On an iterative algorithm to compute the positive stabilizing solution of generalized algebraic Riccati equations

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Abstract:
An iterative algorithm to solve a kind of generalized algebraic Riccati equations (GARE) in LQ stochastic zero-sum game problems is proposed. In our algorithm, we replace the problem of solving a GARE with an indefinite quadratic term by the problem of solving a sequence of GARE with a negative semidefinite quadratic term which can be solved by existing methods. Under some appropriate conditions, we prove that our algorithm is globally convergent.

Key Words: GARE, Iterative, Stochastic

1 Introduction

We consider the following GARE which arises from LQ stochastic zero-sum game problems

$$0 = A^T\Pi + \Pi A + \tilde{A}^T\Pi\tilde{A} - \Pi(B_2B_2^T - B_1B_1^T)\Pi + C^TC, \quad (1)$$

where $A \in \mathbb{R}^{n\times n}$, $\tilde{A} \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times p}$, $B_2 \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{r\times n}$ are given real matrices and $\Pi \in \mathbb{R}^{n\times n}$ is the unique positive semidefinite and stabilizing solution we seek. Compared with traditional H_∞ algebraic Riccati equations (ARE) in deterministic systems, equation (1) includes an additional disturbance term $\tilde{A}^T\Pi\tilde{A}$ which comes from state-dependent noises in the underlying systems. Equation (1) arises in characterizing the solution of a differential game problem for the following system Σ

$$\begin{aligned} x_0 &= x(0) \\ dx(t) &= Ax(t)dt + B_1v(t)dt + B_2u(t)dt + \tilde{A}x(t)dw(t) \\ y &= Cx(t) \end{aligned}$$

where v is the control of an agent seeking to maximize and u is the control of an agent seeking to minimize the value of a performance index (2) given below, w is a Wiener process, and at the saddle point it is required that the associated closed loop system has a stability property and the control inputs u and v are of the form

$$u^* = -B_2^T\Pi x, \quad v^* = B_1^T\Pi x$$

The performance index associated with the system Σ is

$$J(x_0, u, v) = E\int_0^\infty \left(u^Tu + x^TC^TCx - v^Tv\right)dt \quad (2)$$

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Our interest is in providing a new type of solution algorithm to solve GARE (1), built on recent developments for solving algebraic Riccati equations (AREs) with an indefinite quadratic term (see [1, 2]). Such problems arise in some applications such as launch vehicle ascent (see [15]), diesel engine control (see [16]). These applications are important since they are motivated by H_∞ control theory, which plays an important role in modern control theory. Below we discuss past contributions to such problems.

Recall that in [1, 2], the problem of solving an ARE with an indefinite quadratic term is replaced by the problem of solving a sequence of AREs with a negative semidefinite quadratic term and each of them can be solved by some existing algorithms (for example the Kleinman algorithm in [3]); then the solution of the original ARE can be approximated by the sum of the solutions of the AREs with a negative semidefinite term. Since AREs considered in [1, 2] are from linear deterministic systems, the question arise here is:" can we extend the algorithm to linear stochastic systems ?" The immediate answer is yes. However, LQ stochastic game problems are more complicated than LQ deterministic game problems. For example, in LQ stochastic game problems, noises can enter state, control input and unknown disturbance input (see [13, 19] and references therein). Also, Riccati equations with random coefficients arise in Kalman filter and LQG controller design (see [17, 18] and references therein). Recently, stochastic differential games with state-dependent noises have attracted much attention and have been widely applied to various fields (see [14, 20, 21]) (for example to stochastic H_2/H_∞ control). Motivated by these applications, we focus on developing a new algorithm to solve the GARE (1) arising in LQ stochastic differential games with state-dependent noises.

Traditionally, in LQ deterministic game problems, in order to obtain a saddle point solution for each player, one needs to solve algebraic Riccati equations with a sign indefinite quadratic term. For linear time-invariant (LTI) deterministic control systems, to solve optimal control and H_∞ control problems, one needs to solve AREs and many algorithms are available. When the state-dependent noise is introduced into a LQ deterministic game problem, in order to obtain the saddle point solutions for each player, one needs to solve (1). However, equation (1) is typically more difficult to solve than AREs in LQ deterministic game problems because of the additional disturbance term reflected in (1). Newton's method can be used to solve (1) (see [20, 21] for example), however, in Newton's method, one must choose a suitable initial point to implement algorithms and such an initial point is not always straightforward to obtain. In our algorithm, we can always choose a simple initial point $P_0 = 0$ and this is an advantage of our algorithm. In our algorithm, we replace the task of solving a GARE (1) by the task of solving a sequence of GAREs with negative semidefinite quadratic term, then by using the algorithm in [5], we can solve these GAREs recursively. In some sense, our work in this paper is an extension of the work in [1, 2] since it provides an algorithm to solve GARE (1) which is more general than AREs considered in [1, 2].

Another motivation of this paper comes from the work in [5, 6]. In [5, 6], a GARE with a negative semidefinite quadratic term is considered to solve an optimal control problem associated with a kind of linear stochastic systems where the system's parameters are deterministic but state-dependent noises are included. In [5, 6], a sufficient condition is obtained for the existence of unique positive semidefinite and stabilizing solutions of such GAREs; also, an iterative algorithm to solve such GAREs is developed in [5, 6]. Hence the question of how to extend the algorithm in [5, 6] to solve GAREs with a sign indefinite quadratic term arises here. In some sense, the algorithm to solve GAREs with a negative semidefinite quaratic term in [5, 6] can be regarded as an extension of the Kleinman algorithm in [3], since an additional disturbance term is reflected in such GAREs.

There are some conceptual challenges when state-dependent noises enter linear systems. For example, for our algorithm in the LTI case (i.e. the algorithm in [1, 2]), stabilizability and detectability is required to implement our algorithm; so the question arises here of how to define stabilizability and detectability when state-dependent noises enter systems. In this paper, we will review the definitions of these important control concepts in the literature as appropriate for a stochastic framework.

The paper is organized as follows: Section 2 gives some definitions and preliminary results. Section 3 presents our main result. Section 4 states the algorithm. Section 5 establishes our conclusion.

Notation: $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices; $\mathbb{R}^+$ denotes the set of nonnegative real numbers; $\mathbb{Z}$ denotes the set of integers with $\mathbb{Z}_{\geq a}$ denoting the set of integers greater or equal to $a \in \mathbb{R}$. Define function spaces as follows:

$$\mathcal{U} = \left\{ u : \mathbb{R}^+ \to \mathbb{R}^m \,\middle|\, \int_{t_0}^{t_1} \|u(t)\|^2 dt < \infty \forall t_0, t_1 \in \mathbb{R}^+ \right\},$$
$$\mathcal{Y} = \left\{ y : \mathbb{R}^+ \to \mathbb{R}^r \,\middle|\, \int_{t_0}^{t_1} \|y(t)\|^2 dt < \infty \forall t_0, t_1 \in \mathbb{R}^+ \right\}.$$

2 Definitions and Preliminary Results

In this section, we will give some definitions and preliminary results.
To motivate the definitions in this section, we firstly define the following stochastic control system Δ

$$\Delta : \mathcal{U} \to \mathcal{Y}$$

given by the following equations:

$$x(0) = x_0 \quad (3)$$
$$dx(t) = Ax(t) + Bu(t)dt + \tilde{A}x(t)dw(t) \quad (4)$$
$$y(t) = Cx(t) \quad (5)$$

where $t \in \mathbb{R}^+$, $x_0 \in \mathbb{R}^n$ is the initial state, $u(t) \in \mathbb{R}^m$ is the input value, $x(t) \in \mathbb{R}^n$ is the state value, and $y(t) \in \mathbb{R}^p$ is the output value. The matrices $A \in \mathbb{R}^{n\times n}$, $\tilde{A} \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{r\times n}$ are all real. Without loss of generality, throughout this paper, we assume w to be one-dimensional standard Wiener process.
We first recall the concept of stabilizability, which generalizes the stabilizability of deterministic systems to the stochastic context and plays an important role in our algorithm.

Definition 1 *[4, 7] The system Δ is said to be **stabilizable**, briefly $(A, B|\tilde{A})$ is stabilizable, if there exists a feedback control $u(t) = Kx(t)$, such that for any $x_0 \in \mathbb{R}^n$, the closed-loop system*

$$dx(t) = Ax(t) + Bu(t)dt + \tilde{A}x(t)dw(t) \quad (6)$$

is asymptotically mean square stable, i.e.

$$\lim_{t\to\infty} Ex(t)x^T(t) = 0$$

where K is a constant matrix with a suitable dimension.

We remark that for systems of the form $dx(t) = A_1x(t)dt + A_2x(t)dw(t)$, the property of asymptotic mean square stability can be easily checked (for example use the method in [22]).
We now recall the parallel notion of detectability of the stochastic system Δ, which generalizes the detectability of deterministic systems.

Definition 2 *[4, 7] Let $A,\ C,\ \tilde{A}$ be the matrices appearing in the system Δ; if there exists a real matrix H with a suitable dimension, such that the closed-loop system*

$$dx = (A + HC)xdt + \tilde{A}xdw \quad (7)$$

*is asymptotically mean square stable, then $(A,\ C|\tilde{A})$ is called **stochastically detectable**.*

Definition 3 *Let $A,\ \tilde{A},\ B_1,\ B_2,\ C$ be the matrices appearing in equation* (1)*. If there exists a solution Π to* (1) *such that the system*

$$dx(t) = \left(A + B_1 B_1^T \Pi - B_2 B_2^T \Pi\right) x(t)dt + \tilde{A}x(t)dw(t)$$

*is asymptotically mean square stable, then Π is called a **stabilizing solution** of* (1)*.*

We define two functions $\hat{A}$ and $\bar{A}$ in the following. By using these two functions, we can simplify the expressions of Lemma 5 and Lemma 9 below.

Definition 4 *Let $A,\ B_1,\ B_2,\ C$ be the real matrix functions appearing in* (1)*. Suppose there exists a positive semidefinite stabilizing solution Π to* (1)*. Let $P \in \mathbb{R}^{n\times n}$. Let $\hat{A}_P \in \mathbb{R}^{n\times n}$ be defined as*

$$\hat{A}_P \ = \ A + B_1 B_1^T P - B_2 B_2^T P,$$

and let $\bar{A}_P \in \mathbb{R}^{n\times n}$ be defined as

$$\bar{A}_P \ = \ A + B_1 B_1^T P - B_2 B_2^T \Pi$$

We now set up some lemmas which parallel lemmas appearing in [1, 2, 8, 10, 11]. The first of these lemmas establishes some relations that will be very useful in the proof of the main theorem.

Lemma 5 *Let $A,\ \tilde{A},\ B_1,\ B_2,\ C$ be the real matrix functions appearing in* (1)*, and suppose $P,\ Z \in \mathbb{R}^{n\times n}$. Define*

$$\begin{aligned} F: \quad & \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n} \qquad (8)\\ & P \ \longmapsto PA + A^T P + \tilde{A}^T P \tilde{A} - P(B_2 B_2^T \\ & \quad - B_1 B_1^T)P + C^T C. \end{aligned}$$

If $P = P^T$ and $Z = Z^T$, then

$$\begin{aligned} F\left(P + Z\right) =& F\left(P\right) + Z\hat{A}_P + \hat{A}_P^T Z + \tilde{A}^T Z \tilde{A} \\ & - Z\left(B_2 B_2^T - B_1 B_1^T\right) Z \end{aligned} \qquad (9)$$

where $\hat{A}_P$ is defined in Definition 4. Furthermore, if $P = P^T$ and $Z = Z^T$ and they satisfy

$$0 = Z\hat{A}_P + \hat{A}_P^T Z + \tilde{A}^T Z \tilde{A} - Z B_2 B_2^T Z + F\left(P\right) \qquad (10)$$

then

$$F\left(P + Z\right) = Z B_1 B_1^T Z \qquad (11)$$

and

$$\rho[F(P + Z)] = \overline{\sigma}(B_1^T Z)^2. \qquad (12)$$

Proof. The first result can be obtained by direct computations; the second claim is then trivial. □

The next three lemmas (Lemma 6-Lemma 8) are known general results. The first of these gives sufficient conditions for the existence of the unique positive semidefinite and stabilizing solutions of a class of GAREs.

Lemma 6 *[4] Consider the system Δ, and assume that $(A, B|\tilde{A})$ is stabilizable and $(A, C|\tilde{A})$ is stochastically detectable. Then, there exists a positive semidefinite and stabilizing solution Z satisfying the following GARE*

$$0 = A^T Z + ZA + \tilde{A}^T Z \tilde{A} - Z B B^T Z + C^T C. \qquad (13)$$

Furthermore, Z is the unique stabilizing solution of (13) *(i.e. there is no other stabilizing solution to* (13)*).*

Proof. See [4]. □

The next lemma recalls a standard result on the stability of linear stochastic systems.

Lemma 7 *[4, 6, 7] Let $A,\ \tilde{A},\ C$ be the real matrix functions appearing in the system Δ, and suppose that the pair $(A, C|\tilde{A})$ is stochastically detectable. Then $(A, \tilde{A})$ is stable if and only if the following Lyapunov-type equation:*

$$0 = PA + A^T P + \tilde{A}^T P \tilde{A} + C^T C \qquad (14)$$

has a unique positive semidefinite solution P.

Proof. See [4, 6, 7]. □

The next lemma gives a uniqueness result regarding the stabilizing solution of (1).

Lemma 8 *Suppose there exists a stabilizing solution Π to* (1)*; then this solution must be the unique stabilizing solution to* (1) *(i.e. there is no other stabilizing solution to* (1)*). Furthermore, if $\Pi \geq 0$, then the system $dx(t) = \left(A - B_2 B_2^T \Pi\right) x(t) + \tilde{A}x(t)dw_1(t)$ is asymptotically mean square stable.*

Proof. The proof can be obtained by using the argument in [12]. □

The next lemma sets up some basic relationships between the stabilizing solution Π to equation (1) when it exists and the matrix functions P, Z satisfying equation (10). It is not standard, but is reminiscent of similar results in [1, 2, 8, 10, 11]. It plays a crucial role in establishing the validity of the algorithm of the next section.

Lemma 9 *Let $A,\ \tilde{A},\ B_1,\ B_2,\ C$ be the matrices appearing in* (1)*, $P = P^T \in \mathbb{R}^{n\times n}$ and $Z = Z^T \in \mathbb{R}^{n\times n}$ satisfying equation* (10)*, and a stabilizing $\Pi = \Pi^T \in \mathbb{R}^{n\times n}$ satisfying equation* (1)*, and let $\bar{A}_P$ be the function defined in Definition 4. Then*

(i) $\Pi \geq (P + Z)$ if $(\bar{A}_P, \tilde{A})$ is asymptotically mean square stable,

(ii) $(\bar{A}_{P+Z}, \tilde{A})$ is asymptotically mean square stable if $\Pi \geq (P + Z)$.

Proof. The proof can be obtained by using Lemma 6-Lemma 8. Since it is in parallel with the counterparts in [1, 2, 8, 10, 11], for brevity, it is omitted here. □

3 Main Result

In this section, we set up the main theorem by constructing two positive semidefinite matrix series P_k and Z_k, and we also prove that the series P_k is monotonically non-decreasing and converges to the unique positive semidefinite and stabilizing solution Π of GARE (1) if such a solution exists. The statement and proof in the following main theorem are in parallel with the counterparts of the main theorems in [1, 2, 8, 10, 11].

Theorem 10 *Let $A,\ \tilde{A},\ B_1,\ B_2,\ C$ be the real matrix functions appearing in* (1)*. Suppose that $(A, C|\tilde{A})$ is stochastically detectable and $(A, B_2|\tilde{A})$ is stabilizable, and define $F : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ as in* (8)*. Suppose there exists a positive semidefinite stabilizing solution Π of GARE* (1)*. Then*

(I) two square matrix series Z_k and P_k can be defined for all $k \in \mathbb{Z}_{\geq 0}$ recursively as follows:

$$P_0 = 0, \tag{15}$$

$$A_k = A + B_1 B_1^T P_k - B_2 B_2^T P_k, \tag{16}$$

$Z_k \geq 0$ is the unique stabilizing solution of

$$0 = Z_k A_k + A_k^T Z_k + \tilde{A}^T Z \tilde{A} - Z_k B_2 B_2^T Z_k + F(P_k), \tag{17}$$

$$P_{k+1} = P_k + Z_k; \tag{18}$$

(II) the two series P_k and Z_k in part (I) have the following properties:

1) $(A + B_1 B_1^T P_k, B_2|\tilde{A})$ is stabilizable $\forall k \in \mathbb{Z}_{\geq 0}$,

2) $F(P_{k+1}) = Z_k B_1 B_1^T Z_k \ \forall k \in \mathbb{Z}_{\geq 0}$,

3) $(A + B_1 B_1^T P_k - B_2 B_2^T P_{k+1}, \tilde{A})$ is asymptotically mean square stable $\forall k \in \mathbb{Z}_{\geq 0}$,

4) $\Pi \geq P_{k+1} \geq P_k \geq 0 \ \forall k \in \mathbb{Z}_{\geq 0}$;

(III) the limit

$$P_\infty := \lim_{k \to \infty} P_k$$

exists with $P_\infty \geq 0$. Furthermore, $P_\infty = \Pi$ is the unique stabilizing solution of GARE (1), which is also positive semidefinite.

Proof. The proof can be obtained by using Lemma 6-Lemma 9, together with the arguments in [1, 2, 8, 10, 11]. It is omitted here for brevity. □

The following corollary gives a condition under which there does *not* exist a stabilizing solution $\Pi \geq 0$ to $F(\Pi) = 0$. This is useful for terminating the recursion in finite iterations.

Corollary 11 *Given real matrices $A, \ \tilde{A}, \ B_1, B_2, C$ such that $(A, C|\tilde{A})$ is stochastically detectable and $(A, B_2|\tilde{A})$ is stabilizable, and let $\{P_k\}$ and $F : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ be defined as in Theorem 10. If $\exists k \in \mathbb{Z}_{\geq 0}$ such that $(A + B_1 B_1^T P_k, B_2|\tilde{A})$ is not stabilizable, then there does not exist a stabilizing solution $\Pi \geq 0$ to $F(\Pi) = 0$.*

Proof. Restatement of Theorem 10, implication $(II1)$. □

4 Algorithm

Given real matrices $A, \ \tilde{A}, \ B_1, B_2, C$ with compatible dimensions, a specified tolerance $\epsilon > 0$, and supposing $(A, C|\tilde{A})$ is stochastically detectable and $(A, B_2|\tilde{A})$ is stabilizable, an iterative algorithm for finding the positive semidefinite stabilizing solution of equation (1), when it exists, is given as follows:

1. Let $P_0 = 0$ and $k = 0$.
2. Set $A_k = A + B_1 B_1^T P_k - B_2 B_2^T P_k$.
3. Construct (for example using the algorithm in [5]) the unique real symmetric stabilizing solution $Z_k \geq 0$ which satisfies

$$0 = Z_k A_k + A_k^T Z_k + \tilde{A}^T Z_k \bar{A} - Z_k B_2 B_2^T Z_k + F(P_k), \tag{19}$$

where $F : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ is defined in (8).

4. Set $P_{k+1} = P_k + Z_k$.
5. If $\overline{\sigma}(B_1^T Z_k)^2 < \epsilon$, then set $\Pi = P_{k+1}$ and exit. Otherwise, go to step 6.
6. If $(A + B_1 B_1^T P_{k+1}, B_2|\bar{A})$ is stabilizable, then increment k by 1 and go back to step 2. Otherwise, exit as there does not exist a real symmetric stabilizing solution $\Pi \geq 0$ satisfying $F(\Pi) = 0$.

From Corollary 11 we see that if the stabilizability condition in step 6 fails at some $k \in \mathbb{Z}_{\geq 0}$, then there does not exist a stabilizing solution $\Pi \geq 0$ to $F(\Pi) = 0$ and the algorithm should terminate (as required by step 6). But when this stabilizability condition is satisfied $\forall k \in \mathbb{Z}_{\geq 0}$, construction of the series P_k and Z_k is always possible and either P_k converges to Π (which is captured by step 5) or P_k just diverges to infinity, which again means that there does not exist a stabilizing solution $\Pi \geq 0$ to $F(\Pi) = 0$.

Remark 1: For a method to check the stabilizability of a matrix pair in step 6, one can refer to [23].

5 Conclusion

In this paper, an iterative algorithm to compute the stabilizing solution of a GARE with a sign indefinite quadratic term is given. By using our proposed algorithm, we can reduce a GARE with a sign indefinite quadratic term to a sequence of GAREs with a negative semidefinite quadratic term which can be solved by existing methods; then the stabilizing solution of the original GARE can be approximated by the sum of the solutions of these GAREs with a negative semidefinite quadratic term. As for the corresponding deterministic version of the problem, our algorithm has a local quadratic rate of convergence and a natural game theoretic interpretation. However, the proofs directly parallel those of [2, 11] and these two results are not established here.

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