}_T \nu\|_p} \quad (1)$$

is finite for every $T > 0$ and $x \in \mathcal{X}_\Sigma$.

Definition 9: The operator Σ^x is said to be ***smoothing*** if it is weakly Lipschitz and for every $T > 0$, $\gamma > 0$ and $x \in \mathcal{X}_\Sigma$ there exists $t_1 = t_1(T, \gamma, x) \in (0, T)$ such that

$$\|\mathcal{T}_{t+t_1}(\Sigma^x \mathcal{T}_{t+t_1} - \Sigma^x \mathcal{T}_t)\|_L \leq \gamma \quad (2)$$

holds for $\forall t \in [0, T - t_1]$.

Remark 10: [17] The sum (or cascade) of two weakly Lipschitz operators is also weakly Lipschitz.

B. System of Interest

We shall consider a finite dimensional linear time-invariant (FDLTI) dynamical system $P^{x_P} : u \in \mathscr{L}^m_{2e} \mapsto y \in \mathscr{L}^k_{2e}$ described by a state space realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t); \\ y(t) &= Cx(t) + Du(t), \end{aligned} \quad (3)$$

where $x(t) \in \mathcal{X}_P$, $u(t)$ and $y(t)$ are the system state variable, (control) input and system output, respectively with an initial condition $x_P := x(t_0)$. Alternatively, the transfer function $\mathbf{P}(s) : U(s) \mapsto Y(s)$ with zero initial condition ($x(0) = 0$) can be expressed as $\mathbf{P}(s) := C(sI - A)^{-1}B + D$. Note that the system equations in (3) can be written as

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}.$$

Hence, throughout this paper the notation

$$\mathbf{P}(s) : \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

will be used interchangeably.

Definition 11: A square matrix A is said to be ***Hurwitz*** (or stable) if all eigenvalues have negative real part.

Remark 12: A state space realization (3) with a Hurwitz matrix A has a transfer function in $\mathscr{RH}_\infty$.

Definition 13: A pair of real matrices $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ is said to be ***stabilizable*** if $\exists K \in \mathbb{R}^{m \times n}$ (a.k.a. stabilizing state feedback gain) such that $A + BK$ is Hurwitz. Similarly, a pair of real matrices $(C, A) \in \mathbb{R}^{k \times n} \times \mathbb{R}^{n \times n}$ is said to be ***detectable*** if $\exists H \in \mathbb{R}^{n \times k}$ (a.k.a. stabilizing state estimation gain) such that $A + HC$ is Hurwitz.

Many standard optimal control literatures—such as [18], [5]—discuss the state estimation problem and its solution based on the deterministic filtering problem. For the system (3) where (C, A) is detectable (ie, $\exists H$ such that $A + HC$ is Hurwitz), the standard state observer

$$\dot{x_e}(t) = (A + HC)x_e(t) + Bu(t) - Hy(t) \quad (4)$$

takes $y(t)$ and $u(t)$ as its inputs, and produces the state estimate $\hat{x}(t)$ that ensures the error between the true state and its estimate, $e(t) := x(t) - x_e(t)$, converges asymptotically to zero. Furthermore if (A, B) is stabilizable (ie, $\exists K$ such that $A + BK$ is Hurwitz), then the (state feedback) control law $u(t) = Kx(t)$ can be implemented using the estimated state $\hat{x}$ from (4), instead of x which is generally not available. This so-called standard observer/output feedback system can be expressed as

$$\begin{aligned} \dot{x}_e &= (A + BK + HC + HDK)x_e + (B + HD)r - Hy \\ u &= Kx_e + r, \end{aligned} \quad (5)$$

which ensures the error between the true state and its estimate $e(t) := x(t) - x_e(t)$ converges asymptotically to zero, irrespective of the external reference input $r(t)$. There exist techniques to obtain desired gain K and H; see e.g. [5].

III. Observer-form controller implementation

We shall first review observer-form implementation of linear controllers using coprime factorizations. This will include detailed analysis of the relationship between coprime factorizations and the corresponding state space realizations, cited from the existing literature. Second, we will present an analogous observer-form implementations of controllers using kernel representations. In particular, we will establish a relationship between controllers in a kernel representation and its state space realization in the standard state estimation/feedback configuration.

A. Implementing Controllers using Coprime Factorization

An important design objective is to drive the output of the plant to follow a desired reference signal or trajectory, the so-called tracking problem. In the absence of plant uncertainty (ie, either the plant is known or can be modelled completely) and disturbances, this tracking problem can be considered as an open-loop problem [7]. Consider $\mathbf{P}(s) = \mathbf{N}(s)\mathbf{M}^{-1}(s)$, where $(\mathbf{N}(s), \mathbf{M}(s))$ are right coprimes over $\mathscr{RH}_\infty$. A feed-forward controller $\mathbf{M}(s)\mathbf{Q}(s)$, with $\mathbf{Q}(s) \in \mathscr{RH}_\infty$ a bounded operator designed to achieve desired performance, in the open-loop configuration of Fig. 2 can achieve

$$y(t) = \mathbf{P}(s)\mathbf{M}(s)\mathbf{Q}(s)r(t) = \mathbf{N}(s)\mathbf{Q}(s)r(t) \quad (6)$$

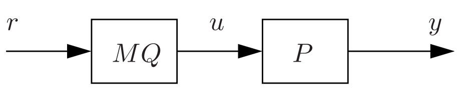

Fig. 2. Open-loop configuration for $y = NQr$.

In practice, however, the plant cannot be modelled completely and/or signals are rather noisy. Hence to control such plant uncertainty and external disturbances, a feedback compensator $\mathbf{C}(s)$ is introduced to form an interconnection as in Fig. 1. A mechanism to implement such controller without affecting the nominal tracking response is shown in Fig. 3. Here one utilizes the Bezout identity $\tilde{V}M - \tilde{U}N = I$, where $\mathbf{C} = \tilde{\mathbf{V}}^{-1}\tilde{\mathbf{U}}$ with $(\tilde{V}, \tilde{U})$ left coprime over $\mathscr{RH}_\infty$.

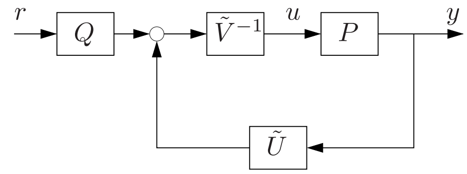

Fig. 3. Feedback implementation of $\mathbf{C}(s)$.

Note that in Fig. 3

$$\begin{aligned} y(t) &= \mathbf{P}(s)[I - \mathbf{C}(s)\mathbf{P}(s)]^{-1}\tilde{\mathbf{V}}^{-1}(s)\mathbf{Q}(s)r(t) \\ &= \mathbf{N}(s)[\tilde{\mathbf{V}}(s)\mathbf{M}(s) - \tilde{\mathbf{U}}(s)\mathbf{N}(s)]^{-1}\mathbf{Q}(s)r(t) \\ &= \mathbf{N}(s)\mathbf{Q}(s)r(t), \end{aligned}$$

which is equivalent to (6). Hence, the disturbance rejection problem can be entirely decoupled from the tracking problem; see e.g. [1].

Remark 14: [7] Notice that if the feedback part of the controller is implemented in any other way then the tracking responses may be compromised. In particular, if the controller $\mathbf{C}(s)$ is non-minimum phase and is implemented as $\tilde{\mathbf{V}}^{-1}(s) = \mathbf{C}(s)$ and $\tilde{\mathbf{U}}(s) = I$ (ie, entirely in forward path), then the RHP zeros of $\mathbf{C}(s)$ will appear in the transfer function from r to y for any $\mathbf{Q}(s)$. Similarly, if the feedback compensator $\mathbf{C}(s)$ has RHP poles (ie. unstable) and is implemented as $\tilde{\mathbf{V}}^{-1}(s) = I$ and $\tilde{\mathbf{U}}(s) = \mathbf{C}(s)$ (entirely in feedback path), then unstable poles will appear as RHP zeros of the tracking responses.

As a special case, when $\mathbf{Q}(s) = I$, we have the feedback implementation in Fig. 4, which is referred as *observer-form implementation* of feedback controller $\mathbf{C}(s)$.

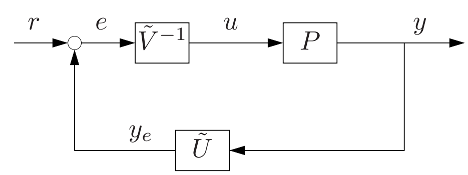

Fig. 4. Observer-form implementation of $\mathbf{C}(s)$.

Note that since $(I + \tilde{V})$ is a strictly proper transfer function, the controller equation can be rewritten as

$$u = [-\tilde{U} \quad I + \tilde{V}]\begin{pmatrix} y \\ u \end{pmatrix} - r, \quad (7)$$

which is shown in Fig. 5.

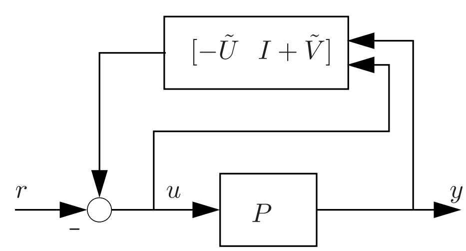

Fig. 5. Observer-form implementation of $\mathbf{C}(s)$.

The following theorem shows how the coprime factorization of $\mathbf{P}(s)$ can be related to a state space realization. The theorem and its proof are not from a single literature, but can be considered as a combined summary of the existing results from [5], [4], [6], [3].

Theorem 15: Consider a plant $P^{x_P} : u(t) \in \mathscr{L}_{2e}^m \mapsto y(t) \in \mathscr{L}_{2e}^k$ with a proper real-rational transfer function $\mathbf{P}(s)$ and its state space realization is given as

$$P : \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{8}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times m}$. Suppose (A, B) is stabilizable and (C, A) is detectable and let $K \in \mathbb{R}^{m\times n}$ and $H \in \mathbb{R}^{n\times k}$ be such that $A + BK$ and $A + HC$ are Hurwitz. Define

$$\begin{bmatrix} \mathbf{M}(s) & \mathbf{U}(s) \\ \mathbf{N}(s) & \mathbf{V}(s) \end{bmatrix} : \left[\begin{array}{c|cc} A+BK & B & -H \\ \hline K & I & 0 \\ C+DK & D & I \end{array}\right], \tag{9}$$

$$\begin{bmatrix} \tilde{\mathbf{V}}(s) & -\tilde{\mathbf{U}}(s) \\ -\tilde{\mathbf{N}}(s) & \tilde{\mathbf{M}}(s) \end{bmatrix} : \left[\begin{array}{c|cc} A+HC & -(B+HD) & H \\ \hline K & I & 0 \\ C & -D & I \end{array}\right]. \tag{10}$$

Then the followings are true:

(a) All eight transfer functions are in $\mathscr{RH}_\infty$;
(b) $\mathbf{M}(s)$ and $\tilde{\mathbf{M}}(s)$ are nonsingular;
(c) $(\mathbf{M}(s), \mathbf{N}(s))$ is right coprime of $\mathbf{P}(s)$ over $\mathscr{RH}_\infty$;
(d) $(\tilde{\mathbf{M}}(s), \tilde{\mathbf{N}}(s))$ is left coprime of $\mathbf{P}(s)$ over $\mathscr{RH}_\infty$;
(e) $\begin{bmatrix} \mathbf{M}(s) & \mathbf{U}(s) \\ \mathbf{N}(s) & \mathbf{V}(s) \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{V}}(s) & -\tilde{\mathbf{U}}(s) \\ -\tilde{\mathbf{N}}(s) & \tilde{\mathbf{M}}(s) \end{bmatrix} = I.$

Proof: (a): Given $A + BK$ and $A + HC$ are Hurwitz, (a) follows immediately from Remark 12.

(b): For a square matrix $X \in \mathbb{R}^{n\times n}$, we have $(sI - X)^{-1} = \text{adj}(sI - X)[\det(sI - X)]^{-1}$, where $\det(sI - X)$ is a monic polynomial in s of degree n. Since each entry of $\text{adj}(sI - X)$ is strictly less than n and $\det(\cdot)$ is a continuous function of its arguments, $\lim_{s\to\infty} \det(\mathbf{M}(s)) = \lim_{s\to\infty} \det(\tilde{\mathbf{M}}(s)) = 1$. Hence, (b) is proved.

(c): Since $\mathbf{M}(s)$ is not singular,

$$\mathbf{M}^{-1}(s) : \left[\begin{array}{c|c} A+BK-BK & -B \\ \hline K & I \end{array}\right] = \left[\begin{array}{c|c} A & -B \\ \hline K & I \end{array}\right] \tag{11}$$

and thus,

$$\mathbf{N}(s)\mathbf{M}^{-1}(s) : \left[\begin{array}{cc|c} A+BK & BK & B \\ 0 & A & -B \\ \hline C+DK & DK & D \end{array}\right].$$

Consider now the state space transformation $T = \left[\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right]$ and after removing uncontrollable modes, we have

$$\mathbf{N}(s)\mathbf{M}^{-1}(s) : \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] : \mathbf{G}(s).$$

Hence (c) is proved.

(d): Similarly, it is easy to show (d) using $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{N}}$ instead.

(e): Computing $\left[\begin{smallmatrix} \mathbf{M}(s) & \mathbf{U}(s) \\ \mathbf{N}(s) & \mathbf{V}(s) \end{smallmatrix}\right] \left[\begin{smallmatrix} \tilde{\mathbf{V}}(s) & -\tilde{\mathbf{U}}(s) \\ -\tilde{\mathbf{N}}(s) & \tilde{\mathbf{M}}(s) \end{smallmatrix}\right]$ gives

$$\left[\begin{array}{c|cc} A+BK & B & -H \\ \hline K & I & 0 \\ C+DK & D & I \end{array}\right] \left[\begin{array}{c|cc} A+HC & -(B+HD) & H \\ \hline K & I & 0 \\ C & -D & I \end{array}\right]$$
$$= \left[\begin{array}{cc|cc} A+HC & BK-HC & B+HD & -H \\ 0 & A+HC & -(B+HD) & H \\ \hline K & K & I & 0 \\ C+DK & C+DK & 0 & I \end{array}\right]$$
$$= \left[\begin{array}{cc|cc} A+HC & 0 & 0 & 0 \\ 0 & A+HC & -(B+HD) & H \\ \hline K & 0 & I & 0 \\ C+DK & 0 & 0 & I \end{array}\right] = I,$$

where the second equality comes with the state space transformation $T = \left[\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right]$, and the third equality comes by removing stable and uncontrollable modes. ■

Using Theorem 15, one can compute left and right coprime factorizations of $\mathbf{P}(s)$ as a state space realization. To do so, the determination of the state feedback gain $K \in \mathbb{R}^{m\times n}$ and the observer gain $H \in \mathbb{R}^{n\times k}$ is required such that $A + BK$ and $A + HC$ are Hurwitz. Appropriate selection of K and H can result in standard observer/output feedback controller $\mathbf{C}(s)$ of the form in (5). Indeed, Theorem 15 part (e), shows that a controller of the form $\mathbf{C}(s) := \mathbf{U}(s)\mathbf{V}^{-1}(s) = \tilde{\mathbf{V}}^{-1}(s)\tilde{\mathbf{U}}(s)$ stabilizes $\mathbf{P}(s)$.

Remark 16: It is straightforward to extract the estimated state x_e of $\mathbf{P}(s)$ from $\tilde{\mathbf{V}}^{-1}(s)$ and $\tilde{\mathbf{U}}(s)$. From (10),

$$\tilde{\mathbf{V}}(s) : \left[\begin{array}{c|c} A+HC & -(B+HD) \\ \hline K & I \end{array}\right], \quad -\tilde{\mathbf{U}}(s) : \left[\begin{array}{c|c} A+HC & H \\ \hline K & 0 \end{array}\right].$$

and hence,

$$\tilde{\mathbf{V}}^{-1}(s) : \left[\begin{array}{c|c} A+BK+HC+HDK & B+HD \\ \hline K & I \end{array}\right].$$

Now, let x_a and x_b be internal states of $\tilde{\mathbf{V}}^{-1}(s)$ and $\tilde{\mathbf{U}}(s)$, respectively, and expand above equations into full differential equations, we have

$$\begin{aligned} \dot{x}_a &= (A + BK + HC + HDK)x_a + (B + HD)(r + y_e); \\ u &= Kx_a + e \end{aligned} \tag{12}$$

and

$$\dot{x}_b = (A + HC)x_b + Hy; \quad y_e = -Kx_b. \tag{13}$$

From differential equations in (12) and (13),

$$\begin{aligned} \dot{x}_a - \dot{x}_b &= [(A+BK+HC+HDK)x_a + (B+HD)(r+y_e)] \\ &\quad - [(A + HC)x_b + Hy] \\ &= [(A+BK+HC+HDK)x_a + (B+HD)r \\ &\quad + (B+HD)(-Kx_b)] - [(A + HC)x_b + Hy] \\ &= (A + BK + HC + HDK)(x_a - x_b) \\ &\quad + (B + HD)r - Hy; \\ u &= Kx_a + (r - Kx_b) = K(x_a - x_b) + r, \end{aligned} \tag{14}$$

which is the state estimator in (5) with $x_e := x_a - x_b$.

B. Nonlinear controller using kernel representation

We aim to generalize the observer-form controller implementation of Fig. 4 for the cases where the controller C is nonlinear. For such nonlinear cases, a left fractional representation (or left factorization) is generally not available. Thus, we shall utilize the kernel representation instead and shall first construct an analogous setting to that in Fig. 4.

Assume that the linear plant $P^{x_P} : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^k$ is strictly proper, but the controller $C^{x_C} : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^m$ can be nonlinear. Suppose the designed controller has a kernel representation $R_C^{x_C} : \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^m$ such that with $r \equiv 0$, $u = C(y) \Leftrightarrow R_C^{x_C}(y,u) = 0$.

Note that in Fig. 4, the reference signal r is fed in between two coprime factors of $C = \tilde{V}^{-1}\tilde{U}$ and

$$u = Cy + \tilde{V}^{-1} r \;\Leftrightarrow\; r = [-\tilde{U} \quad \tilde{V}]\begin{pmatrix} y \\ u \end{pmatrix}. \tag{15}$$

Since we utilize the kernel representation $R_C^{x_C}(y,u)$ as a generalized left factorization, an analogous configuration to that of (15) for the nonlinear C can be expressed as $r = R_C(y,u)$. If R_C is well-defined, by Definition 7 there exists $R_C^{\#} : \mathscr{L}_{2e}^k \times \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^m$ such that

$$r = R_C(y,u) \;\Leftrightarrow\; u = R_C^{\#}(y,r) \tag{16}$$

This leads to advancing a nonlinear observer-form controller implementation as in Fig. 6, which is the nonlinear counterpart of the linear case in Fig. 4.

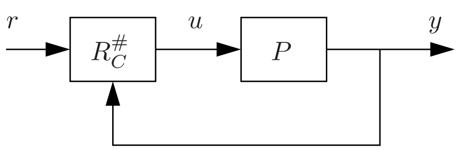

Fig. 6. Observer-form implementation of nonlinear controllers

Note that if P is smoothing and $R_C^{\#}$ is weakly Lipschitz, this nonlinear observer-form is, in fact, well-posed [17]. Since P is LTI and strictly proper, it can be shown via Definition 9 that P is smoothing. To check the weakly Lipschitz condition of $R_C^{\#}$, C can be restricted to be weakly Lipschitz at the design stage. If $r := R_C(y,u) = W(u - C(y))$, where W is a weakly Lipschitz and invertible function, $R_C(y,u)$ and $R_C^{\#}$ are weakly Lipschitz via Remark 10.

Given the well-posedness of the interconnection, a simple manipulation of $r = R_C(y,u)$ will give an expression for $R_C^{\#}$ in terms of (y,u) and r as

$$u = R_C^{\#}(y,r) = R_C(y,u) + u - r, \tag{17}$$

which is shown in Fig. 7. We require an additional assumption here to ensure that the inner loop is well-posed, viz. $R_C(y,u) + u$ is smoothing in u, for any y. This is the analog of requiring $I + \tilde{V}$ to be strictly proper. We will see in the sequel that this assumption is readily satisfied in the situation of interest to us. Next, a relationship between kernel representation and state space realization is established.

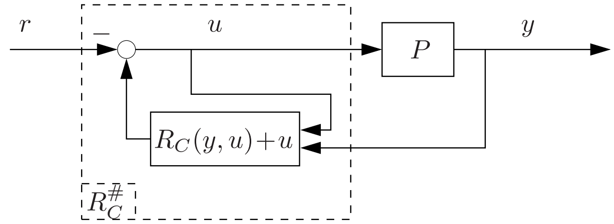

Fig. 7. Alternative observer-form implementation of nonlinear controllers

Theorem 17: Consider a LTI plant $P^{x_P} : u(t) \in \mathscr{L}_{2e}^m \mapsto y(t) \in \mathscr{L}_{2e}^k$ with a proper real-rational transfer function $\mathbf{P}(s)$ and its state space realization in (8). Suppose (A,B) is stabilizable and (C,A) is detectable, and let $C^{x_C} : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^m$ be a nonlinear pre-designed controller with a nonlinear state feedback control law $u_{NL}(x)$. There exists a well-defined kernel operator $R_C^{x_C} : \mathscr{L}_{2e}^{k+m} \to \mathscr{L}_{2e}^m, \forall x_C \in \mathcal{X}_C$ such that $u = [R_C^{x_C}]^{\#}(y,r) \Leftrightarrow u = u_{NL}(x_e) - r$ with the standard state estimator operator, $O : (y,u) \mapsto x_e$, and a state space realization of the form in (4).

Proof: Since (A,B) is stabilizable and (C,A) is detectable, there exist $K \in \mathbb{R}^{m\times n}$ and $H \in \mathbb{R}^{m\times k}$ such that $A + BK$ and $A + HC$ are Hurwitz. Hence, for the LTI plant P one can build the standard state estimator as in (4). In fact, one can compute the transfer function from (4) as $\mathbf{O}(s) : \binom{y}{u} \mapsto x_e = [sI - A - HC]^{-1}[-H \quad B]$.

From (17) and defining $R_C(y,u)$ as $u_{NL} \circ O(y,u) - u$ (as in Fig. 8), we have $u = [R_C^{x_C}]^{\#}(y,r) \Leftrightarrow u = u_{NL}(x_e) - r$. ∎

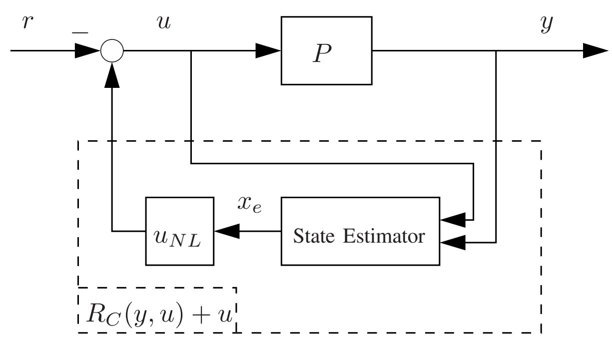

Fig. 8. nonlinear control law.

IV. Numerical Example

We borrow the plant and nonlinear controllers from [19], in which the use of a nonlinear controller to replace a linear controller is motivated. In the setting of [19], a nonlinear controller is specifically designed to decrease the percentage overshoot in the system state as the magnitude of the initial disturbance increases. In particular, the nonlinear state feedback control law of the form

$$u(x) := -Kx + u_{NL}(x) \tag{18}$$

is used, where K is a linear gain and u_{NL} is a nonlinear homogeneous function of the system state x. Consider

$$\mathbf{P}(s) = \frac{1}{s^2 + 2s + 17}$$

with a state-space realization (3), where $A = \left[\begin{smallmatrix} -1 & 4 \\ -4 & -1 \end{smallmatrix}\right]$, $B = [0 \ \ 0.25]'$, $C = [1 \ \ 0]$ and $D = 0$. In [19] exact value for K in (18) is not provided and can be chosen arbitrary as long as it makes the state feedback system with $u(x) = -Kx$ asymptotically stable. Hence, instead of designing K, we can design directly a stable transfer function

$$C^L(s) = \frac{-0.07266s - 1.391}{s^2 + 6.985s + 11.9}$$

with a left coprime factorization $C^L = (\tilde{V})^{-1}\tilde{U}_1$, where $\tilde{V}(s) = \frac{s^2+6.985s+11.9}{s^2+7s+12}$ and $\tilde{U}(s) = \frac{-0.07266s-1.391}{s^2+7s+12}$.

Using Theorem 15, we get

$$\tilde{V}(s) : \left[\begin{array}{cc|c} -7 & -3 & 0.25 \\ 4 & 0 & 0 \\ \hline -0.0586 & -0.103 & 1 \end{array}\right], \ \tilde{U}(s) : \left[\begin{array}{cc|c} -7 & -3 & 0.5 \\ 4 & 0 & 0 \\ \hline -0.145 & -0.696 & 0 \end{array}\right].$$

The nonlinear part is given in [19] as

$$u_{NL}(x) = -\frac{q}{4}(x_1^2 + x_2^2)x_2, \qquad (19)$$

where $x = (x_1, x_2)$ is the system state and q is a design parameter. We implement this using the observer from the linear part; see Remark 16. The standard observer transfer function can be given as

$$O(s) = \begin{bmatrix} \frac{5s-5}{s^2+7s+12} & \frac{1}{s^2+7s+12} \\ \frac{-2.5s-22.5}{s^2+7s+12} & \frac{0.25s-0.5}{s^2+7s+12} \end{bmatrix} \qquad (20)$$

We implement the above closed-loop using our proposed mechanism and achieve the internal state convergence results shown in Fig. 9, which conform with those in [19].

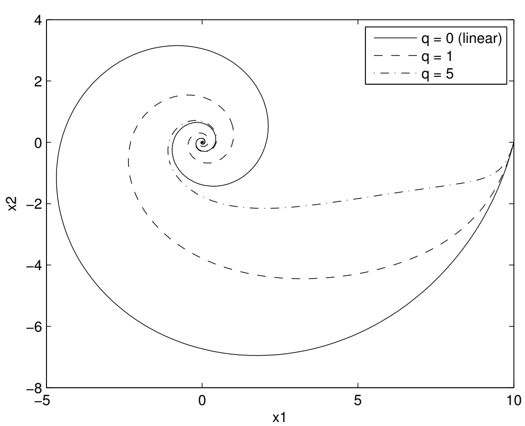

Fig. 9. Internal state x_1 and x_2 convergence results with different q values.

In this example, we have illustrated how easily one can implement a nonlinear controller in a state feedback law as an output feedback (nonlinear) operator using our proposed results. This not only allows more practical implementations, but also reveals the internal structure of nonlinear feedback interconnection.

V. Conclusions

We have established a connection between the state-space realization and observer-form based nonlinear feedback systems. In the absence of left coprime factorizations for nonlinear controllers, we have utilized kernel representations of the nonlinear controller instead. A numerical example has been discussed to show the effectiveness of our results and their applications in practice. The results are of importance in an adaptive control setting for developing data based stability tests to verify closed-loop stability with a proposed controller in advance of its actual insertion in the loop. Our current research attempts focus on extending our proposed results to include nonlinear plants, of our interest.

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