# EXPONENTIAL CONVERGENCE AND PERSISTENT EXCITATION

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ABSTRACT

The notion of robustness of an adaptive algorithm is discussed. A number of simple examples are given of adaptive algorithms behaving in a nonrobust manner, and then conditions for robustness are presented which demand use of persistently exciting signals.

## 1. INTRODUCTION

This paper has one key point, and a number of lesser points flowing from the key point. The key point is as follows. If an adaptive algorithm defined for an ideal situation is to work satisfactorily in practice, it must be robust, i.e. the behaviour in the ideal situation must approximate behaviour in the nonideal practical situation; to ensure this is so, it helps if the ideal algorithm is exponentially stable and this property of exponential stability may or may not result from the externally imposed signals (for example, inputs in an identification problem, trajectories in a reference-trajectory-following adaptive control problems, and so on).

Typical departures from the idealising assumptions that underpin many adaptive algorithms include

- Noise, in both actual plant measurements, and in the implementation of the algorithm. (Paradoxically, noise can sometimes make an otherwise nonrobust algorithm robust.)
- Errors in modelling assumptions. (We might assume a plant is linear, when it has some nonlinearity, or we might assume it has no high frequency modes when this is not the case.)
- Time-variation in plant parameters.
- Changes in the nature of the input (i.e. if the algorithm works satisfactorily for a certain style of input, it should continue to work satisfactorily if the input style is altered.)

Another philosophy can be adopted in coping with such departures - one can make a change to the algorithm, away from that used in the ideal case. This approach to achieving robustness will not be considered in this paper.

There are many adaptive systems problems and many algorithms for solving these problems. There is one class of popular algorithm which we shall immediately note as being nonrobust in the face of time-variation of plant parameters. Many algorithms contain a parameter update equation of the form $\hat{\theta}_{k+1} = \hat{\theta}_k + K_k \times$ error. Here, $\hat{\theta}_k$ is the parameter estimate at time k; the error is a quantity which would be zero were $\hat{\theta}_k$ equal to the correct parameter, with no noise present; $K_k$ is a gain term, depending on for example the choice between stochastic-approximation-type or recursive-least-squares-type algorithms. Statisticians have created a substantial body of literature in which $K_k \to 0$ as $k \to \infty$. The thinking behind this literature is as follows. The models under consideration are time-invariant and there is additive measurement noise. Despite the noise, it is possible (essentially by averaging) to get a better and better knowledge of the unknown parameters as $k \to \infty$. However, the noise itself may be confused with a true signal and lead to misadjustment of the parameters. Hence, as $k \to \infty$ one cuts down on $K_k$, thereby coupling in the noise to a lesser and lesser degree. It turns out one can get an approximate rate of decrease of $K_k$ which both allows learning of the true parameter and ultimate suppression of the noise effects. The difficulty with this approach is that it does not permit a situation where the plant is (slowly) time-varying.

Such an adaptive algorithm is in this sense not robust. (It is debatable whether the word adaptive is even appropriate: everyday usage of this word connotes an ability to adjust in response to changes in the environment, and this ability is slowly switched off in algorithms where $K_k \to 0$. Of course, in practice, the algorithm is often changed, to prevent $K_k \to 0$.)

## 2. EXAMPLES OF NONROBUST AND ROBUST ALGORITHMS

Without giving an illustration of nonrobust behaviour for every conceivable algorithm, we do wish to give a representative number of examples which will convince the reader that the possibility of nonrobust behaviour must continually be considered.

Example 1. Equation error identification, with S-A type algorithm* [1].

The plant is

$$y_k = ay_{k-1} + bu_{k-1} \tag{2.1}$$

with a,b unknown, save that we know $|a| < 1$ and $b \neq 0$. We conceive of an adjustable model

$$\hat{y}_k = \hat{a}_{k-1}y_{k-1} + \hat{b}_{k-1}u_{k-1} \tag{2.2}$$

The equation error algorithm adjusts $\hat{a}_{k-1}$ and $\hat{b}_{k-1}$ to try to bring $y_k - \hat{y}_k$ to zero. The update algorithm is

$$\begin{bmatrix}\hat{a}_k \\ \hat{b}_k\end{bmatrix} = \begin{bmatrix}\hat{a}_{k-1} \\ \hat{b}_{k-1}\end{bmatrix} + \frac{\rho}{y_{k-1}^2+u_{k-1}^2+\nu}\begin{bmatrix}y_{k-1} \\ u_{k-1}\end{bmatrix}[y_k-\hat{y}_k] \tag{2.3}$$

where $\nu > 0$ (usually small) and $\rho \in (0,2)$, but again, $\rho$ is usually small. (It is of course possible to

* In all examples, the algorithm gains given will be prevented from tending to zero.

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contemplate exact computation of a,b given measurements of $y_k$, $u_k$ over several time instants without using (2.3). However, our concern is to obtain algorithms which will be tolerant of noise and other non-idealities, and the equation error algorithm potentially has this property.)

By setting $\tilde{a}_k = a-\hat{a}_k$, $\tilde{b}_k = b-\hat{b}_k$, which are parameter estimation errors, one can derive

$$\begin{bmatrix}\tilde{a}_k\\ \tilde{b}_k\end{bmatrix} = \left[I - \frac{\rho\,\psi_{k-1}\,\psi_{k-1}'}{\psi_{k-1}'\,\psi_{k-1} + \nu}\right]\begin{bmatrix}\tilde{a}_{k-1}\\ \tilde{b}_{k-1}\end{bmatrix} \qquad (2.4)$$

where

$$\psi_{k-1}' = [y_{k-1}\ u_{k-1}] \qquad (2.5)$$

Now consider the ideal equation under several operating conditions.

Case 1 $u_k \equiv 0$. Neglecting initial conditions, $y_k \equiv 0$, and $\hat{a}_k = \hat{a}_{k-1}$, $\hat{b}_k = \hat{b}_{k-1}$ for all k. Also $\hat{y}_k = y_k$. It will look to an observer of the plant and model that the parameter estimates have converged, since there is satisfactory tracking of plant output (2.1) by model output (2.2). But of course, no identification has taken place.

There is an important warning here. Input/output performance (over a finite interval) may mislead.

Case 2 $u_k \equiv 1$. Then, after a possible initial transient, $y_k \equiv b(1-a)^{-1}$. Change the variables in (2.4) to

$$\begin{bmatrix}\tilde{c}_k\\ \tilde{d}_k\end{bmatrix} = \begin{bmatrix}b(1-a)^{-1} & 1\\ 1 & -b(1-a)^{-1}\end{bmatrix}\begin{bmatrix}\tilde{a}_k\\ \tilde{b}_k\end{bmatrix} \qquad (2.6)$$

This leads to

$$\tilde{c}_k = \left(1 - \frac{\rho\,\psi_{k-1}'\psi_{k-1}}{\psi_{k-1}'\psi_{k-1}+\nu}\right)\tilde{c}_{k-1} = \lambda\tilde{c}_{k-1} \qquad (2.7)$$

$$\tilde{d}_k = \tilde{d}_{k-1}$$

where $\lambda\in(0,1)$. Now we see that one linear functional of $[\tilde{a}_k\ \tilde{b}_k]$ converges exponentially fast to zero, while one does not converge, assuming it is unitially nonzero.

From (2.6) and (2.7)

$$\tilde{a}_k = \frac{b(1-a)}{b^2+(1-a)^2}\,\tilde{c}_k + \frac{(1-a)^2}{b^2+(1-a)^2}\,\tilde{d}_k$$

$$\tilde{b}_k = \frac{(1-a)^2}{b^2+(1-a)^2}\,\tilde{c}_k - \frac{b(1-a)}{b^2+(1-a)^2}\,\tilde{d}_k \qquad (2.8)$$

Thus, $\tilde{a}_k$ and $\tilde{b}_k$ converge exponentially fast to nonzero values , i.e. identification of the true plant parameters a and b does not occur. However, note that $y_k-\hat{y}_k \to 0$ exponentially fast. Once again, the input/output performance of the plant is apparently identified.

Case 3 $u_{4k} = u_{4k+1} = 1$, $u_{4k+2} = u_{4k+3} = -1$. This leads to $y_{4k} = -y_{4k+2} = -b(1+a)(1+a^2)^{-1}$ and $y_{4k+1} = -y_{4k+3} = b(1-a)(1+a^2)^{-1}$.

Define

$$\psi_{\pm} = \begin{bmatrix}\dfrac{-ab \pm b}{1+a^2} & 1\end{bmatrix}'$$

and

$$A_{\pm} = I - \frac{\rho\,\psi_{\pm}\,\psi_{\pm}'}{\psi_{\pm}'\,\psi_{\pm} + \nu} \qquad (2.9)$$

then, from (3.4)

$$\begin{bmatrix}\tilde{a}_{2k+2}\\ \tilde{b}_{2k+2}\end{bmatrix} = A_+ A_- \begin{bmatrix}\tilde{a}_{2k}\\ \tilde{b}_{2k}\end{bmatrix} \qquad (2.10)$$

It is easy to see that

$\|A_+ A_- x\| < \|x\|$ for any $x\neq 0$, as follows. First observe that $\|A_\pm x\| \leq \|x\|$ (from (2.9)), with equality if and only if $\psi_\pm' x=0$ (i.e. $A_\pm x=x$). Consequently, $\|A_+ A_- x\| \leq \|A_- x\| \leq \|x\|$ with equalities if and only if $A_+(A_- x) = A_- x$ and $A_- x = x$. But then $\psi_+' x=0$ and $\psi_-' x=0$ must hold (from 2.9). This is impossible since $\psi_+$ and $\psi_-$ are linearly independent.

What this shows is that (from (2.10)) over a time interval of length two, there is a guaranteed contraction of the size $\|[\tilde{a}_j\ \tilde{b}_j]\|$. Thus for this example it is certain that $[\tilde{a}_j\ \tilde{b}_j]'$ converges to zero exponentially fast. Also, $y_k-\hat{y}_k \to 0$ exponentially fast.

In Case 1, if $u_k$ is known to be zero, common sense dictates that no identification is possible. But one could be misled in Case 2. What else could happen in practice?

- round off noise in the implementation of (2.7) is likely to make $\tilde{d}_k$ behave as a Wiener process so that one linear functional of $\tilde{a}_k$, $\tilde{b}_k$ will actually diverge. This means that the idealised algorithm will not satisfactorily approximate the practical situation
- likewise, if (2.1) is only an approximation of a real plant, the behaviour of $d_k$ will be modified.

On the other hand, in Case 3 round off noise will do no more than cause minor fluctuations to the behaviour of $\hat{a}_k$, $\hat{b}_k$ because of the convergence mechanism inherent in (2.10). A similar consideration applies to inaccuracy in (2.1).

Even if neither of the above noted difficulties is considered a problem with the Case 2 example, a problem still arises in respect of a change of style of input. Thus, suppose $u_j=1$ for $j=1, \ldots, 1000$, and then $u_j=10$. Since almost certainly $\tilde{d}_k$ will not be zero, one can check that the tracking of the true plant output by the model output after the change of input style may be very poor.

To get satisfactory behaviour of the algorithm, it is basically necessary that $u_k$ be sufficiently complicated - a notion that will be made more precise subsequently. In retrospect, it is obvious that for the uncomplicated $u_k$ of Case 1 and Case 2 one cannot expect satisfactory identification. [In Case 2, all one can expect to identify is the DC gain of (2.1), viz $b(1-a)^{-1}$.]

One way $u_k$ can become complicated is if it consists of noise. This illustrates the important point that noise in the right place can promote satisfactory behaviour, while as noted above, noise in the wrong place may spoil it.

Example 2 Reference trajectory following (Goodwin-Ramadge-Caines approach [2]) using SA-type algorithm.

The plant is as in (2.1), with a,b unknown but possibly now $|a| > 1$. The reference trajectory is denoted by $y_k^*$. The general approach is to estimate a,b with estimates $\hat{a}_k$, $\hat{b}_k$ assumed available on measuring $y_k$. Then $u_k$ is chosen to secure the right $y_{k+1}^*$ acting as if $\hat{a}_k$, $\hat{b}_k$ were the correct a,b. Thus

$$y_{k+1}^* = \hat{a}_k y_k + \hat{b}_k u_k \tag{2.11}$$

(If $\hat{b}_k$ is very small, the algorithm is adjusted.)

Now the identification algorithm will ensure that $y_k - y_k^* \to 0$ (for a proof, see [2]). Suppose $y_k^* \equiv 1$. Then (2.11) implies

$$u_k \to \frac{1-\hat{a}_k}{\hat{b}_k}$$

and (2.1) implies

$$1 - a - b\,\frac{1-\hat{a}_k}{\hat{b}_k} \to 0$$

Suppose a=2, b=1. Then

$$\hat{a}_k - \hat{b}_k \to 1 \tag{2.12}$$

With this particular $y_k^*$ sequence then, it appears as if $\hat{a}_k$, $\hat{b}_k$ cannot be separately identified. Indeed this is easy to check. Suppose in fact that with $y_k^* \equiv 1$, $\hat{a}j - \hat{b}j = 1$ for some j, while a=2, b=1. Suppose also that $y_j = 1$. Then (2.11) implies

$$u_j = \frac{1 - \hat{a}_j}{\hat{b}_j} = -1$$

and (2.1) ensures that $y_{j+1} = 1$. Further, since $y_{j+1} - y_{j+1}^* = 0$, there will be no update of $\hat{a}_k$, $\hat{b}_k$, i.e. $\hat{a}_{k+1} = \hat{a}_k$, $\hat{b}_{k+1} = \hat{b}_k$. Everything looks fine. But what are we actually implementing? Combining (2.1) and (2.11) with a=2, b=1, we have

$$y_{k+1} = (2 - \frac{\hat{a}_k}{\hat{b}_k})\, y_k + \frac{1}{\hat{b}_k}\, y_{k+1}^* \tag{2.13}$$

If $\hat{a}_k = 1.001$, $\hat{b}_k = 0.001$, then

(2.13) is unstable, and small errors would grow into large errors. But this means that the idealised algorithm is not robust in the sense that it is not approximating satisfactorily behaviour which will be encountered in practice.

The problem with change of input style ($y_k^* \equiv 1$ for $k = 1, \ldots, j$ and $y_{j+1}^* = 10$) can be illustrated also with $\hat{b}_k = 1000$, $\hat{a}_k = 1001$. We shall have $y_k = 1$ for $k = 1, \ldots, j$ and $y_{k+1} = 1.009$. The difficulty here is not that there is a significant difference between the trajectories of the ideal algorithm, and those encountered in practice; rather the difficulty is that the convergence which apparently occurs or exists over [1,j] is illusory. This is a different form of nonrobustness. Of course, the change of value of $y^*$ will itself promote identification - but immediately after the change, that identification will still not have taken place.

In summary then, the trajectory $y_k^* \equiv 1$ has the potential to lead to nonrobustness of the algorithm itself (trajectories predicted by the ideal algorithm are nothing like those in practice, because of instability), or to nonrobustness in the sense that there is a spurious appearance of convergence.

What of the effects of noise? Noise in $y_k^*$ could stabilise the algorithm, or help make it robust. On the other hand, while noise in the parameter estimation algorithm is likely to cause $\hat{a}_k + \hat{b}_k$ to behave like a Wiener process, until (2.13) proves unstable (when instability is encountered in (2.13), this could promote identification). In any case, the behaviour of the algorithm under ideal (here, noise-free) circumstances will mislead concerning its behaviour in the noisy situation.

There is incidentally an entirely distinct practically undesirable aspect of the above algorithm. Normally, one hopes that finite intervals at least of trajectories obtained with an algorithm should depend continuously in initial conditions. The algorithm of [2] does not have this property, and thus in a new sense is not robust. (In particular, the ideal algorithm is not continuous at $\hat{b}_o = 0$). There is then a potential for round off error in a simulation to change the initial condition so as to cause a major change in the resulting trajectory.

Example 3 Reference trajectory following (Widrow approach [3,4]) using SA-type algorithm. The idea behind this reference trajectory following algorithm is to identify an inverse of the plant, and drive a copy of this inverse using the $y_k^*$ sequence to generate the required plant input $u_k$. Figure 1 illustrates the general arrangement for the plant of (2.1). The plant must be stable for this scheme to work, and in its exact form, the plant must also have no zeros. However, an approximate form of the algorithm (not considered here) allows the plant to have zeros, even nonminimum phase zeros.

Suppose now that $y_k^* \equiv 1$, $\hat{c}_o + \hat{d}_o = 0$ and initial conditions of the plant are zero. Then $u_1 = 0$, $y_2 = 0$ and the identifier with zero inputs causes $\hat{c}_1 = \hat{c}_o$, $\hat{d}_1 = \hat{d}_o$. Then $\hat{c}_1 + \hat{d}_1 = 0$, $u_2 = 0$, $y_3 = 0$ and so on. No identification takes place, and in this instance, we do not even have $y_k - y_k^* \to 0$. Again, suppose that $y_k^* \equiv 1$ and $\hat{c}_j + \hat{d}_j = (1-a)b^{-1}$.

Then $u_j = (1-a)b^{-1}$. Suppose also that $y_j = 1$. Then $y_{j+1} = 1$. It can be checked that the identifier will give $\hat{c}_{j+1} + \hat{d}_{j+1} = (1-a)b^{-1}$. So we see that $u_k = (1-a)b^{-1}$ for all k, and we would expect $\hat{c}_k - \hat{d}_k$ to drift. So long as $y_j^*$ remains at 1, $y_j$ would track it. But if $y_j^*$ changed to 10 after a long interval, i.e. there is a change of "input" style, the tracking of it by $y_j$ may be very poor, depending on the value to which $\hat{c}_k - \hat{d}_k$ had drifted.

Example 4 Adaptive pole positioning with SA-type identification.

Suppose the plant is as in (2.1), without necessarily $|a| < 1$ and we have to find a feedback law $u_k = fy_k + u_k^*$ so that $a + bf = d$, where $0<d<1$ with d prescribed. Because a, b are unknown, we

estimate them via $\hat{a}_k$, $\hat{b}_k$, choose $\hat{f}_k$ so that $\hat{a}_k + \hat{b}_k \hat{f}_k = d$, and implement $u_k = \hat{f}_k y_k + u_k^*$.

One worry with this scheme is the problem of what one should do in case $\hat{b}_k$ is zero, or very small. Let us leave aside this question for the moment and consider a situation where nonrobust behaviour can arise.

The closed-loop plant will be

$$y_k = (a+b\frac{d-\hat{a}_{k-1}}{\hat{b}_{k-1}})\, y_{k-1} + b u_{k-1}^* \qquad (2.14)$$

and the prediction of $y_k$ which the identifier will make will be

$$\hat{y}_k = d y_{k-1} + \hat{b}_{k-1} u_{k-1}^* \qquad (2.15)$$

In all treatments of this problem, not surprisingly the allegedly satisfactory algorithms require $y_k - \hat{y}_k \to 0$.

If $u_k^* \equiv 1$, it is easily checked that this convergence implies

$$1 - \hat{a}_k - \hat{b}_k \left(\frac{1-a}{b}\right) \to 0 \qquad (2.16)$$

All this suggests the following converse agrument. Suppose that initially

$$1 - \hat{a}_o - \hat{b}_o \left(\frac{1-a}{b}\right) = 0 \qquad (2.17)$$

and that

$$y_o = \frac{\hat{b}_o}{1-d} \qquad (2.18)$$

Then it is easily verified that $y_1 = \hat{y}_1 = \hat{b}_o (1-d)^{-1}$, and so $\hat{a}_1 = \hat{a}_o$, $\hat{b}_1 = \hat{b}_o$. More generally, $y_k$, $\hat{y}_k$, $\hat{a}_k$ and $\hat{b}_k$ will apparently remain constant for all time.

Suppose in particular that a=0, b=1, d=0.9. Then (3.17) implies $\hat{a}_o + \hat{b}_o = 1$. From (2.14) the actual closed-loop pole will then be at

$$\left[0 + 1\,\frac{0.9 - (1-\hat{b}_o)}{\hat{b}_1}\right] = 1 - \frac{0.1}{\hat{b}_o}$$

Obviously, if $\hat{b}_o$ is small, the actual closed-loop can be unstable, far from 0.9. The algorithm in practice then will exhibit very different trajectories to those suggested by the idealised case, which, accordingly is nonrobust. If $\hat{b}_o$ is 0.1 say, this type of nonrobustness will not be observed. However, there will be a spurious indication of convergence, and a change of style of input will at once illustrate that the closed-loop pole is not at 0.9. Also, algorithm noise, modelling errors and the like can be expected to cause $\hat{b}_k$ to drift away from $\hat{b}_o$.

## 3. PERSISTENCY OF EXCITATION AND ROBUSTNESS

Suppose that one faced the task of estimating the coefficients of the transfer function of a system by using sine wave excitation at different frequencies, evaluating the transfer function at these different frequencies and inferring from these data the coefficients of the transfer function numerator and denominator. It is clear that a certain minimum number of sinusoids would have to be applied, depending on the number of unknown coefficients. One could also conceive of applying all the sinusoids at once, analysing the output into its different components and in this way identifying the transfer function. As well as requiring a certain minimum complexity of input for satisfactory identification, however, one would normally also need indefinite persistence of the input signal. It would not be satisfactory if the coefficients were slowly time-varying to apply the input signal for a finite interval only.

Roughly the same ideas, viz. that the input must be complex, to an extent dictated by the number of unknown coefficients and that it must persist, turn out also to be relevant in adaptive identification. The results go back at least to the mid-seventies [6-8], but for a more comprehensive treatment, see [9]. The results are also well recognized by practitioners. Now it is not of course necessary that an input be a linear combination of sinusoids, or periodic, for identification to take place. Accordingly, a time-domain condition is preferred for the speficication of the input.

For a system with denominator degree n and with m numerator coefficients, we shall say that an input $u_k$ satisfies a persistency of excitation condition if for all j, some S, some $\alpha_1 > 0$ and some $\alpha_2 > 0$.

$$\alpha_1 I \leq \sum_{k=j}^{j+S} \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_{k-m-n+1} \end{bmatrix} [u_k\ u_{k-1} \cdots u_{k-m-n+1}] \leq \alpha_2 I \qquad (3.1)$$

(If $u_k$ is a linear combination of at least $\frac{1}{2}(m+n)$ sinusoids this condition is guaranteed to hold. It will also hold for pseudorandom binary sequences of a sufficiently long period. It will not hold if $u_k$ is a linear combination of fewer than $\frac{1}{2}(m+n)$ sinusoids where a DC level counts as half a sinusoid.)

The significance of this condition for identification is as follows: it is the key condition which ensures for equation error and output error identification, using SA-type or RLS-type algorithms with gains not going to zero, that the parameter estimation error approaches zero exponentially fast [5,9] (the time constant depends on e.g. $\alpha_1$, $\alpha_2$ and the system under consideration). As discussed in the next section, and as is not so well recognized, this implies various robustness properties. It certainly also gives robustness with respect to change of input style and ensures that the error between the true plant output and model output goes to zero.

The condition is actually also necessary for exponentially fast convergence of the parameter error, which implies that there is no point in trying to relax it.

The condition is also relevant in adaptive pole positioning, where $u_k$ should be replaced by the external input $u_k^*$, see [10]; also, as developed by C R J Johnson, Jr and the author in unpublished work, it is relevant to model reference adaptive control.

In using (3.1) to establish various convergence results, it is usually necessary to show that because $u_k$ is persistently exciting, some other signal or signals have this property. This can be quite challenging, see [9] for example. As a general rule, provided signals are known to be bounded, one can establish that if a signal is persistently exciting at one point of a linear system, a signal at some other

point will be persistently exciting, provided also that the system is time-invariant, or slowly time-varying. Various techniques are used to ensure the boundedness properties, e.g. assumption of open-loop or closed-loop stability, minimum phase character, uniform observability, see e.g. [4,5,9,10].

Recently, a continuous-time equivalent to (3.1) has been derived in unpublished works of the author and colleagues. This is

$$\alpha_1 I \leq \int_s^{s+\delta} U(t)U'(t)dt \leq \alpha_2 I \qquad (3.2)$$

for all s, and some $\delta$, $\alpha_1$, $\alpha_2$ all positive, where U(t) is an (m+n) vector with

$$U_i(t) = L^{-1}\left\{\frac{1}{(s+\beta)^i} L[u(t)]\right\} \qquad (3.3)$$

where $\beta$ is any positive number. Once again, such a condition implies a related condition involving other signals than those in $U_i(t)$, with these signals relevant in identification [11].

What now of adaptive control problems such as reference trajectory following? It might be held that in order to solve an adaptive control problem one must have a model of the plant and therefore (3.1) is needed. Perhaps so, but the difficulty with (3.1) is that it involves the sequence $\{u_k\}$ which is not known a priori. It takes more work to obtain the result [9,12] that in lieu of (3.1), one wants for all j, some S, some $\alpha_1>0$, some $\alpha_2>0$,

$$\alpha_1 I \leq \sum_{k=j}^{j+S} \begin{bmatrix} y_k^* \\ y_{k-1}^* \\ \vdots \\ y_{k-m-n+1}^* \end{bmatrix} [y_k^* \; y_{k-1}^* \cdots y_{k-m-n+1}^*] \leq \alpha_2 I \qquad (3.4)$$

(There are additional conditions also needed - for example the Goodwin-Ramadge-Caines algorithm [2] requires the plant to have entirely stable zeros, while the Widrow algorithm [3,4] requires the plant to be stable and, in the first instance, to have no zeros, though this latter assumption can be relaxed.)

From an applications point of view (3.4) is disappointing. After all, one will often have reference trajectories $y_k^*$ which comprise long intervals of constant signals and then (3.4) will not hold (unless perhaps S is taken to be extremely large). It appears to be standard folklore among users of adaptive controllers, that the adaption mechanism must be switched off in such long intervals, else unacceptable drifting of parameter estimates will occur which will manifest themselves as a possibly serious problem when a level change in $y_k^*$ is affected. This is consistent with (3.4) being necessary as well as sufficient.

Equation (3.4), in retrospect, is entirely reasonable. Assuming that the control scheme converges, we must have $y_k \to y_k^*$ and so (3.4) would hold with $y_k^*$ replaced by $y_k$. Then (3.4) amounts to the appropriate statement of (3.1) for the inverse of the plant rather than the plant itself. The inverse of course has the same number of unknown coefficients in its transfer function as the plant. It will therefore hardly be surprising to see (3.4) arising in the conditions for the adaptive linear-quadratic problem.

It is interesting to note that most, but not all, adaptive algorithms in their ideal forms are susceptible to the proof of a convergence result (typically to the effect that some tracking error goes to zero), irrespective of assumptions such as (3.1) or (3.4). However, there are two adaptive problems where no such result has been obtained i.e. a persistency of excitation assumption is not just the way an algorithm is made exponentially convergent, but it is essential even to the ideal algorithm. These are the Widrow type reference trajectory following adaptive control scheme [4] and global adaptive pole placement (without persistency of excitation, only local results have been obtained, see e.g. [10].

## 4. CONSEQUENCES OF THE PERSISTENCY OF EXCITATION CONDITIONS

We have just indicated that the persistency of excitation condition ensures, for the identification problem, exponentially fast convergence of the parameter estimates to their true values. In effect, the same is true for adaptive control problems. This conclusion of course applies to the ideal algorithm. What if the actual set-up is non-ideal in one of the ways described in the first section? The short answer is that if the departure from the ideal assumption is not too great, the departure from the ideal performance will not be too great also. However, when the ideal assumptions are departed from significantly, performance may collapse, e.g. by the appearance of some instability. In this section, we aim to indicate the general thinking behind this conclusion, which is a conclusion from stability theory, not of adaptive system theory.

If one writes down a set of state variable equations for the system comprising the unknown plant and the adaptive identifiers and controllers, assuming no departures from ideality, one gets a set of equations, virtually always time-varying, usually nonlinear, of the form

$$\xi_{k+1} = f(\xi_k, z_k^*, p, k) \qquad (4.1)$$

where $\xi_k$ is the state vector, $z_k^*$ denotes a persistently exciting signal (it may be an input or a reference trajectory) and p denotes a parameter vector (typically the coefficients of the plant transfer function).

Equation (4.1) is known to have an exponential convergence property. Now if one postulates departures from ideality, one can build these into the equations with the result that (4.1) may be replaced by

$$\xi_{k+1} = \tilde{f}(\xi_k, \tilde{z}_k^*, p_k, k) + \tilde{g}(\xi_k, \tilde{z}_k^*, p_k, k) \qquad (4.2)$$

where now $\tilde{z}_k^*$ is like $z_k^*$, $\tilde{f}$ is like f and is zero when $\xi_k=0$, and $\tilde{g}(\xi_k, \tilde{z}_k^*, p_k, k)$ is small. (Of course, one can be more precise about what one means by 'like' and 'small').

Now these variations on (4.1) are like a number considered in texts on stability [13,14]. In particular, $\tilde{f}$ close enough to f will not disturb exponential stability, while small $\tilde{g}$, if small enough, will cause $\xi_k$ to decay to a small ball containing the origin, the radius of which depends on $\tilde{g}$. Also, if p is time-varying and the time-variation is slow enough on the average, exponential stability is retained.

In outline, we are saying that if there are mistakes in the modelling assumptions (nonlinearity or high order modes being overlooked), or measurement

noise introduced, or plant parameter variation taking place, one can appeal to known results on stability theory or extensions of them to conclude that, when the ideal algorithm is exponentially convergent, the nonideal algorithm will behave satisfactorily if the departure from the ideal is not great. By "behave satisfactorily" we mean that asymptotic parameter estimates will not be too much in error, the error in tracking a reference trajectory will not be too great, etc. The details of the calculation are often long and involved, see e.g. [15-19].

We emphasise that once the basic stability theorems are understood, there are no significant differences in dealing with time-varying plant parameters as opposed to, e.g. reduced order models.

It might well be conjectured that mere asymptotic stability rather than exponential stability is all that will be needed to secure robust behaviour. This is certainly not so (and can be proven).

## 5. CONCLUDING REMARKS

If adaptive algorithms are to be used in practice, it is vital that practitioners be informed ahead of time, rather than discover through mishap, that some proposed algorithms will, at least some of the time, not work. If there is one plea that should be made, it is that algorithms not be advanced without a warning to prospective users, and preferably with some indication of their robustness.

Setting aside this main conclusion of the paper, one can ask in what directions the ideas might profitably be developed. First, we have said nothing concerning the amount of robustness that can be expected with a persistency of excitation condition. In relation to plant parameter variations for example, we would expect that plant parameters could not vary with a shorter time constant than that associated with convergence of the ideal algorithm: what then is this rate of convergence, in terms of the inputs and nominal plant parameters?

Another totally different direction of extending the ideas currently under examination involves plants characterised by state-variable equations in which most, but not all, parameters are known. The equations governing the error performance of the adaptive identification algorithm now have nonlinearities of a kind not encountered in other previously studied adaptive problems.

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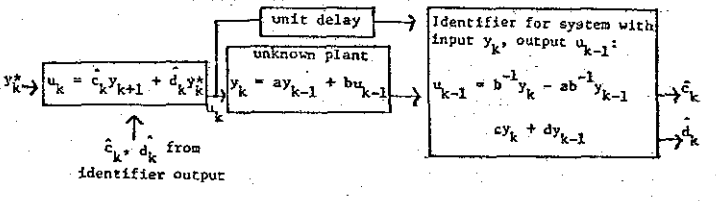


FIGURE 1: Illustration of reference trajectory following adaptive controller