Noisy Localization on the Sphere: a Preliminary Study

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***Abstract*—The problem of sensor network localization or target localization based on distance measurements has been well studied. In previous work, the problem is considered in the plane or in 3-dimensional space. This work deals with the problem of great distance localization on the surface of the earth when the planar assumption becomes invalid, but there remain the constraint that the points lie in a 2-dimensional manifold. The challenge lies with finding an appropriate technique to cope with noisy measurements when the conventional formulation for a planar model cannot be used. To this end, we adopt a tool recently applied to the planar model, the Cayley-Menger matrix. Simulation results show that the proposed method is effective and robust to noise. Open questions are also identified.**

I. Introduction

In network sensor localization, sensors at known positions use measurements from an emitter or target to localize that target. The measurements may be of different kinds, e.g. range or time or arrival, time difference of arrival, angle of arrival and so on. We consider here the use of range measurements. Our interest will be localization on the earth without GPS, with long distances involved. The interesting problems are those where there is noise contaminating the range measurements. To avoid ambiguities, three or more measurements are needed, and then the question arises of how to allow for the presence of noise in those measurements.

For conventional localization problems in 2- and 3-dimensional space, a recent paper [2] has shown that an entity formed from the distances between the sensors and the sensors and the target, termed the Cayley-Menger determinant (CMD), can be used to formulate certain geometric relations among these distances in the noiseless case,. This fact can be exploited in the noisy case, so that, as illustrated in [2], the effect of errors in noisy distance measurements can be reduced, thereby obtaining a better estimate of the target position as compared to other approaches to using the noisy distance measurements, see e.g. [6]–[10]. However, because over sufficiently large distances, the earth cannot be assumed to be flat, the Cayley-Menger determinant approach to localization with noisy range measurements cannot be used without some modification. A companion paper [3] investigates the range of distances when the planar assumption fails.

In this work, we shall attempt to examine the following problem: *Given three anchors and one target on the surface of a sphere, is it possible to localize the target with noisy measurements using the Cayley-Menger determinant method; if so, what kind of performance can we obtain as compared to other methods?*

For the purpose of analysis and reducing complexity in this paper, we make the overall assumption that *the earth is a perfect sphere and all sensors and target lie on the surface of the sphere. We comment briefly near the end of the paper on ways by which one could allow for the ellipsoidal shape of the earth.*

The paper is organized as follows. In section II, we provide some background information. We state the definition of the Cayley-Menger determinant in 3-dimensional space. In section III, we look at how it is possible to formulate a Cayley-Menger determinant on a sphere with three sensors and one target in the noiseless measurement case as well as in the noisy case. In section IV, we show how the errors in noisy measurements can be estimated and subsequently reduced by solving an optimization problem; then we gives the algorithm to locate the coordinates of an unknown point on the surface of the sphere, using noisy or noiseless distance measurements. In section V, we provide computation examples for the noiseless measurements and noisy cases, respectively. The paper ends with concluding remarks and directions for future work in Section VI. In fact, as this section makes clear, many loose ends still need to be examined; this is essentially a work-in-progress.

II. Background Concepts

The basic problem of target localization can be formally defined below: given a set of sensors at known positions, and a set of distance measurements from these sensors to the single unknown target, determine the position of the target. The problem could have many variations: for example, when multiple targets are present, when measurements are noisy, when the anchor (sensor) positions are noisy (see e.g. [11]), etc.

A. Localization on the plane using range measurements

The localization problem on the plane is simple. In the noiseless case, with two sensors, one can determine the position of a target up to binary ambiguity, and with one extra sensor, uniquely (provided that these sensors are not collinear).

When the measurements are noiseless, a conventional multilateration (trilateration in this case) method will solve the problem. One can imagine drawing circles centered at each sensor with radii equal to the associated range measurement, and determining a common point of intersection. In the absence of sensor collinearity, there is a single such point, being the target.

In the noisy measurement situation, an additional step has to be performed to compensate the effect of noise. Various approaches have been proposed in [2], [6]–[10]. Cao et al [2] explored an approach based on using an underlying geometric relationship, expressed using the Cayley-Menger matrix, to formulate a optimization problem to estimate the noises contained in each of the three sensor to target distance measurements. The proposed method is effective for small noise and when the sensors and/or target are not collinear or close to being collinear. It appeals to an underlying geometric constraint on the true distances.

In 2 dimensions, more than three sensors can of course be used. The method of [2] deals with this. In 3 dimensions, a minimum of four sensors is required. This is not hard to see, since it is a straightforward generalization of the 2-dimensional case.

B. The Cayley-Menger Determinant

The Cayley-Menger matrix of n points in an $m-$dimensional space is defined as per [1]

$$M = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & d_{0,1}^2 & \cdots & d_{0,n-2}^2 & d_{0,n-1}^2 \\ 1 & d_{1,0}^2 & 0 & \cdots & d_{1,n-2}^2 & d_{1,n-1}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & d_{n-2,0}^2 & d_{n-2,1}^2 & \cdots & 0 & d_{n-2,n-1}^2 \\ 1 & d_{n-1,0}^2 & d_{n-1,1}^2 & \cdots & d_{n-1,n-2}^2 & 0 \end{bmatrix}$$

where $d_{i,j} = d_{j,i}$, $i, j = 0, \dots, n-1, i \neq j$ is the Euclidean distance between the points p_i and p_j. The following is a the key theorem which we shall use:

Theorem 1: [1] Consider an n-tuple of points $p_0, ..., p_{n-1}$ in m-dimensional space. If $n \geq m+2$ then the $(n+1)\times(n+1)$ Cayley-Menger matrix $M(p_0, ...p_{n-1})$ has rank $m + 2$.

Given five points in 3-dimensional space, this theorem is equivalent to requiring a single relationship among the distances, namely, that the determinant of the matrix M of (1) is zero:

$$\det\left(M(p_0, p_1, p_2, p_3, p_4)\right) = 0 \tag{1}$$

In the next section, we shall attempt to develop a variant of the concept to use on the sphere. The 2-dimensional case has already been discussed in [2].

III. Formulation of Cayley-Menger Constraint on a Sphere

A. Localization on a sphere with great circle distances

Given two sensors on a sphere, and noiseless great circle distances to an emitter or target, the target evidently can only be localized with binary ambiguity. On the other hand, measurements from three or more sensors will in general resolve the ambiguity, provided that the sensors are not all located on a common great circle, i.e. they are not coplanar with the center of the sphere. In the noisy case, similar remarks will apply as for the case of localization in the plane, and it is clear that one needs a way of handling the noise. Our approach will be first to introduce a Cayley-Menger matrix and determinant appropriate for the sphere, with an analog of Theorem 1 applying to noiseless measurements. Then we will show how to handle the presence of noise.

Consider 3 sensor nodes 1, 2 and 3, with known positions p_1, p_2, p_3 and a further node 0, the target, with unknown position p_0. All these four points lie on the surface of the sphere. Sensor to target distances on the surface of the sphere in general are given as great circle distances. Measurements based on VHF signals or even HF signals along the surface of the earth (a perfect sphere under the global assumption) yield great circle distances.

Since a Cayley-Menger determinant involves the Euclidean distances, it is therefore essential to convert each great circle distance into its corresponding Euclidean distance, or the distance along the chord formed by the pair of end points. Assuming points on the surface of the sphere are represented by vectors with the sphere center at the origin, the great circle distance between two points a and b with position vectors $\mathbf{a}, \mathbf{b}$ in 3-dimensional Euclidean coordinates is obtainable via the following equations [4]:

$$\cos\theta = \frac{<\mathbf{a}, \mathbf{b}>}{|\mathbf{a}||\mathbf{b}|} \tag{2}$$

$$\widehat{d_{a,b}} = r\theta \tag{3}$$

Here, $<\mathbf{a}, \mathbf{b}>$ is the dot product of the position vectors of the two points, r is the radius of the sphere, θ is the angle subtended at the origin, and $\widehat{d_{a,b}}$ is the (accurate) great circle distance between points a and b.

The Euclidean distance d_{ab} can be found by any one of the following equations:

$$d_{ab} = 2r\sin(\frac{\theta}{2}) = 2r\sin(\frac{\widehat{d_{ab}}}{2r}) \tag{4}$$

$$d_{ab}^2 = r^2 + r^2 - 2r^2\cos\theta \tag{5}$$

From a numerical point of view, (4) is preferable, especially for small θ.

To handle the noisy measurement problem, our first goal is to derive an analog of a Cayley-Menger determinant condition, which applies for true distances. However, as noted previously, we would apparently need five points in 3-dimensional space to do this. An additional point on the surface of the sphere would resolve the problem, but in practical situations, it would increase the cost of localization. Adding a ground beacon involves large amounts of infrastructure cost as well as maintenance of the beacons, which does not make sense, if it is just to apply the Cayley-Menger determinant method. We limit our solution to one with only three sensors and propose the following novel result which is a Corollary of Theorem 1, simple in retrospect, but perhaps not so obvious until it has been stated:

Corollary 1: Let p_0, p_1, p_2, and p_3 be four points on the surface of a sphere of radius r, and suppose that d_{ij} denotes the (exact) Euclidean distance between points p_i and p_j. Then with the definition of the Spherical Cayley-Menger matrix (SCM) as

$$SCM = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{01}^2 & d_{02}^2 & d_{03}^2 & r^2 \\ 1 & d_{01}^2 & 0 & d_{12}^2 & d_{13}^2 & r^2 \\ 1 & d_{02}^2 & d_{12}^2 & 0 & d_{23}^2 & r^2 \\ 1 & d_{03}^2 & d_{13}^2 & d_{23}^2 & 0 & r^2 \\ 1 & r^2 & r^2 & r^2 & r^2 & 0 \end{bmatrix} \quad (6)$$

there holds

$$\det(SCM) = 0$$

Proof: With the four points p_0, p_1, p_2, p_3, associate a fifth point p_4, which is the center of the sphere. The Euclidean distance from this fifth point to each of the first four points is r. Therefore, we can form the Cayley-Menger matrix associated with these five points, and it is (6), and because all five points lie in 3-dimensional space, the determinant is zero by Theorem 1. ■

Remark 1: The problem identified prior to the theorem statement of finding a fifth point is bypassed, by not requiring the fifth point to lie on the surface of the sphere. Choosing it at the sphere center gives us the relevant distances, including that from the fifth point to the target, whose position though unknown is known to be on the sphere's surface. We comment in the final section on what might be done when the sphere is replaced by an ellipsoid.

Remark 2: One should make the distinction between this definition of SCM, as a special case of a 3-dimensional CM when four points are co-spherical and one point is the center of that sphere, with another special CM matrix given in [5] for the case when all the 5 points are co-spherical.

Remark 3: It can be easily verified that the determinant of SCM becomes the determinant of a CM matrix for a problem in which p_0, p_1, p_2 and p_3 are coplanar when r goes to infinity, i.e. ideas of [2] are recovered.

B. Noisy Case: Spherical Cayley-Menger Matrix

Let d_{ij} denote the accurate Euclidean distance between nodes i and j with $i, j \in \{0, 1, 2, 3\}, i \neq j$. Suppose 0 corresponds to the target, and nodes 1, 2 and 3 to the sensors. If the great circle distance measurements from sensor to target, denoted by $\widetilde{d_{0i}}$, are corrupted by noise, we can find the corresponding noisy squared Euclidean distance, denoted by $\overline{d_{0j}}^2$, using equation (4) or (5) . We can postulate the existence of an error variable ϵ_i relating the true values d_{0i}^2 to the noisy values according to

$$\overline{d_{0i}}^2 = d_{0i}^2 - \epsilon_i \quad (7)$$

for $i \in \{1, 2, 3\}$. We shall now utilize (7) in conjunction with the geometric constraint condition associated with the Spherical Cayley-Menger Matrix.

Substituting $\overline{d_{0i}}^2 + \epsilon_i$ in place of d_{0i}^2 in SCM yields a form for the Spherical Cayley-Menger matrix in which noisy measurement values explicitly appear, as the three unknowns $\epsilon_1, \epsilon_2, \epsilon_3$. Call this form of the matrix $SCM*$, to emphasize the dependence on the ϵ_i.

By evaluating the determinant of the matrix SCM^* and setting it to zero, we will then arrive at an equation which will provide a relationship among the errors $\epsilon_1, \epsilon_2, \epsilon_3$ and it is a relationship which includes the measured data. Whatever the errors are, they must satisfy this relationship. The proof of the theorem is largely parallel to that of Theorem 3 of [2]; however, an important noncoplanarity property has to be argued here.

Theorem 2: Let p_0, p_1, p_2 and p_3 be four points on the surface of a sphere of radius r, suppose p_1, p_2, p_3 are not coplanar with the sphere center, let d_{0i} and $\overline{d_{0i}}$ denotes the exact and noisy Euclidean distances between points p_0 and p_i, and let ϵ_i denote the associated error between the squares as in (7). Then the errors $\epsilon_i, i \in \{1, 2, 3\}$ satisfy a single algebraic equality which is quadratic though not homogeneous in the ϵ_i's, i.e. for some $\mathbf{A}$, $\mathbf{b}$ and c, there holds

$$\epsilon^T \mathbf{A} \epsilon + \epsilon^T \mathbf{b} + c = 0 \quad (8)$$

where

$$\epsilon = [\epsilon_1, \epsilon_2, \epsilon_3]^T$$

and where $\mathbf{A}, \mathbf{b}, c$ are given in the proof below.

Proof:

Rewrite the SCM^* by permuting the rows and columns so that rows and columns with indices (1,2,3,4,5,6) becomes rows and columns with indices (6,1,2,3,4,5) respectively; denote the new matrix as $\overline{SCM}$:

$$\overline{SCM} = \begin{bmatrix} 0 & \overline{d_{01}}^2 + \epsilon_1 & \overline{d_{02}}^2 + \epsilon_2 & \overline{d_{03}}^2 + \epsilon_3 & r^2 & 1 \\ \overline{d_{01}}^2 + \epsilon_1 & 0 & d_{12}{}^2 & d_{13}{}^2 & r^2 & 1 \\ \overline{d_{02}}^2 + \epsilon_2 & d_{12}{}^2 & 0 & d_{23}{}^2 & r^2 & 1 \\ \overline{d_{03}}^2 + \epsilon_3 & d_{13}{}^2 & d_{23}{}^2 & 0 & r^2 & 1 \\ r^2 & r^2 & r^2 & r^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Then the determinantal equation yields

$$\det(SCM^*) = \det(\overline{SCM}) = 0 \quad (9)$$

Partition $\overline{SCM}$ as follows

$$\overline{SCM} = \begin{bmatrix} z_{11} & \mathbf{z_{12}} \\ \mathbf{z_{21}} & \mathbf{Z_{22}} \end{bmatrix}$$

where z_{11} is zero, $\mathbf{z_{12}}$ is a row vector and $\mathbf{z_{21}}$ is a column vector. The Schur formula yields

$$\det(\overline{SCM}) = \det(\mathbf{Z_{22}})[z_{11} - \mathbf{z_{12}} \mathbf{Z_{22}^{-1}} \mathbf{z_{21}}] = 0 \quad (10)$$

Observe that $\mathbf{Z_{22}}$ is actually the standard Cayley-Menger matrix associated with the three sensors p_1, p_2, p_3 (lying on the surface of the sphere) and the sphere center. Since they are not co-planar by hypothesis, $\det(\mathbf{Z_{22}})$ is nonzero by a converse of Theorem 1, see [5]. This ensures that $\mathbf{Z_{22}^{-1}}$ exists while z_{11} is zero. Hence, from (10)

$$\mathbf{z_{12}}\mathbf{Z_{22}^{-1}}\mathbf{z_{21}} = 0$$

Given that

$$\begin{aligned}\mathbf{z_{12}} = \mathbf{z_{21}^T} = [\ \overline{d_{01}}^2 \quad \overline{d_{02}}^2 \quad \overline{d_{03}}^2 \quad r^2 \quad 1\ ] \\ +[\ \epsilon_1 \quad \epsilon_2 \quad \epsilon_3 \quad 0 \quad 0\ ]\end{aligned}$$

we obtain

$$\epsilon^T \mathbf{A} \epsilon + \epsilon^T \mathbf{b} + c = 0$$

where $\epsilon = [\epsilon_1, \epsilon_2, \epsilon_3]^T$, with $\mathbf{A}$ being the top left 3×3 block of $\mathbf{Z_{22}}^{-1}$, $\mathbf{b} = 2\mathbf{A}[\ \overline{d_{01}}^2 \quad \overline{d_{02}}^2 \quad \overline{d_{03}}^2\ ]^T$ and

$$\begin{aligned}\mathbf{c} = [\ \overline{d_{01}}^2 \quad \overline{d_{02}}^2 \quad \overline{d_{03}}^2 \quad r^2 \quad 1\ ]\mathbf{Z_{22}^{-1}} \\ [\ \overline{d_{01}}^2 \quad \overline{d_{02}}^2 \quad \overline{d_{03}}^2 \quad r^2 \quad 1\ ]^T\end{aligned}$$

■

Remark 4: In the event that four or more sensors, or multiple measurements from same set of three sensors, are available with noisy measurements to the target, one such constraint equation can be found for each selection of three. With N sensors, only $N-2$ of these constraint equations are independent. One could consider constraint equations using $\{1,2,3\}, \{1,2,4\}, \ldots, \{1,2,N\}$ for example.

Remark 5: The condition that p_1, p_2 and p_3 not be coplanar with the sphere center is indeed essential for unique localization. If they were coplanar, there would be two positions for p_0, on each side of the plane, consistent with the distance constraints.

IV. Localizing the Target

A. Determining Target Location

This subsection explains how to estimate the position of a target from the sensors' positions and the great circle distances from each sensor to the target. Note we have assumed that the sensor position are accurate and the great circle distance measurements may be noisy.

Consider temporarily the case when the great circle distances are noiseless, i.e. $\widehat{d_{0i}}$ are used. We can write down the following four equations:

$$\widehat{d_{01}} = r \arccos\left(\frac{<\mathbf{p_0} \cdot \mathbf{p_1}>}{|\mathbf{p_0}||\mathbf{p_1}|}\right) \tag{11}$$

$$\widehat{d_{02}} = r \arccos\left(\frac{<\mathbf{p_0} \cdot \mathbf{p_2}>}{|\mathbf{p_0}||\mathbf{p_2}|}\right) \tag{12}$$

$$\widehat{d_{03}} = r \arccos\left(\frac{<\mathbf{p_0} \cdot \mathbf{p_3}>}{|\mathbf{p_0}||\mathbf{p_3}|}\right) \tag{13}$$

$$(p_{0x})^2 + (p_{0y})^2 + (p_{0z})^2 = r^2 \tag{14}$$

The set (11) to (14) provides four equations for three unknowns. In the noiseless case, there will exist a unique solution to the equations. Now suppose that the great circle distances are noisy. If we simply insert the noisy distances $\widetilde{d_{0i}}$ into the equations above, there will in general no longer be any solution to the equations, because they are an overdetermined set.

Let us now indicate an algorithm for obtaining a solution to the equations in the noiseless case, which has the property that if noisy measurements replace noiseless ones in the algorithm, the algorithm can still be executed and it will yield a target position estimate (though not of course one which satisfies (11) through (14) simultaneously, which will be impossible). In the next subsection, we will indicate an improvement to the algorithm for the noisy case.

The algorithm is motivated by what has been suggested for planar localization with three noisy distance measurements [9]. Take the cosine of both sides of equations (11), (12) and (13). Then subtract the transformed (11) from the transformed (13) and the transformed (12) from the transformed (13) to obtain two equations for p_0. These equations are then combined with (14) and the resulting three equations are solved for the three unknowns.

In the noiseless case, the above method must deliver a correct answer for p_0 due to geometric consistency. This motivates us to utilize the same consistency requirement embedded in the Cayley-Menger determinantal condition to handle the noisy measurements: *one replaces the direct measured noisy great circle distances by a set of estimated great circle distances that have geometric geometric consistency (which is enforced by the use of a Cayley-Menger determinant constraint) to obtain a target estimate.* We can use the optimization method outlined in the next subsection to obtain estimated great circle distances.

B. Optimization and error reduction

In this subsection, we will see how the errors in the noisy measurements can be estimated, subsequently leading to estimates of the Euclidean distances between the sensors and the target, which are consistent with the geometrical constraint embodied in the Cayley-Menger determinant being zero. The estimates then allow estimation of the target position. The analysis is analogous to that in [2].

It is generally accepted in the localization literature that, on the basis that errors are often associated with gaussian random variables, one should minimize the mean square error. This is what we seek to do here, but subject to requiring that the geometric constraint equation (8) also holds.

Thus we aim to minimize:

$$J = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 \tag{15}$$

subject to the quadratic equality constraint (8). This is actually a reasonably standard problem of numerical analysis. If there happen to be more constraints, on account of having more sensors, the problem is less standard, but nevertheless well posed.

For the single constraint case and using the Lagrangian multiplier method, we obtain the following objective function H:

$$H(\epsilon_1, \epsilon_2, \epsilon_3, \lambda_1) = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \lambda f(\epsilon_1, \epsilon_2, \epsilon_3) \qquad (16)$$

where $f(\epsilon_1, \epsilon_2, \epsilon_3)$ has the quadratic form (8).

By differentiating the objective function H with respect to $\epsilon_1, \epsilon_2, \epsilon_3$, and λ, and setting the result to zero we can obtain four equations. One of these is (8).

On solving these four equation numerically, we will, often, end up with multiple solutions, with some sometimes being complex numbers. Hence, we need to eliminate any complex solution or non-optimal real stationary point solution, i.e. solutions corresponding to other than the global minimum. The global minimum must be one of the solutions. The solution for the least squares problem is then the set of ϵ_i's which satisfy the condition set out below:

$$\begin{aligned} &\min\ {\epsilon_1}^2 + {\epsilon_2}^2 + {\epsilon_3}^2 \\ &\text{s.t. } f(\epsilon_1, \epsilon_2, \epsilon_3) = 0 \end{aligned}$$

As we now have the values for $\epsilon_1, \epsilon_2, \epsilon_3$, we then obtain the estimated Euclidean distances and subsequently convert them into the estimated great circle distance for each of the sensor to target pairs. The Euclidean distances are consistent with the Cayley-Menger condition, in that if substituted into the determinant, will result in the determinant being zero. This also enforces the geometric consistency of estimated great circle distances, i.e. the four equations (11) to (14) will now have a solution.

In effect, in this subsection we have almost described how to compute a maximum likelihood estimate of the target. We have done this by computing a maximum likelihood estimate of the errors associated with the squares of the Euclidean distances between the sensors and the target which is consistent with the inherent geometrical constraint that links these errors. The entire derivation is very reminiscent of the 2-dimensional result of [2]. The argument that Z_{22} is nonsingular is peculiar to this problem.

V. Computational Examples

In this section, we will give two related computational examples to demonstrate the steps introduce in the previous sections. The first subsection describes the setup of both examples, which deal with the same geometric arrangement of sensors and target. In the first example, we directly use the noiseless great circle distances from each sensor to the target. while in the second case (noisy), we introduce errors into the great circle distance measurements.

A. Setup

- radius of sphere in the two cases is set to unity.
- actual target position $\mathbf{p_0} = [0.4118, 0.7180, 0.5612]$, denoted by the diamond.
- sensor nodes 1, 2, 3 are designated as the sensor beacons, with known position vectors, denoted by the 3 round dots:
- sensor position $\mathbf{p_1} = [0.0802, 0.3574, 0.9305]$
- sensor position $\mathbf{p_2} = [0.2661, 0.8731, 0.4085]$
- sensor position $\mathbf{p_3} = [0.1698, 0.9844, 0.0458]$
- the origin serves as a pseudo sensor $\mathbf{p_4}$
- for the ten distances that are required in the 3-dimensional Cayley-Menger matrix, only the 3 great circle distance measurements between target and sensors, namely $[\widehat{d_{01}}, \widehat{d_{02}}, \widehat{d_{03}}]$ will change, as a result of varying the target position $\mathbf{p_0}$.

B. Noiseless Case

In this example, we shall show the performance of the Cayley-Menger determinant method in a noiseless situation by using the matrix SCM of section III. The noiseless great circle distances measurements between the actual target and sensor positions are:

$\widehat{d_{01}} = 0.6236$, $\widehat{d_{02}} = 0.2627$, $\widehat{d_{03}} = 0.6395$

By converting the values of $\widehat{d_{0i}}$ into their corresponding Euclidean distances d_{0i} using (4), we can then substitute the corresponding Euclidean distances into the matrix SCM. Evaluating the determinant of SCM results in a value of 0 (as expected).

This indicates that the Euclidean distances are consistent with the set of points $(p_0, p_1, p_2, p_3, p_4)$ in 3-dimensional space and hence there is no correction needed. As independent verification of this, we note that solutions for the errors obtained from MATLAB simulation using the algorithm in subsection IV-B are as follows:

$\epsilon_1^* = 0.000$, $\epsilon_2^* = 0.000$, $\epsilon_3^* = 0.000$

Using these accurate great circle distances and following the method in subsection IV-A, we obtain the estimated target position $\mathbf{p_0'} = [0.4118, 0.7180, 0.5612]$, which is the same as the true target position as expected.

C. Noisy Case

Let us now consider the case where the measurements are noisy. All nodes and parameters in this case remain the same as those in the first case apart from the 3 great circle distance measurements from sensors to the target. Suppose the 3 great circle distance measurements are corrupted with certain error percentages, leading to noisy great circle distance measurements:

$$\begin{aligned} \widetilde{d_{01}} &= 0.6236 \times 0.96 = 0.5986(-4\%) \\ \widetilde{d_{02}} &= 0.2627 \times 0.98 = 0.2575(-2\%) \\ \widetilde{d_{03}} &= 0.6395 \times 1.03 = 0.6587(+3\%) \end{aligned}$$

By converting each great circle distance $\widetilde{d_{0i}}$ into its corresponding Euclidean distance $\overline{d_{0i}}$, we can then substitute the values of $\overline{d_{0i}}$ into the matrix SCM^*. By evaluating the determinant of the matrix SCM^*, we obtain one quadratic equality constraint, defined by (8). Subsequently, we followed the procedures as mentioned in subsection IV-B and obtained

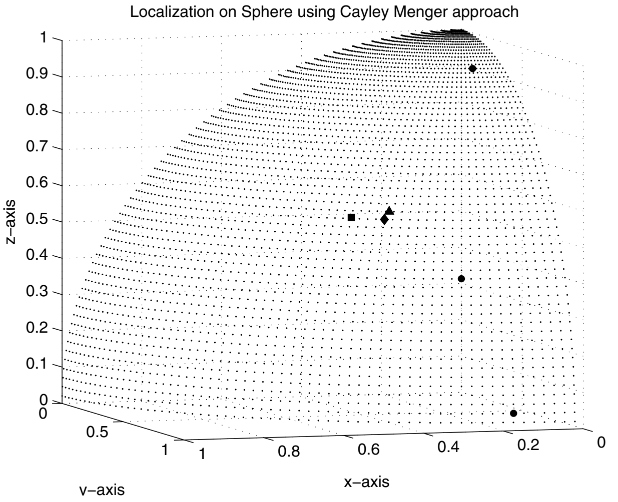

Fig. 1. Noisy Case : Round dots denote sensors, Diamond denotes actual target, Triangle denotes optimized target estimate based on using SCM-optimized distance estimates, Square denotes a target estimate based on using noisy distances directly.

the following solution to the constrained least squares problem: $\epsilon_1^+ = -1.235 \times 10^{-3}$, $\epsilon_2^+ = 7.886 \times 10^{-3}$, $\epsilon_3^+ = -4.216 \times 10^{-3}$.

Following the method in subsection IV-A, we can solve for the optimized target estimate and we obtain the position $[0.3973, 0.7099, 0.5816]$.

For a direct comparison with a non-optimized estimate, the noisy great circle distance measurements $\widetilde{d_{01}}$, $\widetilde{d_{02}}$ and $\widetilde{d_{03}}$ are used directly in (11)-(14) and we obtain a target estimate at position $[0.4804, 0.6713, 0.5643]$ whose error is clearly substantial.

As depicted in Fig. 1, for this example which is reasonably generic, the Cayley-Menger determinant method of estimating errors results in a better estimation of the unknown target location on the surface of the sphere as compared to just using the noisy measurements for localization without utilizing the geometric constraints.

VI. Concluding Remarks

In this paper, we have dealt with the problem of localization on earth using range measurements, when the planar assumption becomes invalid. We use the geometrical constraints for compensation of the effect of noisy measurements, by formulation of a Spherical Cayley-Menger matrix. Although 3-dimensional ideas are being used (for which normally four sensors would be expected), localization can be achieved with but three sensors, as for the case of planar localization. This simple yet effective idea is verified using simulation examples.

A number of issues remain to be addressed. The Spherical Cayley-Menger matrix (SCM) is expressed using Euclidean distances for simplicity and following convention. An expression using great circle distances (and trigonometric functions) is an easy extension. A different variation is obtained by adding altitude measures into the formulation; the problem can be easily extended to the case when the sensors and the target are at different heights, though some a priori estimate of target height would be required, to be able to record a Euclidean distance from the earth's center. Note that ellipsoidal models are in fact available for certain large areas of the earth, e.g. of the size of Australia; one could imagine an iterative localization scheme, in which a height value was assumed based on the current position estimate, and then a new position estimate would be obtained.

A projection method might be used to localize points on sphere. Sensor to target distance measurements could be projected onto the plane formed by the three sensors, and then a planar model can be used. However, the complexity and the effectiveness of this approach are yet to be determined.

The *numerical range of validity* of the planar assumption for practical problems of localization on the earth's surface was recently studied by some of the authors and reported in [3]. Clearly for small enough distances, a planar approximation will be satisfactory.

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