

## An iterative procedure to solve HJBI equations in nonlinear $H_\infty$ control<sup>\*</sup>

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**Abstract:** In this paper, an iterative algorithm to solve a special class of Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations is proposed. By constructing two series of nonnegative functions, we replace the problem of solving an HJBI equation by the problem of solving a sequence of Hamilton-Jacobi-Bellman (HJB) equations whose solutions can be approximated recursively by existing methods. The local convergence of the algorithm is guaranteed. A numerical example is provided to demonstrate the accuracy of the proposed algorithm.

Keywords: HJBI; iterative algorithm; locally exponentially stable.

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### 1. INTRODUCTION

Traditionally, in linear  $H_2$  control, one needs to solve LQ-type Algebraic Riccati Equations (AREs) with a negative semidefinite quadratic term; in linear  $H_\infty$  control, one needs to solve AREs with an indefinite quadratic term. Some iterative procedures to solve such AREs were proposed in Kleinman [1968] and Lanzon et al. [2007] respectively. In Kleinman [1968], a sequence of monotonically non-increasing matrices is constructed by solving Lyapunov equations to obtain the unique stabilizing solution of an ARE with a negative semidefinite quadratic term. In Lanzon et al. [2007], an ARE with an indefinite quadratic term is replaced by a sequence of AREs with a negative semidefinite quadratic term and each of them can be solved by the Kleinman algorithm; then the solution of the original ARE can be approximated by the sum of the solutions of the AREs with a negative semidefinite term. In some sense, the iteration scheme in Lanzon et al. [2007] is an extension to the one in Kleinman [1968], since both algorithms enjoy similar characteristics such as high numerical reliability, local quadratic rate of convergence (see Kleinman [1968], Lanzon et al. [2007]) and, as noted, the algorithm in Lanzon et al. [2007] can be applied in more general cases (i.e. solving AREs with an indefinite quadratic term).

Although linear optimal control theory, as well as linear  $H_\infty$  control theory, has been well developed in the past decades, matters become more complicated when

a nonlinear control system is considered. For example, in nonlinear optimal control, HJB equations need to be solved to obtain an optimal control law. However, HJB equations are first-order, nonlinear partial differential equations that have been proven to be impossible to solve in general and are often very difficult to solve for specific nonlinear systems. Since these equations are difficult to solve analytically, there has been much research directed toward approximating their solutions. For example, the technique of successive approximation in policy space (see Bellman [1971, 1957], Bellman et al. [1965]) can be used to approximate the solutions of HJB equations iteratively. In fact, it can be shown (see Leake et al. [1967]) that the technique of policy space iteration can be used to replace the problem of solving a nonlinear HJB partial differential equation by the problem of solving a sequence of linear partial differential equations. Also, in some sense, the iterative procedure to solve HJB equations in Leake et al. [1967] is a generalization of the Kleinman algorithm in Kleinman [1968], since both of them obtain solutions by constructing a sequence of monotonic functions or matrices while the algorithm in Leake et al. [1967] can be used in more general cases than just the LQ problem.

In nonlinear  $H_\infty$  control, given a disturbance attenuation level  $\gamma > 0$ , in order to solve the  $H_\infty$  suboptimal control problem, one needs to solve Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations. It is clear that HJBI equations are generally more difficult to solve than HJB equations, since the disturbance inputs are additionally reflected in HJBI equations. Recall the iterative algorithm in Lanzon et al. [2007]: an ARE with an indefinite quadratic term is reduced to a sequence of AREs with a negative semidefinite quadratic term, which are more easily solved by an existing algorithm (e.g. the Kleinman algorithm).

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If we regard HJB equations as the general version of AREs with a negative semidefinite quadratic term and HJBI equations as the general version of AREs with an indefinite quadratic term, then the question arising here is: "can we approximate the solution of an HJBI equation by obtaining the solutions of a sequence of HJB equations and thereby extend the algorithm in Lanzon et al. [2007] to nonlinear control systems?" In this paper, we will answer this question to some degree; that is, we extend the algorithm in Lanzon et al. [2007] to a special class of nonlinear control systems and develop an iterative procedure to solve a special class of HJBI equation associated with the nonlinear  $H_\infty$  control problem.

One method of finding solutions to HJBI equations is developed in Wise et al. [1994], where the method of characteristics is used to form integral expressions, the solutions of which can be approximated by successive approximations. We compare our algorithm to this one in a numerical example at the end of the paper.

The structure of the paper is as follows. Section 2 introduces the steady-state HJBI equation we treat in this paper. Section 3 recalls some existing definitions and results, and then establishes some preliminary results which will be used in the main theorem. Section 4 presents the main result. Section 5 states the algorithm. Section 6 gives a numerical example. Section 7 contains concluding remarks. Space limitations preclude the inclusion of most proofs.

## 2. PROBLEM SETTING

In this section, we introduce the steady-state HJBI equation we want to solve.

We begin with some notation:  $\mathbb{R}$  denotes the set of real numbers;  $\mathbb{R}^+$  denotes the set of nonnegative real numbers;  $(\cdot)^T$  denotes the transpose of a vector or a matrix;  $\bar{\sigma}(\cdot)$  denotes the maximum singular value of a matrix;  $\mathbb{Z}$  denotes the set of integers with  $\mathbb{Z}_{\geq a}$  denoting the set of integers greater or equal to  $a \in \mathbb{R}$ ;  $\mathbb{R}^n$  denotes an  $n$ -dimensional Euclidean space. For a given control system, define  $\mathbb{X}_0$  as a neighborhood of the origin in  $\mathbb{R}^n$ ,  $\mathbb{U}_0$  as a neighborhood of the origin in  $\mathbb{R}^m$ ,  $\mathbb{W}_0$  as a neighborhood of the origin in  $\mathbb{R}^q$ , and  $\mathbb{Y}_0$  as a neighborhood of the origin in  $\mathbb{R}^p$ . Define function spaces as follows:

$$\begin{aligned} \mathcal{X}_0 &= \left\{ x : \mathbb{R}^+ \rightarrow \mathbb{X}_0 \mid \int_{t_0}^{t_1} \|x(t)\|^2 dt < \infty \quad \forall t_0, t_1 \in \mathbb{R}^+ \right\}, \\ \mathcal{U}_0 &= \left\{ u : \mathbb{R}^+ \rightarrow \mathbb{U}_0 \mid \int_{t_0}^{t_1} \|u(t)\|^2 dt < \infty \quad \forall t_0, t_1 \in \mathbb{R}^+ \right\}, \\ \mathcal{W}_0 &= \left\{ w : \mathbb{R}^+ \rightarrow \mathbb{W}_0 \mid \int_{t_0}^{t_1} \|w(t)\|^2 dt < \infty \quad \forall t_0, t_1 \in \mathbb{R}^+ \right\}, \\ \mathcal{Y}_0 &= \left\{ y : \mathbb{R}^+ \rightarrow \mathbb{Y}_0 \mid \int_{t_0}^{t_1} \|y(t)\|^2 dt < \infty \quad \forall t_0, t_1 \in \mathbb{R}^+ \right\}. \end{aligned}$$

A matrix is said to be **Hurwitz** if all of its eigenvalues have negative real part.

We work with the nonlinear control system

$$\Gamma : \mathcal{U}_0 \times \mathcal{W}_0 \rightarrow \mathcal{Y}_0 \quad (1)$$

given by the following equations:

$$x(0) = x_0 \quad (2)$$

$$\dot{x}(t) = f(x(t)) + g_1(x(t))w(t) + g_2(x(t))u(t) \quad (3)$$

$$y(t) = h(x(t)) \quad (4)$$

where  $x \in \mathcal{X}_0$  is the state;  $x_0 \in \mathbb{X}_0$  is the initial state;  $u \in \mathcal{U}_0$  is the input;  $w \in \mathcal{W}_0$  is the disturbance;  $y \in \mathcal{Y}_0$  is the output.  $f : \mathbb{X}_0 \rightarrow \mathbb{R}^n$ ,  $g_1 : \mathbb{X}_0 \rightarrow \mathbb{R}^{n \times q}$ ,  $g_2 : \mathbb{X}_0 \rightarrow \mathbb{R}^{n \times m}$  and  $h : \mathbb{X}_0 \rightarrow \mathbb{R}^p$  are smooth functions with  $f(0) = 0$  and  $h(0) = 0$ . It is assumed further that  $f, g_1, g_2$  are such that (3) has a unique solution for any  $u \in \mathcal{U}_0$ ,  $w \in \mathcal{W}_0$ , and  $x_0 \in \mathbb{X}_0$ . Throughout this paper, it is further assumed that the functions  $f, g_1, g_2, h$  defined in the system  $\Gamma$  can be represented in the following form:

$$f(\tilde{x}) = F\tilde{x} + f_r(\tilde{x}), \quad (5)$$

$$g_1(\tilde{x}) = G_1 + g_{1r}(\tilde{x}), \quad (6)$$

$$g_2(\tilde{x}) = G_2 + g_{2r}(\tilde{x}), \quad (7)$$

$$h(\tilde{x}) = H\tilde{x} + h_r(\tilde{x}), \quad (8)$$

where  $F, G_1, G_2, H$  are real constant matrices with suitable dimensions and  $f_r(\tilde{x}), g_{1r}(\tilde{x}), g_{2r}(\tilde{x}), h_r(\tilde{x})$  are higher order terms.

The steady-state HJBI equation associated with the system  $\Gamma$  we treat in this paper is

$$\begin{aligned} 0 &= 2 \left( \frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}} \right)^T f(\tilde{x}) + \left( \frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}} \right)^T \\ &\quad (g_1(\tilde{x})g_1^T(\tilde{x}) - g_2(\tilde{x})g_2^T(\tilde{x})) \\ &\quad \left( \frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}} \right) + h^T(\tilde{x})h(\tilde{x}), \\ \Pi(0) &= 0 \end{aligned} \quad (9)$$

where  $f, g_1, g_2, h$  are real functions in the system  $\Gamma$ ,  $\tilde{x} \in \mathbb{X}_0$  is the state vector of the system  $\Gamma$  and  $\Pi : \mathbb{X}_0 \rightarrow \mathbb{R}^+$  is the unique local nonnegative stabilizing solution we seek. Here, a solution of (9) is called a local **stabilizing solution** if this solution is such that the closed-loop of the system  $\Gamma$  is locally exponentially stable under the feedback inputs  $u^*(t) = -g_2^T(x(t)) \frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}} \Big|_{\tilde{x}=x(t)}$  and  $w^*(t) = g_1^T(x(t)) \frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}} \Big|_{\tilde{x}=x(t)}$ .

## 3. DEFINITIONS AND PRELIMINARY RESULTS

In this section, we firstly give some definitions, and then set up some lemmas. There are five lemmas in this section and they can be divided into the following two categories:

- Lemma 5 and Lemma 6: Lemma 5 and Lemma 6 give sufficient conditions for the existence and uniqueness of the local nonnegative stabilizing solutions of a special class of HJB and HJBI equations we treat in this paper.
- Lemma 7-9: Lemma 7 gives a basic formula (see (16)) which will be used in our proposed algorithm; Lemma 8 sets up an iterative scheme based on the local exponential stability of two vector fields which can be recursively defined in our proposed algorithm; Lemma 9 constructs three matrix sequences which are used in our main result.

The system  $\Delta : \mathcal{U}_0 \rightarrow \mathcal{Y}_0$  is used in this section, and it is defined by letting  $w(t) = 0$  for all  $t \geq 0$  in the system  $\Gamma$ .

*Definition 1.* (Lukes [1971]) Let  $f, g_2$  be the real functions defined in the system  $\Delta$  and suppose that they have the representation (5) and (7) respectively. The pair  $(f, g_2)$  is called **stabilizable** if and only if  $(F, G_2)$  is stabilizable.

We now define a function, motivated by the right-hand side of the HJBI equation, that will be useful throughout the paper.

*Definition 2.* Let  $f, g_1, g_2, h$  be the real vector functions defined in the system  $\Gamma$ , and  $\tilde{x} \in \mathbb{X}_0$  be the state value of  $\Gamma$ . Let  $\mathbb{T}$  be the set which includes all smooth mappings from  $\mathbb{X}_0$  to  $\mathbb{R}$ , and define  $\Theta : \mathbb{T} \rightarrow \mathbb{T}$  as

$$(\Theta(V))(\tilde{x}) = 2 \left( \frac{\partial V(\tilde{x})}{\partial \tilde{x}} \right)^T f(\tilde{x}) + \left( \frac{\partial V(\tilde{x})}{\partial \tilde{x}} \right)^T (g_1(\tilde{x})g_1^T(\tilde{x}) - g_2(\tilde{x})g_2^T(\tilde{x})) \left( \frac{\partial V(\tilde{x})}{\partial \tilde{x}} \right) + h^T(\tilde{x})h(\tilde{x}), \quad (10)$$

for all  $V \in \mathbb{T}$ ,  $\tilde{x} \in \mathbb{X}_0$ .

We define two functions that will later be useful.

*Definition 3.* Let  $f, g_1, g_2, h$  be the real vector functions defined in the system  $\Gamma$ . Let  $\Pi \in \mathbb{T}$  be the local nonnegative stabilizing solution of (9). Let  $\mathbb{T}, \Theta$  be defined as in Definition 2. Given  $V \in \mathbb{T}$ , we then define  $\hat{f}_V : \mathbb{X}_0 \rightarrow \mathbb{R}$  as

$$\hat{f}_V(\tilde{x}) = f(\tilde{x}) + g_1(\tilde{x})g_1^T(\tilde{x}) \frac{\partial V(\tilde{x})}{\partial \tilde{x}} - g_2(\tilde{x})g_2^T(\tilde{x}) \frac{\partial V(\tilde{x})}{\partial \tilde{x}}$$

for all  $\tilde{x} \in \mathbb{X}_0$ , and define  $\bar{f}_V : \mathbb{X}_0 \rightarrow \mathbb{R}$  as

$$\bar{f}_V(\tilde{x}) = f(\tilde{x}) + g_1(\tilde{x})g_1^T(\tilde{x}) \frac{\partial V(\tilde{x})}{\partial \tilde{x}} - g_2(\tilde{x})g_2^T(\tilde{x}) \frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}}$$

for all  $\tilde{x} \in \mathbb{X}_0$ .

We define a function which will be used in Lemma 9.

*Definition 4.* Let  $F, G_1, G_2, H$  be the real matrices appearing in (5)-(8). Define  $U : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  as

$$U(Q) = QF + F^T Q - Q(G_2 G_2^T - G_1 G_1^T)Q + H^T H. \quad (11)$$

In the remainder of this section, we will set up some preliminary results regarding the smooth stabilizing solutions of a special class of HJB (HJBI) equations.

The following lemma sets up the results regarding the existence and uniqueness of the local smooth stabilizing solutions of a special class of HJB equations.

*Lemma 5.* Consider the system  $\Delta$ , suppose (5)-(8) hold, and let  $\tilde{x} \in \mathbb{X}_0$  be the state value of  $\Delta$ . Let  $x_0 \in \mathbb{X}_0$  be the initial state of the system  $\Delta$ . Let  $f, g_2, h$  be the real vector functions defined in  $\Delta$  and let  $F, G_2, H$  be the real matrices appearing in (5)-(8). If  $(F, G_2)$  is stabilizable and  $(H, F)$  is detectable, then

- (i) there exists a unique stabilizing solution  $P \geq 0$  (i.e. there is no other stabilizing solution) to the following ARE:

$$0 = PF + F^T P - PG_2 G_2^T P + H^T H, \quad (12)$$

here, a solution  $P$  of (12) is called a **stabilizing solution** of (12) if it is such that the matrix  $F - G_2 G_2^T P$  is Hurwitz;

- (ii) there exists at least locally a solution  $Z(\tilde{x}) \geq 0$  to the following equations:

$$0 = f^T(\tilde{x}) \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} - \frac{1}{2} \left( \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} \right)^T \quad (13)$$

$$g_2(\tilde{x})g_2^T(\tilde{x}) \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} + \frac{1}{2} h^T(\tilde{x})h(\tilde{x}),$$

$$0 = Z(0),$$

$$0 = \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} \Big|_{\tilde{x}=0},$$

$$P = \frac{\partial^2 Z(\tilde{x})}{\partial \tilde{x}^2} \Big|_{\tilde{x}=0},$$

where  $P \geq 0$  is the unique stabilizing solution to (12);

- (iii) the solution  $Z(\tilde{x})$  appearing in (ii) is also the unique local nonnegative stabilizing solution to (13) (i.e. there is no other local nonnegative stabilizing solution to (13)). Here, a solution of (13) is called the local **stabilizing solution** of (13) if it is such that the system  $\Delta$  is locally exponentially stable under the input  $u^*(t) = -g_2^T(x(t)) \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} \Big|_{\tilde{x}=x(t)}$

**Proof.** See Zhou et al. [1996] for (i). See van der Schaft [1999] for (ii). The proof for (iii) is omitted for brevity.  $\square$

The next lemma gives an existence and uniqueness result regarding the steady-state HJBI equation (9).

*Lemma 6.* Consider the system  $\Gamma$ , and let  $F, G_1, G_2, H$  be the real matrices appearing in (5)-(8). If  $(H, F)$  is detectable, and there exists a stabilizing solution  $K \geq 0$  to the following ARE:

$$0 = KF + F^T K - K(G_2 G_2^T - G_1 G_1^T)K + H^T H, \quad (14)$$

then there exists a local solution  $\Pi(\tilde{x}) \geq 0$  to the steady-state HJBI equation (9) for  $\tilde{x} \in \mathbb{X}_0$  with  $\Pi(0) = 0$ ,  $\frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}} \Big|_{\tilde{x}=0} = 0$  and  $K = \frac{\partial^2 \Pi(\tilde{x})}{\partial \tilde{x}^2} \Big|_{\tilde{x}=0}$ . Furthermore, such  $\Pi(\tilde{x})$  is also the unique stabilizing solution to (9) (i.e. there is no other stabilizing solution to (9)).

**Proof.** See van der Schaft [1999] for the existence of  $\Pi(\tilde{x})$ . The rest of the proof is omitted for brevity.  $\square$

The next lemma establishes some relations that will be very useful in the proof of the main theorem.

*Lemma 7.* Let  $f, g_1, g_2, h$  be the real vector functions defined in the system  $\Gamma$ , and  $\tilde{x} \in \mathbb{X}_0$  be the state value of  $\Gamma$ . Let  $\mathbb{T}$  and  $\Theta$  be as defined in Definition 2, and let addition in  $\mathbb{T}$  be defined in the obvious way. Given  $V, Z \in \mathbb{T}$ , then

$$(\Theta(V + Z))(\tilde{x}) = (\Theta(V))(\tilde{x}) + 2 \left( \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} \right)^T \hat{f}_V(\tilde{x}) - \left( \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} \right)^T (g_2(\tilde{x})g_2^T(\tilde{x}) - g_1(\tilde{x})g_1^T(\tilde{x})) \frac{\partial Z(\tilde{x})}{\partial \tilde{x}}.$$

Furthermore, if  $V, Z \in \mathbb{T}$  satisfy

$$0 = 2 \left( \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} \right)^T \hat{f}_V(\tilde{x}) - \left( \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} \right)^T g_2(\tilde{x})g_2^T(\tilde{x}) \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} + (\Theta(V))(\tilde{x}) \quad (15)$$

for all  $\tilde{x} \in \mathbb{X}_0$ , then

$$(\Theta(V + Z))(\tilde{x}) = \left( \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} \right)^T g_1(\tilde{x}) g_1^T(\tilde{x}) \frac{\partial Z(\tilde{x})}{\partial \tilde{x}} \quad (16)$$

for all  $\tilde{x} \in \mathbb{X}_0$ .

**Proof.** The first result can be obtained by direct computations; the second claim is then trivial.  $\square$

The next lemma sets up some basic relationships between the local nonnegative stabilizing solution of equation (9) and the functions  $V, Z \in \mathbb{T}$  satisfying equation (15).

*Lemma 8.* Let  $f, g_1, g_2, h$  be the real vector functions defined in the system  $\Gamma$ . Let  $V \in \mathbb{T}$  be of the form  $V(\tilde{x}) = \frac{1}{2} \tilde{x}^T A \tilde{x} + V_r(\tilde{x})$  for all  $\tilde{x} \in \mathbb{X}_0$ , where  $V_r(\tilde{x})$  are terms of higher order than quadratic and  $A \geq 0$  is a constant matrix with suitable dimensions. Let  $Z(\tilde{x})$  be the local nonnegative stabilizing solution of (13). Suppose  $V, Z \in \mathbb{T}$  satisfy equation (15). Let  $\Pi \in \mathbb{T}$  be the local nonnegative stabilizing solution to equation (9). Let  $\Sigma : \mathbb{X}_0 \rightarrow \mathbb{R}$  and  $\Sigma(\tilde{x}) = \Pi(\tilde{x}) - V(\tilde{x}) - Z(\tilde{x})$  for all  $\tilde{x} \in \mathbb{X}_0$ . Then

- (i)  $\Pi(\tilde{x}) \geq V(\tilde{x}) + Z(\tilde{x})$  for all  $\tilde{x} \in \mathbb{X}_0$  if  $x^* = 0$  is a locally exponentially stable equilibrium point of the vector field  $\hat{f}_V(\tilde{x})$ ,
- (ii)  $x^* = 0$  is a locally exponentially stable equilibrium point of vector field  $\hat{f}_{V+Z}(\tilde{x})$  if  $\Pi(\tilde{x}) \geq V(\tilde{x}) + Z(\tilde{x})$  for all  $\tilde{x} \in \mathbb{X}_0$ .

**Proof.** The proof of this lemma is omitted for brevity.  $\square$

The next lemma constructs three matrix sequences which will be used in our main result. In some sense, we can say that this lemma is the linear version of our proposed algorithm in this paper.

*Lemma 9.* (Lanzon et al. [2007]) Let  $F, G_1, G_2, H$  be the real matrices appearing in (5)-(8). Let  $U$  be the function defined by (11). Suppose  $(H, F)$  is detectable,  $(F, G_2)$  is stabilizable and there exists a stabilizing solution  $K \geq 0$  to equation (14). Then

- (I) three square matrix sequences  $J_k, F_k$ , and  $D_k$  can be defined for all  $k \in \mathbb{Z}_{\geq 0}$  which satisfy

$$J_0 = 0, \quad (17)$$

$$F_k = F + G_1 G_1^T J_k - G_2 G_2^T J_k, \quad (18)$$

$D_k$  is the unique positive semidefinite and stabilizing solution of

$$0 = D_k F_k + F_k^T D_k - D_k G_2 G_2^T D_k + U(J_k),$$

and then

$$J_{k+1} = J_k + D_k; \quad (19)$$

- (II) the series defined in part (I) have the following additional properties:

- 1)  $(F + G_1 G_1^T J_k, G_2)$  is stabilizable,  $\forall k \in \mathbb{Z}_{\geq 0}$ ,
- 2)  $D_k \geq 0 \quad \forall k \in \mathbb{Z}_{\geq 0}$ ,
- 3)  $U(J_{k+1}) = D_k G_1 G_1^T D_k, \quad \forall k \in \mathbb{Z}_{\geq 0}$ ,
- 4)  $F + G_1 G_1^T J_k - G_2 G_2^T J_{k+1}$  is Hurwitz  $\forall k \in \mathbb{Z}_{\geq 0}$ ,
- 5)  $\Pi \geq J_{k+1} \geq J_k \geq 0, \quad \forall k \in \mathbb{Z}_{\geq 0}$ ,
- 6)  $(G_1^T D_k, F_{k+1})$  is detectable,  $\forall k \in \mathbb{Z}_{\geq 0}$ ;

- (III) let

$$J_\infty := \lim_{k \rightarrow \infty} J_k \geq 0,$$

then the limit exists and  $J_\infty = K$ .

**Proof.** See Lanzon et al. [2007].  $\square$

#### 4. MAIN RESULT

In this section, we set up the main theorem by constructing two nonnegative function series  $Z_k(\tilde{x})$  and  $V_k(\tilde{x})$  for all  $\tilde{x} \in \mathbb{X}_0$ , and we also assert (the proof is omitted for brevity in this paper) that  $V_k(\tilde{x})$  is monotonically increasing and converges to the unique local nonnegative stabilizing solution  $\Pi(\tilde{x})$  of the HJBI equation (9) if such a solution exists.

*Theorem 10.* Consider the system  $\Gamma$ , and let  $F, G_1, G_2, H$  be the real matrices appearing in (5)-(8). Let  $\tilde{x} \in \mathbb{X}_0$  be the state of the system  $\Gamma$ . Define  $\Theta : \mathbb{T} \rightarrow \mathbb{T}$  as in (10). Suppose  $(H, F)$  is detectable,  $(F, G_2)$  is stabilizable and there exists a stabilizing solution  $K \geq 0$  to equation (14). Let  $F_k, D_k$  and  $J_k$  be the matrix sequences appearing in Lemma 9. Then

- (I) there exists a unique local nonnegative stabilizing solution  $\Pi(\tilde{x})$  to the equation (9);

- (II) two unique real function sequences  $Z_k(\tilde{x})$  and  $V_k(\tilde{x})$  for all  $k \in \mathbb{Z}_{\geq 0}$  can be defined recursively as follows:

$$V_0(\tilde{x}) = 0 \quad \forall \tilde{x} \in \mathbb{X}_0, \quad (20)$$

$Z_k(\tilde{x})$  is the unique local nonnegative stabilizing solution of

$$0 = 2 \hat{f}_{V_k}^T(\tilde{x}) \frac{\partial Z_k(\tilde{x})}{\partial \tilde{x}} - \left( \frac{\partial Z_k(\tilde{x})}{\partial \tilde{x}} \right)^T g_2(\tilde{x}) g_2^T(\tilde{x}) + \Theta(V_k)(\tilde{x}), \quad \forall \tilde{x} \in \mathbb{X}_0 \quad (21)$$

$$0 = Z_k(0),$$

$$0 = \left. \frac{\partial Z_k(\tilde{x})}{\partial \tilde{x}} \right|_{\tilde{x}=0},$$

and then

$$V_{k+1} = V_k + Z_k; \quad (22)$$

- (III) the two series  $V_k(\tilde{x})$  and  $Z_k(\tilde{x})$  in part (II) have the following properties:

- 1)  $(f(\tilde{x}) + g_1(\tilde{x}) g_1^T(\tilde{x}) \frac{\partial V_k(\tilde{x})}{\partial \tilde{x}}, g_2(\tilde{x}))$  is stabilizable  $\forall k \in \mathbb{Z}_{\geq 0} \quad \forall \tilde{x} \in \mathbb{X}_0$ ,

- 2)  $(\Theta(V_{k+1}))(\tilde{x}) = \left( \frac{\partial Z_k(\tilde{x})}{\partial \tilde{x}} \right)^T g_1(\tilde{x}) g_1^T(\tilde{x}) \frac{\partial Z_k(\tilde{x})}{\partial \tilde{x}} \quad \forall k \in \mathbb{Z}_{\geq 0} \quad \forall \tilde{x} \in \mathbb{X}_0$ ,

- 3)  $f(\tilde{x}) + g_1(\tilde{x}) g_1^T(\tilde{x}) \frac{\partial V_k(\tilde{x})}{\partial \tilde{x}} - g_2(\tilde{x}) g_2^T(\tilde{x}) \frac{\partial V_{k+1}(\tilde{x})}{\partial \tilde{x}}$  is locally exponentially stable at the origin  $\forall k \in \mathbb{Z}_{\geq 0} \quad \forall \tilde{x} \in \mathbb{X}_0$ ,

- 4)  $\Pi(\tilde{x}) \geq V_{k+1}(\tilde{x}) \geq V_k(\tilde{x}) \geq 0 \quad \forall k \in \mathbb{Z}_{\geq 0} \quad \forall \tilde{x} \in \mathbb{X}_0$ ,

- 5)  $Z_k(\tilde{x}) = \frac{1}{2} \tilde{x}^T D_k \tilde{x} + O_0(\tilde{x}) \quad \forall k \in \mathbb{Z}_{\geq 0} \quad \forall \tilde{x} \in \mathbb{X}_0$ ,  
 $V_k(\tilde{x}) = \frac{1}{2} \tilde{x}^T J_k \tilde{x} + O_1(\tilde{x}) \quad \forall k \in \mathbb{Z}_{\geq 0} \quad \forall \tilde{x} \in \mathbb{X}_0$ ,  
where  $D_k$  and  $J_k$  are the matrix sequences appearing in Lemma 9, and  $O_0(\tilde{x})$  and  $O_1(\tilde{x})$  are terms of higher order than quadratic.

- (IV) For all  $\tilde{x} \in \mathbb{X}_0$ , the limit

$$V_\infty(\tilde{x}) := \lim_{k \rightarrow \infty} V_k(\tilde{x})$$

exists with  $V_\infty(\tilde{x}) \geq 0$ . Furthermore,  $V_\infty = \Pi$  is the unique local nonnegative stabilizing solution to (9).

**Proof.** See Lemma 6 for (I). Results (II) and (III) can be shown together by an inductive argument by using Lemmas 5-9. Result (IV) can be shown by using (III)

and Lemma 6. For brevity, the full proof is omitted here and will be published elsewhere.  $\square$

From Theorem 10 (III1), we know that we can check the existence of the local nonnegative stabilizing solution of (9) by checking the stabilizability of a matrix function pair. By Definition 1, we can check the stabilizability of a matrix function pair by checking their linear parts. Meanwhile, from (10) we note that the HJBI equation (9) can be expressed equivalently by  $\Theta(\Pi) = 0$ . Hence we have the following corollary which gives a condition under which there does *not* exist a local nonnegative stabilizing solution  $\Pi$  to  $\Theta(\Pi) = 0$ . This is useful for terminating the recursion in a finite number of iterations.

*Corollary 11.* Let  $F, G_1, G_2, H$  be the real matrices appearing in (5)-(8). Let  $J_k$  be the matrix sequence appearing in Lemma 9. Suppose that  $(H, F)$  is detectable and  $(F, G_2)$  is stabilizable. Let  $x \in \mathbb{X}_0$  be the state of the system  $\Gamma$ . Define  $\Theta : \mathbb{T} \rightarrow \mathbb{T}$  as in (10). If  $\exists k \in \mathbb{Z}_{\geq 0}$  such that  $(F + G_1 G_1^T J_k, G_2)$  is not stabilizable, then there does not exist a local nonnegative stabilizing solution to  $\Theta(\Pi) = 0$ .

**Proof.** If  $\exists k \in \mathbb{Z}_{\geq 0}$  such that  $(F + G_1 G_1^T J_k, G_2)$  is not stabilizable, then since (5)-(7) hold and  $V_k(\tilde{x}) = \frac{1}{2} \tilde{x}^T J_k \tilde{x} + O_1(\tilde{x}) \quad \forall \tilde{x} \in \mathbb{X}_0$  by (III5), we conclude that  $(f(\tilde{x}) + g_1(\tilde{x}) g_1^T(\tilde{x}) \frac{\partial V_k(\tilde{x})}{\partial \tilde{x}}, g_2(\tilde{x}))$  is not stabilizable  $\forall k \in \mathbb{Z}_{\geq 0}, \forall \tilde{x} \in \mathbb{X}_0$  by Definition 1. Then by Theorem 10 (III1), there does not exist a local nonnegative stabilizing solution to  $\Theta(\Pi) = 0$ .

## 5. ALGORITHM

Let  $f, g_1, g_2, h$  be the real functions defined in the system  $\Gamma$  and suppose (5)-(8) hold. Let  $J_k$  be the matrix sequence appearing in Lemma 9. Suppose  $(F, G_2)$  is stabilizable and  $(H, F)$  is detectable; an iterative algorithm for finding the local nonnegative stabilizing solution of equation (9) is given as follows:

- (1) Let  $V_0 = 0$  and  $k = 0$ .
- (2) Construct (for example using the algorithm in Leake et al. [1967], though this is not necessary) the unique local nonnegative stabilizing solution  $Z_k(\tilde{x})$  which satisfies

$$0 = 2\hat{f}_{V_k}^T(\tilde{x}) \frac{\partial Z_k(\tilde{x})}{\partial \tilde{x}} - \left( \frac{\partial Z_k(\tilde{x})}{\partial \tilde{x}} \right)^T g_2(\tilde{x}) g_2^T(\tilde{x}) + \frac{\partial Z_k(\tilde{x})}{\partial \tilde{x}} + (\Theta(V_k))(x), \quad (23)$$

$$0 = Z_k(0),$$

$$0 = \left. \frac{\partial Z_k(\tilde{x})}{\partial \tilde{x}} \right|_{\tilde{x}=0},$$

where  $\Theta$  is defined by (10).

- (3) Set  $V_{k+1} = V_k + Z_k$ .
- (4) Rewrite  $Z_k(\tilde{x}) = \frac{1}{2} \tilde{x}^T D_k \tilde{x} + O_0(\tilde{x})$  (note that this is always possible from Theorem 10 if  $Z_k(\tilde{x})$  exists), where  $O_0(\tilde{x})$  are terms of higher order than quadratic and  $D_k \geq 0$  is the matrix sequence appearing in Lemma 9.
- (5) If  $\bar{\sigma}(G_1^T D_k) < \epsilon$  where  $\epsilon$  is a specified accuracy, then set  $\Pi = V_{k+1}$  and exit. Otherwise, go to step 6.
- (6) If  $(F + G_1 G_1^T J_k, G_2)$  is stabilizable, then increment  $k$  by 1 and go back to step 2. Otherwise, exit as there

does not exist a local nonnegative stabilizing solution  $\Pi$  satisfying  $\Theta(\Pi) = 0$ .

From Corollary 11 we see that if the stabilizability condition in step 6 fails for some  $k \in \mathbb{Z}_{\geq 0}$ , then there does not exist a local nonnegative stabilizing solution  $\Pi$  to  $\Theta(\Pi) = 0$  and the algorithm should terminate (as required by step 5). But when this stabilizability condition is satisfied  $\forall k \in \mathbb{Z}_{\geq 0}$ , construction of the series  $V_k(\tilde{x})$  and  $Z_k(\tilde{x})$  is always possible and  $V_k(\tilde{x})$  converges to  $\Pi(\tilde{x})$  (which is captured by step 5).

## 6. A NUMERICAL EXAMPLE

In this section, a numerical example is given to demonstrate the algorithm proposed in this paper. To analyze the performance, we intentionally choose an example for which we can find an analytic solution to the HJBI equation, and also compare to the method of characteristics (see Wise et al. [1994]). This initial test looks to be very promising.

### Example 1

In Wise et al. [1994], the method of characteristics is used to solve HJBI equations recursively. The following example comes from van der Schaft [1992], and it illustrates the proposed algorithm outperforming the method of characteristics in Wise et al. [1994] when solving HJBI equations. The comparison is possible because in this particular case, we are able to obtain the exact solution of the HJBI equation. The scalar system is given by

$$\dot{x}(t) = u(t) + x(t)w(t) \quad (24)$$

with output  $y(t) = x(t)$ . For this example, we have  $f(\tilde{x}) = 0, g_1(\tilde{x}) = \tilde{x}, g_2(\tilde{x}) = 1, h(\tilde{x}) = 1, F = 0, G_1 = 0, G_2 = 1, H = 1$  and it is clear that  $(F, G_2)$  is stabilizable and  $(H, F)$  is detectable. Now the steady-state HJBI equation becomes

$$\tilde{x}^2 - \left( \frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}} \right)^2 (1 - \tilde{x}^2) = 0 \quad (25)$$

with  $\Pi(0) = 0$ . We have (without any approximation)

$$\frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}} = \pm \frac{\tilde{x}}{\sqrt{1 - \tilde{x}^2}}, \quad \Pi(0) = 0, \quad (26)$$

for  $-1 < \tilde{x} < 1$ . However, since  $\Pi(0) = 0$  and we seek the solution for which  $\Pi(\tilde{x}) \geq 0$  in a neighborhood of the origin, we have

$$\frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}} = \frac{\tilde{x}}{\sqrt{1 - \tilde{x}^2}} \quad (27)$$

for  $-1 < \tilde{x} < 1$ . Now the closed-loop saddle point solution for the system (24) is  $u^*(\tilde{x}) = -\frac{\tilde{x}}{\sqrt{1 - \tilde{x}^2}}, w^*(\tilde{x}) = \frac{\tilde{x}^2}{\sqrt{1 - \tilde{x}^2}}$  and the closed-loop of the system (24) under the saddle point inputs  $u^*$  and  $w^*$  is

$$\dot{\tilde{x}} = -\tilde{x} \sqrt{1 - \tilde{x}^2} \quad (28)$$

for  $-1 < \tilde{x} < 1$ . Then it is clear that  $x^* = 0$  is a local stable equilibrium point for the system (28). We approximate the value of  $\Pi(\tilde{x})$  by approximating the value of  $\frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}}$ . From (27), we know that the value of  $\Pi(\tilde{x})$  is symmetric about the origin. In view of this, we only approximate the value of  $\frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}}$  for  $0 \leq \tilde{x} < 1$  in the following.

The exact solution of  $\frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}}$  in (25) can be approximated by both our algorithm and the method of characteristics in Wise et al. [1994].

To approximate  $\frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}}$  in (25), we carry out our proposed algorithm from Section 5.

For convenience, we denote  $(\cdot)_{k,\tilde{x}} = \frac{\partial(\cdot)_k}{\partial \tilde{x}}$  in the following for  $k = 0, 1, 2, 3$ . After a straightforward computation, we obtain the first three approximations  $V_{1,\tilde{x}}, V_{2,\tilde{x}}, V_{3,\tilde{x}}$  of  $\frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}}$  in (25) as follows:

$$V_{1,\tilde{x}} = Z_{0,\tilde{x}} = \tilde{x}, \quad (29)$$

$$Z_{1,\tilde{x}} = \tilde{x}^3 - \tilde{x} + \tilde{x}\sqrt{\tilde{x}^4 - \tilde{x}^2 + 1}, \quad (30)$$

$$V_{2,\tilde{x}} = \tilde{x}^3 + \tilde{x}\sqrt{\tilde{x}^4 - \tilde{x}^2 + 1}, \quad (31)$$

$$Z_{2,\tilde{x}} = f_2 + \sqrt{f_2^2 + \tilde{x}^2 Z_{1,\tilde{x}}^2}, \quad (32)$$

$$V_{3,\tilde{x}} = \tilde{x}^5 + \tilde{x}^3\sqrt{\tilde{x}^4 - \tilde{x}^2 + 1} + \sqrt{f_2^2 + \tilde{x}^2 Z_{1,\tilde{x}}^2}, \quad (33)$$

where  $f_2 = \tilde{x}^5 - \tilde{x}^3 + (\tilde{x}^3 - \tilde{x})\sqrt{\tilde{x}^4 - \tilde{x}^2 + 1}$ . If we use the method in Wise et al. [1994] to approximate the local nonnegative stabilizing solution  $\Pi(\tilde{x})$  to the HJBI equation (17), the first three approximations  $\bar{V}_{1,\tilde{x}}, \bar{V}_{2,\tilde{x}}, \bar{V}_{3,\tilde{x}}$  of  $\frac{\partial \Pi(\tilde{x})}{\partial \tilde{x}}$  in (25) are

$$\bar{V}_{1,\tilde{x}} = \tilde{x}, \quad (34)$$

$$\bar{V}_{2,\tilde{x}} = \tilde{x} + \frac{1}{2}\tilde{x}^3, \quad (35)$$

$$\bar{V}_{3,\tilde{x}} = \tilde{x} + \frac{1}{2}\tilde{x}^3 + \frac{7}{16}\tilde{x}^5 + \frac{9}{80}\tilde{x}^7 + \frac{437}{53760}\tilde{x}^9. \quad (36)$$

We plot these approximations together in Figure 1 (we ignore the first approximations for both algorithms since they are identical) to compare their convergence to the "Exact Solution", which is given by (27).

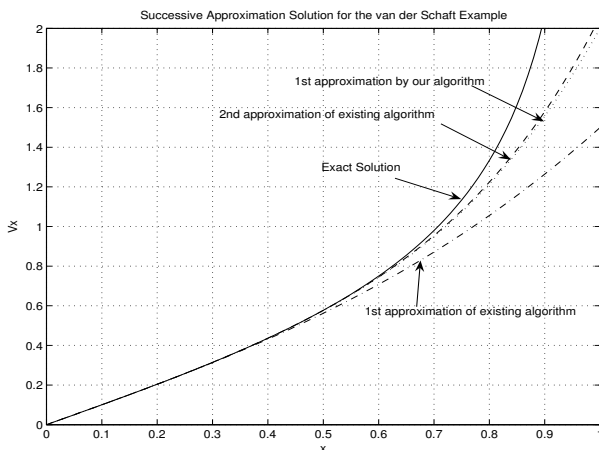


Fig. 1. Demonstration and comparison of algorithm

From Figure 1, we can see that our algorithm has better accuracy than the method of characteristics in Wise et al. [1994], noting in particular the following points:

1. For both the 2nd approximation and the 3rd approximation, our algorithm is more accurate than the method in Wise et al. [1994].

2. The 2nd approximation (dotted line) of our algorithm is very close to the 3rd approximation (dashed line) of the method in Wise et al. [1994].
3. The 3rd approximation of our algorithm (thin solid line) is very close to the exact solution (thick solid line).

## 7. CONCLUDING REMARKS

In this paper, we have developed an iterative procedure to solve a special class of HJBI equations. Under some suitable assumptions, we can compute the local nonnegative stabilizing solutions of HJBI equations recursively by constructing a monotone non-decreasing function series. Our algorithm is an extension of the algorithm in Lanzon et al. [2007] to nonlinear control. Our algorithm has an additional property, not established here; that is, as for the  $H_\infty$  linear algorithm of Lanzon et al. [2007], convergence of the approximation to the limiting value is quadratic.

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