Self-localization of Mobile Agents in the Plane

Iman Shames, Barış Fidan, Brian D. O. Anderson
Department of Information Engineering
Australian National University and National ICT Australia
Canberra, Australia
{Iman.Shames,Baris.Fidan,Brian.Anderson}@anu.edu.au

Hatem Hmam
Electronic Warfare Radar Division
Defence Science & Technology Organisation
Edinburgh, Australia
Hatem.Hmam@dsto.defence.gov.au

***Abstract*—This paper considers initially the problem of localizing three agents moving in the plane when the inter-agent distances are known, and in addition, the angle subtended at each agent by lines drawn from two landmarks at known positions is also known. In addition, it is shown that there are in general a finite number greater than one of possible sets of positions for the three agents. Later, generalization of the result for more than three agents is presented.**

I. INTRODUCTION

The problem of localizing the position of one or more mobile agents given sensed data containing some information relevant to the position is an old problem that comes in many varieties. To name a few, we can consider localization of unmanned aerial vehicles in open space, or localization of groups of ground mobile agents. The use of global positioning system (GPS) has made the task of localization easier, but still in the possible events of loss or corruption of GPS signals [1], or when the agents are operating indoors, use of GPS for localization purposes may be infeasible or limited [2]. In this paper, we consider a particular problem, which has arisen in the context of small groups of agents sensing the angle subtended at each of them by two landmarks at known positions.

The problem considered here is very similar to the problems considered in [3], [4], [5]. While the solution of the problems treated in the above references need at least three landmarks, in this paper we consider having access to only two. In order to provide enough information to compensate for the lack of information about a third landmark, a cooperative localization scheme is considered . Cooperative localization is achieved when the agents use relative distance or bearing measurements to other agents to do the localization task, similar to [6]. This problem originated from flight trials with real unmanned aerial vehicles (UAVs) conducted by the Defence Science & Technology Organisation (DSTO) of Australia. Loss or corruption of GPS signal is a reality, and a method was needed to localize the formation in that situation. The position information of the two landmarks is available, as are inter-agent distances.

The arrangement we consider is depicted in Fig. 1. Three mobile agents, designated T_1, T_2 and T_3 are to be localized; the inter-agent distances are known. Additionally, the agents detect two landmarks located at positions L_1 and L_2, which are known to the agents. The landmarks can be scanning radars, non-scanning radars, RF beacons or other visible features if an imaging sensor is used. A priori information is assumed to be available of which positions all agents on the same known side of the line joining the two landmarks. Each agent collects the bearing angle information to each of the landmarks. However, with no GPS information, there is no absolute heading reference for each agent, and the bearing angle information cannot be used directly for localization purposes, nevertheless, using the angle difference the need for knowing the heading is removed [6]. This angle difference is the angle subtended at each agent by the two landmarks (λ_i, $i = 1, 2, 3$), see Fig. 1, and it can therefore be concluded that each agent i is located on a circle of known centre, A_i, and radius, r_i, (determined by λ_i) and passing through the two landmarks. The centres, A_i ($i = 1, 2, 3$) lie on the perpendicular bisector of the line joining the two landmarks.

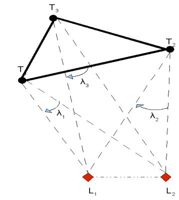

Fig. 1. The arrangement we consider in this paper.

The cooperative localization task is then to put the pieces of information together, i.e. inter-agent distances, subtended angles and landmark positions, and localize the agents.

The closest result to that of this paper is achieved by a geometric analysis that reduces the localization problem to one of solving a set of simultaneous polynomial equations in several variables [6]. A detailed analysis carried out in [6] shows that the solution count is at most 12. The analysis in [6], however, is restricted to collinear anchors.

The paper is organized as follows, Section 2 reviews the basic material on rigid formations and sensor network/formation localization. Section 3 formally describes and analyses the main problem described in the beginning of this section using the tools introduced in Section 2. This analysis enables us to argue immediately that the localization problem generically has more than one solution. In Section 4, we review the results in the mechanical engineering literature regarding four-bar linkage mechanisms and application of these results to the localization problem defined in Section 3. In Section 5, we illustrate the results with some numerical examples. Section 6 contains concluding remarks.

978-1-4244-1653-0/08/$25.00 ©2008 IEEE

II. Rigid Formations and Localization

In this section, we review some aspects of the problem of localizing, i.e. determining the positions of, agents in a formation where a number of inter-agent distances are known, and also some absolute position data is available. We draw on literature on rigid graph theory and its application to sensor network localization [7], [8], [9], [10].

A formation of point agents in the plane, $\mathcal{F}$, can be represented by a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, where the vertices in $\mathcal{V}$ correspond to the agents, and there is an edge in the graph between two vertices $v_1, v_2 \in \mathcal{V}$ just when the distance between the corresponding agents of the formation is known. We call $\mathcal{G}$ the underlying graph of $\mathcal{F}$.

Assigning coordinate values to each vertex of a graph, $\mathcal{G}(\mathcal{V},\mathcal{E})$, so that the Euclidean distance between any two vertices joined by an edge is equal to the length of this edge, is the graph realization problem. Given one solution to the graph realization problem, it is trivial that any translation, rotation or reflection of this solution is another solution. All solutions of this sort are called congruent. When there can be only one family of congruent solutions, we say that the graph realization problem has a unique solution.

Hendrickson in [9] presents necessary conditions for a graph to be uniquely realizable in $\mathbb{R}^2$, i.e. with one family of congruent solutions, and the conditions were proved later by Jackson and Jordan, [10], to be necessary and sufficient.

These conditions involve two concepts, namely redundant rigidity of a graph, and three-connectedness of a graph. The concept of redundant rigidity requires a prior concept of rigidity. A formation $\mathcal{F}$ is called *rigid* if by explicitly maintaining distances between all the pairs of agents whose representative vertices are connected by an edge in $\mathcal{E}$, the distances between all other pairs of agents in $\mathcal{F}$ are consequentially held fixed as well. The reader may refer to [11], [12], [13] for detailed information on rigid formations and rigidity.

A *redundantly rigid* formation is one which remains rigid when any single edge constraint is removed. By contrast, and we shall use the concept in the next section, a *minimally rigid* formation is one which ceases to be rigid when any single edge constraint is removed. The underlying graph of a formation is called rigid, redundantly rigid, and minimally rigid if the formation is, respectively, rigid, redundantly rigid, and minimally rigid.

The notion of a 3-connected graph $\mathcal{G}$ is standard, see [14]. Such a graph has the property that between any two vertices, three non-intersecting paths can be found. Jackson and Jordan's result [10] is as follows:

Theorem 1: Consider a two dimensional formation $\mathcal{F}$ with underlying graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$. Then the graph realization problem is uniquely solvable for generic values of the formation edge lengths (inter-agent distances) if and only if $\mathcal{G}$ is redundantly rigid and 3-connected.

A graph $\mathcal{G}$ with the two properties in Theorem 1, i.e. redundant rigidity and 3-connectedness, is termed *globally rigid*. For a formation which is rigid but not globally rigid, one at least of two ambiguities known as flip ambiguity or discontinuous flex ambiguity occurs in the realization problem[9]. The reader may refer to [9], [12] and references therein for further information on these ambiguities.

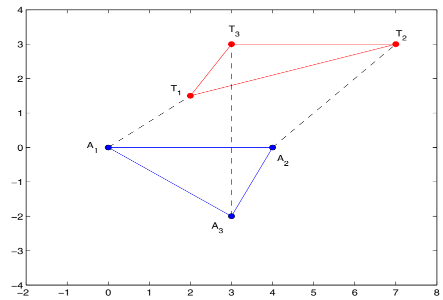

Fig. 2. The graph described in Problem 1

III. The Formation Localization Problem

In this section, we consider a formation with a particular structure and show it is minimally rigid. Accordingly, even with the knowledge of three agent positions, it will not be uniquely localizable. We shall later relate this formation and the associated result to the localization problem presented in the introduction. The problem of this section is formulated as follows.

Problem 1: Consider a formation $\mathcal{F}$ with the underlying graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, where

$$\mathcal{V} = \mathcal{T} \cup \mathcal{A}$$

is the set of vertices with $\mathcal{T} = \{T_1, T_2, T_3\}$, $\mathcal{A} = \{A_1, A_2, A_3\}$. The agents in $\mathcal{A}$ are known as anchor agents, and those in $\mathcal{T}$ as target agents. Furthermore, $\mathcal{E}$ is the set of edges and is defined as,

$$\mathcal{E} = \{T_1T_2, T_1T_3, T_2T_3, A_1A_2, A_1A_3, A_2A_3, A_1T_1, A_2T_2, A_3T_3\}$$

Knowing the length of all edges in $\mathcal{E}$, and the exact positions of the anchor agents:

(i) Can one localize the target agents, uniquely, or to one of a finite number of sets of positions?
(ii) If so, what are the possible localization solutions?

An example of the formation $\mathcal{F}$ described in Problem 1 is depicted in Fig. 2.

A crucial fact pertinent to answering the localization problem is that the formation described in Problem 1, is a minimally rigid formation. One way is presented in the following paragraphs to see this fact. Laman's Theorem provides a combinatorial way to check rigidity, and indeed minimal rigidity. It requires the idea of an induced subgraph of a graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$. Let $\mathcal{V}'$ be a subset of $\mathcal{V}$. Then the subgraph of $\mathcal{G}$ induced by $\mathcal{V}$ is the graph $\mathcal{G}' = (\mathcal{V}',\mathcal{E}')$ where $\mathcal{E}'$ includes all those edges of $\mathcal{E}$ which are incident on a vertex pair in $\mathcal{V}$.

Theorem 2 (Laman's Theorem [13]): A graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ in $\mathbb{R}^2$ of $|\mathcal{V}|$ vertices and $|\mathcal{E}|$ edges is rigid if and only if there

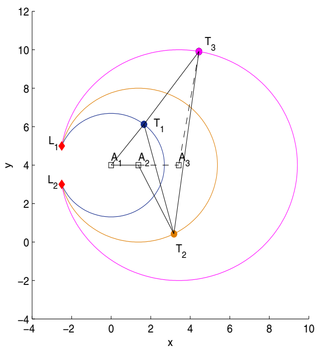

Fig. 3. The loci of agents, T_i, $i = 1, 2, 3$, for $\measuredangle L_1T_iL_2 = \lambda_i$, and L_1 and L_2 are landmarks with known positions. The solid lines construct the four-bar linkage mechanism with the coupler point. Note that the loci are NOT full circles, but only arcs.

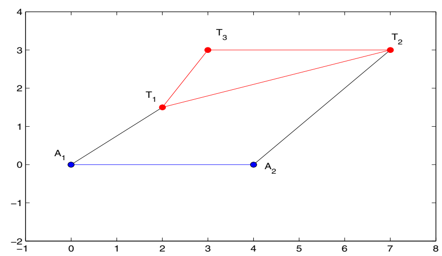

Fig. 4. The four-bar linkage mechanism obtained after deletion of A_3T_3.

exists a subgraph $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$ with $2|\mathcal{V}|-3$ edges such that for any subset $\mathcal{V}''$ of $\mathcal{V}$, the induced subgraph $\mathcal{G}'' = (\mathcal{V}'', \mathcal{E}'')$ of $\mathcal{G}'$ obeys $|\mathcal{E}''| \leq 2|\mathcal{V}''|-3$. It is minimally rigid if $|\mathcal{E}| = 2|\mathcal{V}|-3$. It is easy to check for the graph of Fig. 2 that $|E| = 2|V|-3$; one takes $\mathcal{G}' = \mathcal{G}$ and can verify the counting condition for all induced suggraphs. In the light of the minimal rigidity of the formation of Fig. 2, there will be noncongruent formations meeting the distance constraints. Then, even though the positions of A_1, A_2 and A_3 are fixed, the positions of T_1, T_2 and T_3 will not be uniquely determinable.

We shall now explain how the problem posed in the introduction fits into the framework we have just described. It has been assumed that the agents form a triangular formation, where, T_i is the the i-th agent. The separation distance between two agents i and j is known and equal to $R_{ij} \quad (or \quad R_{ji})$. For a given agent, T_i, and two landmarks with known position, L_1 and L_2, the locus for the position of T when $\measuredangle L_1T_iL_2 = \lambda_i$, is a part of a circle with a radius equals to $d/(2\sin(\lambda_i)$ and a center at $A_i = (x_i, y_i) = (d/(2\tan\lambda_i) \quad, 0)$, assuming that the origin of coordinates is the middle of L_1L_2, $d = \overline{L_1L_2}$ and the x-axis coincides with the perpendicular bisector of L_1L_2. In Fig. 3 each mobile platform, T_i, $i = 1, 2, 3$, and the associated circles are depicted. In this case the centres of the circles, A_i, serve as the virtual anchors, since we know their exact positions in the plane. Hence, the agents, T_i $(i \in \{1, 2, 3\})$, in the formation and these virtual anchors, A_j $(j \in \{1, 2, 3\})$, form a graph which satisfies the conditions presented in Problem 1. We aim to exhibit, using the concept of four-bar linkage mechanism, which is discussed in the next section, an upper bound for the number of localization possibilities is presented.

IV. Four-Bar Linkage Mechanisms And Application to The Localization Problem

A. Four-Bar Linkage Mechanisms

A *four-bar linkage* mechanism is one of the simplest closed loop kinematic linkage mechanisms. These mechanisms perform a wide variety of motions with a few simple parts.An example of such mechanisms is presented in Fig. 4. If in Fig. 2, we delete the edge connecting A_3 to T_3, $A_1A_2T_1T_2$ can be considered as a four-bar linkage mechanism (see Fig. 4). In this mechanism, T_3 is termed the *coupler point*. The curve that this coupler point moves on is called the *coupler curve*. Generally coupler curves are closed curves. A coupler curve K_C may comprise either a single part or a bipartite curve (A bipartite curve is one with two branches, like a hyperbola.). In the case that K_C is bipartite we denote the branches as K_{C_1} and K_{C_2}.

For a given four-bar linkage mechanism, in Fig. 4, the equation of the coupler curve (bipartite or single part), K_C, when the center of cartesian coordinates system is placed on A_1, and A_1 and A_2 are placed on the x-axis, is,

$$\begin{aligned} &a^2\left((x-k)^2+y^2\right)(x^2+y^2+b^2-r^2)^2- \\ &2ab\left((x^2+y^2-kx)\cos\gamma+ky\sin\gamma\right)(x^2+y^2+b^2-r^2) \\ &\left((x-k)^2+y^2+a^2-R^2\right)+b^2(x^2+y^2) \\ &\left((x-k)^2+y^2+a^2-R^2\right)^2- \\ &4a^2b^2\left((x^2+y^2-kx)\sin\gamma-ky\cos\gamma\right)^2=0 \end{aligned} \tag{1}$$

where,r and R are the lengths of the side links, here $\overline{T_1A_1}$ and $\overline{T_2A_2}$ respectively, k is the length of the frame link, $\overline{A_1A_2}$. In addition, $a = \overline{T_2T_3}$, $b = \overline{T_1T_3}$, and $\gamma = \measuredangle T_1T_3T_2$.

In addition another coupler curve, K'_C, can be obtained from the reflection of K_C, when A_1A_2 is the image axis. In general the equation describing K'_C can be obtained by substituting $-y$ for y in (1). In the case of a bipartite K_C we denote the branches of K'_C as K'_{C_1} and K'_{C_2}. As a result the locus of the coupler point is made up of two polynomial curves each with degree of six.

B. Application to The Formation Localization Problem

From our analysis of Problem 1 we know that, T_3 is placed on a circle with A_3 as its center and R_3 as its radius. So the possible solutions for the localization problem, can be obtained from the calculation of intersections of the circle, C_{A_3,R_3} : $(x-x_3)^2+(y-y_3)^2 = R_3^2$ and the two coupler curves. One might then expect that for each coupler curve we would have 12 intersection points (real and complex), and as a result 24 localization solutions in all. Somewhat surprisingly perhaps, the following theorem states that the maximum number of localization solutions is 12.

Lemma 1: The circle,

$$C_{A_3,R_3} : (x - x_3)^2 + (y - y_3)^2 = R_3^2 \quad (2)$$

and the coupler curve described by (1) have at most six real points of intersection.

Proof. Replacing y by $y_3 \pm \sqrt{R_3^2 - (x - x_3)^2}$ in (1), and then eliminating the radicals, we obtain,

$$a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0. \quad (3)$$

The number (necessarily non-zero for this problem) of real solutions for this equation can be 2, 4 or at most 6. ■

Note also that, unless a solution is repeated, there are necessarily an even number of distinct real solutions. In summary, we have established the following theorem,

Theorem 3: The maximum number of (real) localization solutions for Problem 1 is 12. For generic values of distances and angles, the minimum number of localization solutions is 4.

Proof. Equation (1) corresponds to the coupler curves for the four-bar likages mechanism depicted in Fig. 4 corrosponding to Problem 1 [15]. Since there are 2 (single part or bipartite) coupler curves (the second one is the image of the first one when the frame link is the image axis) and for each coupler curve based on Lemma 1, there are a maximum of 6 and a minimum of 2 possible solutions, we have at most 12 possible solutions, and at least 4 localization solutions. □

Returning to the problem presented in Section 1, based on the procedure introduced earlier in this section for constructing a four-bar mechanism by deleting edge A_3T_3, we can have the linkage mechanism depicted by solid lines in Fig. 3. In addition, here for the coupler curve equation we have, $k = (d/(2\tan\lambda_2)) - d/((2\tan\lambda_1))$, $r = R_1 = d/(2\sin\lambda_1)$, $R = R_2 = d/(2\sin\lambda_2)$, $b = R_{13}$,and $a = R_{23}$. From Theorem 3 we can have up to 12 localization solutions.

Remark 1: Terming intersection points that occur on that part of the circle for which the angle subtended by the landmarks does not equal the measured value for the angle, i.e. the undrawn shorter arcs joining L_1L_2 in Fig. 3, as *invalid solutions*, for some cases where there are 4 intersections and the two of the intersection points are invalid solutions, there are only 2 (mirror image) localization solutions.

Remark 2: For generic values of distance and angles the number of solutions for Problem 1 can be 4, 8, or at most 12.

Remark 3: Theorem 3 holds for the cases that the distance and angle information can be associated with a real scenario. For instance for the cases that inter-agent distances are not feasible to construct a triangle, seeking a localization solution is irrelevant.

Remark 4: The explicit form of (3) and a localization algorithm to be used in real-time is presented in [16].

C. Localization of Larger Formations

The following theorem extends the current idea of localization of three agents to localization of a formation with a globally rigid underlying graph.

TABLE I
THE ANGLE AND DISTANCE VALUES IN EACH SCENARIO.

Scenario	λ_1	λ_2	λ_3	$\overline{T_1T_2}$	$\overline{T_1T_3}$	$\overline{T_2T_3}$
Fig. 5	0.3805	0.2526	0.1674	3.1623	5.099	2.8284
Fig. 6	0.2487	0.2487	0.2487	2.0859	2.7552	4.6188
Fig. 7	$\pi/3$	$\pi/3.5$	$\pi/4$	1.5	0.9765	1.25
Fig. 8	$\pi/4$	$\pi/5$	$\pi/6$	3	2	2

TABLE II
THE NUMBER OF LOCALIZATION SOLUTIONS IN EACH SCENARIO AND IMPORTANT CHARACTERISITCS OF EACH SCENARIO.

Scenario	No. of Distinct Solutions	Characteristic
Fig. 5	8	generic
Fig. 6	infinite	non-generic/ infinite ambiguity
Fig. 7	2	non-generic/ repeated solutions
Fig. 8	2	generic/ invalid solutions

TABLE III
THE LEGENDS USED IN SIMUALTION RESULTS

agent	small solid red circle
landmark	solid red diamond
formation	solid red triangles
agent locus	dashed blue circles
coupler curve	solid black curves

Theorem 4: Consider a formation $\mathcal{F}$ with the underlying globally rigid graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, and the three agents, T_1, T_2 and T_3 in the formation which form a triangle. Assuming that these three agents are the only agents capable of measuring the angle subtended at them by the two landmarks L_1 and L_2, with known positions, then there are at most twelve possible localization solutions for the formation.

Proof. Theorem 3 states that the upper bound for the number of localization solutions of a triangular formation using the value of the angles subtended at each agent by two landmarks, is 12. On the other hand, in [10] it has been shown that the necessary and sufficient condition for unique localization of a formation is that the associated graph is globally rigid and there are 3 nodes with exactly known positions. As a result, for each possible localization of three agents there is a localization solution for the whole formation, so there are up to 12 possible localization possibilities for the formation. □

V. SIMULATION RESULTS

In this section several simulation results are presented to show different localization solutions in different scenarios. The angle and distance values used in each simulation scenario are presented in Table I. In addition, it is worth mentioning that after running several simulations, a case with twelve localization solutions was never encountered. The important characteristics of each simulation result and the number of localization solution in each scenario are presented in Table II. The legends used in simulation results are described in Table III.

VI. FUTURE WORK AND CONCLUDING REMARKS

In this paper an upper bound of 12 is obtained for the number of possible localization solutions of a triangular agent

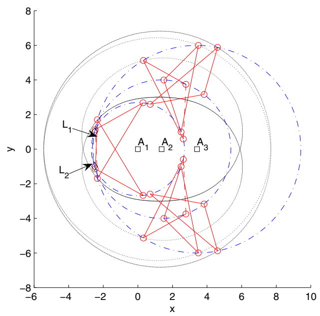

Fig. 5. The possible localization solutions where $d = 2$, $\lambda_1 = 0.3805$, $\lambda_2 = 0.2526$, $\lambda_3 = 0.1674$, $\overline{T_1T_2} = 3.1623$, $\overline{T_1T_3} = 5.0990$, and $\overline{T_2T_3} = 2.8284$.

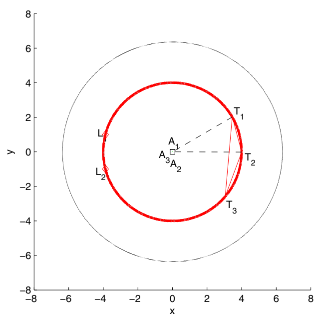

Fig. 6. Infinite number of localization solutions when $\lambda_1 = \lambda_2 = \lambda_3$. The locus of T_3 coincides with one of the branches of the coupler curve.

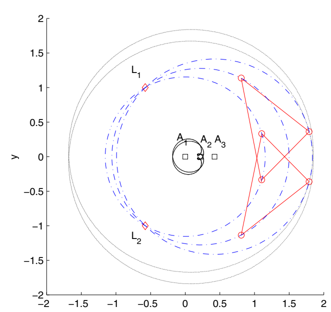

Fig. 7. Repeated localization solutions. Here $d = 2$, $\lambda_1 = \pi/3$, $\lambda_2 = \pi/3.5$, $\lambda_3 = \pi/4$, $\overline{T_1T_2} = 1.5$, $\overline{T_1T_3} = 0.9765$, and $\overline{T_2T_3} = 1.25$.

formation, collecting bearing measurements to two landmarks. Furthermore, a more general theorem regarding localization of globally rigid formations, taking advantage of a priori knowledge about the position of two landmarks, is presented. In addition, a bad geometry is identified, and it is when all angles subtended at each of the three agents by the two landmarks are equal, or equivalently, the three agents and the two landmarks are placed on the same circle. In this case there are an infinite number of localization solutions. In the end some simulation results are presented. A future research direction is to tackle localization with noisy measurements using geometric constraints.

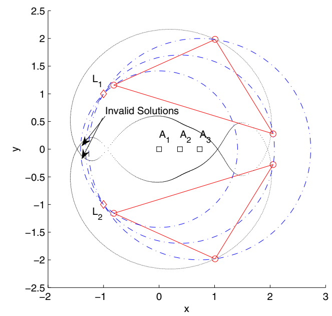

Fig. 8. Occurrence of invalid localization solutions. Here $d = 2$, $\lambda_1 = \pi/4$, $\lambda_2 = \pi/5$, $\lambda_3 = \pi/6$, $\overline{T_1T_2} = 3$, $\overline{T_1T_3} = 2$, and $\overline{T_2T_3} = 2$.

ACKNOWLEDGMENTS

This work is supported by NICTA, which is funded by the Australian Government as represented by the Department of Broadband, Communications and the Digital Economy and the Australian Research Council through the ICT Centre of Excellence program.

REFERENCES

[1] Ward P. W., "GPS receiver RF interference monitoring, mitigation and analysis techniques," *Navigation*, vol. 41, pp. 367–391, winter, 1995-1994.

[2] G. Dedes and A. G. Dempste, "Indoor GPS positioning: challenges and opportunities," in *IEEE Conference on Vehicular Technology*, Dallas, USA, Sep. 2005, pp. 412–415.

[3] M. Betke and L. Gurvits, "Mobile robot localization using landmarks," *IEEE Transactions on Robotics and Automation*, vol. 13, pp. 251–263, 1997.

[4] I. Shimshoni, "On mobile robot localization from landmark bearings," *IEEE Transactions on Robotics and Automation*, vol. 13, pp. 971–976, 2002.

[5] Hmam H., "Mobile platform self-localisation," in *Proceedings of Information Decision and Control*, Adelaide, Australia, Feb. 2007, pp. 242–247.

[6] Hmam H., "Cooperative sensor self-localization in the plane," submitted to IEEE Transactions on Aerospace and Electronic Systems.

[7] Eren T., Goldenberg D., Whiteley W., Yang Y., Morse A. S., and Anderson B. D. O., "Rigidity, computation and randomization in network localization," in *Proceedings of Network Localization, Joint Conference of IEEE Computer and Communication Societies*, Hong Kong, March 2004, pp. 2673–2684.

[8] Mao G., Fidan B., and Anderson B. D. O., *Sensor Networks and Configuration: Fundamentals, Techniques, Platforms and Experiments*, chapter 13, Localization, pp. 281–315, Springer-Verlag, 2006.

[9] B. Hendrickson, "Conditions for graph unique realizations," *SIAM Journal on Computing*, vol. 21, pp. 65–84, Feb. 1992.

[10] B. Jackson and T. Jordan, "Connected rigidity matroids and unique realizations of graphs," *J Combinatorial Theory Series B*, vol. 94, pp. 1–29, 2005.

[11] Tay T. and Whiteley W., "Generating isostatic frameworks," *Structural Topology*, vol. 11, pp. 21–69, 1985.

[12] B. D. O. Anderson, C. Yu, B. Fidan, and J. M. Hendrickx, "Control and information architectures for formations," in *Proceedings of the 2006 IEEE International Conference on Control Applications*, Munich, Germany, Oct. 2006, pp. 1127–1138.

[13] Laman G., "On graphs and rigidity of plane skeletal structures," *J. Engrg. Math.*, vol. 11, pp. 331–340, 1970.

[14] Tutte W. T., "A theory of 3-connected graphs," *Indagationes Mathematicae*, vol. 23, pp. 441–455, 1961.

[15] Beyer R., *Kinematic Synthesis of Mechanisms*, Chapman and Hall Ltd., 37 Essex Street, London, 1963, Translated from german by H. Kuenzel.

[16] I. Shames, B. Fidan, B. D. O. Anderson, and H. Hmam, "Cooperative self-localization of mobile agents moving in planar formations," Preprint.