Optimality Analysis of Sensor-Target Geometries in Passive Localization: Part 2 - Time-of-Arrival Based Localization

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Abstract

In this paper we characterize the relative sensor-target geometry in $\mathbb{R}^2$ in terms of potential localization performance for time-of-arrival based localization. Our aim is to characterize those relative sensor-target geometries which minimize the relative Cramer-Rao lower bound.

1. Introduction

Currently the two most common passive measurement technologies available for localization and tracking are bearing measurements [1], [2] and time-of-arrival measurements [3] (or range-difference measurements). The received signal strength (RSS) also permits passive localization of an emitter whose transmission power is known. However, in this paper we focus on range-difference based localization systems in $\mathbb{R}^2$. This is part 2 of a pair of papers dealing with geometric characterizations of potential passive localization accuracy. Part 1 [4] deals with bearing-based localization and the reader is referred to [4] for a general introduction and related materials and references.

The primary motivation for studying the relative sensor-target geometry arises from the realization that the relative geometry has a significant effect on the performance of localization estimators. The performance bounds, e.g. variance lower bounds etc., placed on a localization algorithm are inherently derived in terms of the relative sensor-target geometry. Indeed, in this paper we examine such accuracy lower-bounds and explicitly optimize the geometry such that we can achieve the smallest error bounds possible (under some given estimator assumptions). This involves solving an optimization problem for the relative sensor positioning requirements that permit the smallest variance bound. We provide a rigorous analysis of the time-of-arrival (or time-difference) based localization problem and we provide a necessary and sufficient condition for optimal sensor placement. We also show that an equi-angular arrangement of the sensors around a single target is an optimal configuration of sensors regardless of the individual sensor-target ranges.

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The remainder of the paper is organized as follows. In Section 2 we outline some notation and conventions and define the time-of-arrival (and related time-difference-of-arrival) based localization problem. In Section 3 we introduce the Cramer-Rao inequality and the related Fisher information matrix. Moreover, in Section 3 we discuss the Fisher information matrix relationship to geometric characterizations for localization and we derive the Fisher information matrix and determinant for time-of-arrival based localization with an arbitrary number of sensors. In Section 4 we characterize the localization geometry in detail and provide a number of illustrative examples. In Section 5 we provide a useful discussion on the practicality of these results and in Section 6 we give our conclusion.

2. Notation and Related Conventions

We consider a single stationary target and multiple sensors all located in $\mathbb{R}^2$. The single target's location is given by $\mathbf{p} = [x_p\ y_p]^T$. Consider a number of sensors labeled $1, \ldots, N \geq 2$ with the location of the i^{th} sensor given by $\mathbf{s}_i = [x_{si}\ y_{si}]^T$. Let the range between the i^{th} sensor $\mathbf{s}_i$ and the target $\mathbf{p}$ be given by $r_i = \|\mathbf{p}-\mathbf{s}_i\|$. For simplicity we denote the angle subtended at the target by two sensors i and j by $\vartheta_{ij} = \vartheta_{ji} \in [0, \pi)$.

A. On Time-of-Arrival and Time-Difference-of-Arrival

Consider a target emitter located $\mathbf{p} = [x_p\ y_p]^T \in \mathbb{R}^2$ which transmits a signal at a specific time τ. Let the location of the event characterized by $\mathbf{p}$ and τ be denoted by $\mathbf{x} = [x_p\ y_p\ \tau]^T \in \mathbb{R}^3$. Suppose that each sensor can measure the time of arrival of the transmitted signal at the sensor. This time of arrival is denoted by t_i. Then t_i obeys the following relationship:

$$t_i(\mathbf{x}) = \frac{\|\mathbf{p} - \mathbf{s}_i\|}{c} + \tau, \quad \forall i \in \{1, \ldots, N\} \tag{1}$$

where c is the signal propagation speed. We normalize such that $c \equiv 1$. Generally the measurement is assumed to be noisy, so that $\widehat{t_i} = t_i(\mathbf{x}) + e_i$ where $t_i(\mathbf{x})$ is the true time of signal arrival and e_i is the measurement error. The errors e_i, $\forall i \in \{1, \ldots, N\}$ are assumed to be mutually independent and Gaussian distributed with zero mean and the same variance σ_t^2. Stacking the measurements from N sensors results in the the following measurement vector

$$\widehat{\mathbf{y}}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) + \mathbf{e} = [t_1 \ \ldots \ t_n]^{\mathrm{T}} + [e_1 \ \ldots \ e_n]^{\mathrm{T}} \qquad (2)$$

where now we assume that $\widehat{\mathbf{y}}(\mathbf{x}) \sim \mathcal{N}(\mathbf{y}(\mathbf{x}), \mathbf{R}_t)$ where $\mathbf{R}_t = \sigma_t^2 \mathbf{I}$ is the covariance of $\widehat{\mathbf{y}}$. The problem of estimating $\mathbf{p}$ from the given noisy measurements $\widehat{\mathbf{y}}$ is known as the time-of-arrival localization problem. The time-of-arrival localization problem also results in an estimate of the time of signal transmission τ (although this parameter is not always required).

An alternative approach to estimate the location $\mathbf{p}$ from the given timing measurements $\widehat{t_i}(\mathbf{x}) = t_i(\mathbf{x}) + e_i$, $\forall i \in \{1, \ldots, N\}$ involves taking the time differences. The true time-difference $d_{ij} = (t_j - t_i)c$ between sensor i and j where $i \neq j$ results in the following range-difference equations

$$d_{ij}(\mathbf{p}) = \|\mathbf{p} - \mathbf{s}_j\| - \|\mathbf{p} - \mathbf{s}_i\|, \quad \forall i, j \in \{1, \ldots, N\}$$

with $c \equiv 1$. Note that there are only $(N-1)$ independent range-difference equations that can actually be formed. In this formulation we have eliminated the unknown τ. Without loss of generality we only consider range-difference equations between sensor 1 and sensor i. If we take the time difference $\widehat{t_i} - \widehat{t_1}$ then we obtain the following range-difference measurements

$$\widehat{d}_{1i} = d_{1i}(\mathbf{p}) + \epsilon_{i-1}, \ \forall i \in \{2, \ldots, N\}$$

where ϵ_{i-1} is the range-difference measurement error. Writing the range difference equations in vector form gives

$$\widehat{\mathbf{d}} = \mathbf{d}(\mathbf{p}) + \boldsymbol{\epsilon} = [d_{12} \ d_{13} \ \ldots \ d_{1N}]^{\mathrm{T}} + [\epsilon_1 \ \epsilon_2 \ \ldots \ \epsilon_{N-1}]^{\mathrm{T}}$$

Note that the covariance matrix of $\boldsymbol{\epsilon}$ is now given by

$$\mathbf{R}_d = 2\sigma_t^2 \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{2} \\ \frac{1}{2} & \cdots & \frac{1}{2} & 1 \end{bmatrix}$$

so that $\widehat{\mathbf{d}} \sim \mathcal{N}(\mathbf{d}(\mathbf{p}), \mathbf{R}_d)$. The problem of estimating $\mathbf{p}$ from the given noisy measurements $\widehat{\mathbf{d}}$ is known as the time-difference-of-arrival localization problem (or the range-difference based localization problem).

Clearly the information available for solving the two localization problems is equivalent, and generally both require at least four sensors in order to uniquely solve for an estimate of $\mathbf{p}$. The time-difference (or range-difference) based localization problem is more common in the literature. The time-of-arrival localization problem explicitly yields an estimate of the time of signal transmission τ and the simultaneous estimation of $\mathbf{p}$ and τ is commonly known as event localization. We point out that the time-difference based localization problem can indirectly yield a similar estimate of τ if desired.

In this paper we want to explicitly characterize the geometrical relationships between the sensors and the target in terms of the lower bounds on the potential localization accuracy. Since both the time-of-arrival and the time-difference-of-arrival (or range-difference) localization problems are in essence equivalent (albeit different algorithms of varying quality can be designed separately for the two problem formulations) we can choose which formulation yields the simplest analysis with the geometrical results being equally applicable to both problems. In this paper we use the time-of-arrival based formulation due to its simplicity of formulation.

B. Comment on Relative Geometric Configurations

Without loss of generality we will always restrict the sensor indexing such that the true bearings obey $\phi_j \geq \phi_i$ when $j > i$ and $\forall i, j \in \{1, \ldots, N\}$. Each bearing is modeled by the following equation

$$\phi_i(\mathbf{p}) = \operatorname{arctan2}(x_p - x_{si}, \ y_p - y_{si}) \qquad (3)$$

where the $\operatorname{arctan2}$ function is defined such that $\phi_i(\mathbf{p}) \in [0, 2\pi)$ (note that $\operatorname{arctan2}$ is related to the standard $\arctan$ function and is common in many computer programming languages). In the subsequent sections we explore the optimal geometric configurations of a number of sensors relative to a single target in terms of the angular relationships between the target and the sensors. Requiring $\phi_j \geq \phi_i$ when $j > i$ and $\forall i, j \in \{1, \ldots, N\}$ will greatly simplify the subsequent presentation with no loss of generality resulting.

3. The Cramer-Rao Inequality and Fisher Information For Range-Difference Based Localization in $\mathbb{R}^2$

In this section we give the Cramer-Rao bound for the time-of-arrival localization problem in $\mathbb{R}^2$. Considering an *unbiased* estimate $\widehat{\mathbf{x}}$ of $\mathbf{x} = [x_p \ y_p \ \tau]^T \in \mathbb{R}^3$ the Cramer-Rao bound states that

$$E\left[(\widehat{\mathbf{x}} - \mathbf{x})(\widehat{\mathbf{x}} - \mathbf{x})^{\mathrm{T}}\right] \geq \mathcal{I}^{-1}(\mathbf{x}) \triangleq \mathcal{C}(\mathbf{x}) \qquad (4)$$

where $\mathcal{I}(\mathbf{x})$ is the Fisher information matrix. In general, if $\mathcal{I}(\mathbf{x})$ is singular then no unbiased estimator for $\mathbf{x}$ exists with a finite variance [5], [6]. If $\mathcal{I}(\mathbf{x})$ is nonsingular then the existence of an unbiased estimator of $\mathbf{x}$ with finite variance is theoretically possible. If (4) holds with equality then the estimator is called *efficient* and the parameter estimate $\widehat{\mathbf{x}}$ is unique [6]. Finally, the condition (4) says nothing about the performance and realizability of biased estimators.

Consider the set of independent measurements from N sensors given by $\widehat{\mathbf{y}} = \mathbf{y}(\mathbf{x}) + \mathbf{e}$ with each time-of-arrival measurement modeled as in (1). The observable measurements obey $\widehat{\mathbf{y}} \sim \mathcal{N}(\mathbf{y}(\mathbf{x}), \mathbf{R}_t)$ where $\mathbf{R}_t$ is defined by $\sigma_t^2 \mathbf{I}$. The Fisher information matrix in this case quantifies the amount of information that the observable random vector $\widehat{\mathbf{y}}$ carries about the unobservable parameter $\mathbf{p}$. It is a matrix with the $(i, j)^{th}$ element given by

$$(\mathcal{I}(\mathbf{x}))_{i,j} = E\left[\frac{\partial}{\partial \mathbf{x}_i} \ln\left(f_{\widehat{\mathbf{y}}}(\widehat{\mathbf{y}}; \mathbf{x})\right) \frac{\partial}{\partial \mathbf{x}_j} \ln\left(f_{\widehat{\mathbf{y}}}(\widehat{\mathbf{y}}; \mathbf{x})\right)\right]$$

where $\mathbf{x}_i$ is the i^{th} element of the event location vector $\mathbf{x}$, e.g. $\mathbf{x}_1 = x_p$ and $\mathbf{x}_2 = y_p$ and $\mathbf{x}_3 = \tau$. Here $f_{\widehat{\mathbf{y}}}(\widehat{\mathbf{y}};\mathbf{x})$ is the likelihood function of $\mathbf{x}$ given fixed measurements $\widehat{\mathbf{y}}$ and the natural logarithm of $f_{\widehat{\mathbf{y}}}(\widehat{\mathbf{y}};\mathbf{x})$ is given by

$$\ln\left(f_{\widehat{\mathbf{y}}}(\widehat{\mathbf{y}};\mathbf{x})\right) = \frac{1}{2}(\widehat{\mathbf{y}} - \mathbf{y}(\mathbf{x}))^T \mathbf{R}_t^{-1}(\widehat{\mathbf{y}} - \mathbf{y}(\mathbf{x})) + c$$

where c is a constant independent of $\mathbf{x}$. Following a simple calculation we can determine that

$$\left(\mathcal{I}\left(\mathbf{x}\right)\right)_{i,j} = \frac{\partial \mathbf{y}(\mathbf{x})}{\partial \mathbf{x}_i}\mathbf{R}_t^{-1}\frac{\partial \mathbf{y}(\mathbf{x})}{\partial \mathbf{x}_j}$$

for $i, j \in \{1, 2\}$ and $\mathbf{x}_1 = x_p$, $\mathbf{x}_2 = y_p$ and $\mathbf{x}_3 = \tau$. Thus, the entire Fisher information matrix is simply given by

$$\mathcal{I}(\mathbf{x}) = \nabla_{\mathbf{x}}\mathbf{y}(\mathbf{x})^{\mathrm{T}}\mathbf{R}_t^{-1}\nabla_{\mathbf{x}}\mathbf{y}(\mathbf{x}) \tag{5}$$

where $\nabla_{\mathbf{x}}\mathbf{y}(\mathbf{x})$ is the Jacobian

$$\nabla_{\mathbf{x}}\mathbf{y}(\mathbf{x}) = \begin{bmatrix} \sin(\phi_1) & \cos(\phi_1) & 1 \\ \vdots & \vdots & \vdots \\ \sin(\phi_N) & \cos(\phi_N) & 1 \end{bmatrix} \tag{6}$$

where again ϕ_i is defined as in (3). When one sensor measures the time-of-arrival $\widehat{t_1} = t_1 + e_1$ then the Fisher information matrix is given by

$$\mathcal{I}\left(\mathbf{x}\right) = \frac{1}{\sigma_t^2}\begin{bmatrix} \sin^2(\phi_1) & \sin(\phi_1)\cos(\phi_1) & \sin(\phi_1) \\ \sin(\phi_1)\cos(\phi_1) & \cos^2(\phi_1) & \cos(\phi_1) \\ \sin(\phi_1) & \cos(\phi_1) & 1 \end{bmatrix}$$

Clearly $\det(\mathcal{I}(\mathbf{x})) \equiv 0$ is identically satisfied for any $\mathbf{x}$. Hence, no unbiased estimator with finite variance exists for the location $\mathbf{p}$ or τ when $N = 1$. The variance of the sum of independent random variables is equal to the sum of the variances. This immediately implies that the general Fisher information matrix for N time-of-arrival measurements is simply given by

$$\mathcal{I}\left(\mathbf{x}\right) = \frac{1}{\sigma_t^2}\sum_{i=1}^{N}\begin{bmatrix} \sin^2(\phi_i) & \frac{\sin(2\phi_i)}{2} & \sin(\phi_i) \\ \frac{\sin(2\phi_i)}{2} & \cos^2(\phi_i) & \cos(\phi_i) \\ \sin(\phi_i) & \cos(\phi_i) & 1 \end{bmatrix} \tag{7}$$

where i indexes the timing measurement from the i^{th} sensor. Independent measurements from additional sensors in general positions cannot decrease the total information. It is straightforward to show that when $N = 2$ the determinant $\det(\mathcal{I}(\mathbf{x}))$ vanishes for all $\mathbf{x}$. Hence, no unbiased estimator with finite variance exists for the location $\mathbf{p}$ or τ when $N = 2$. In general we need at least $N \geq 3$ sensors in order to estimate the value of $\mathbf{x}$ and due to the nonlinearity in the equations for t_i we generally require $N > 3$ sensors for the estimate of $\mathbf{x}$ to be uniquely defined (i.e. no ambiguity in the solution for $\mathbf{x}$).

Note that $\det\left(\mathcal{I}(\mathbf{x})\right)$ is inversely proportional to the uncertainty volume of an unbiased estimate of $\mathbf{x}$ [6]. We use $\det\left(\mathcal{I}(\mathbf{x})\right)$ to analyze the sensor-emitter geometry and establish which sensor configurations minimize the variance (or mean-square-error) achievable by an efficient estimator.

In this paper we are not constructing estimators but rather characterizing the effect of the localization geometry on the performance of a generic unbiased and efficient estimator. In practice this analysis can only serve as a guide for sensor placement with biased estimation algorithms. Indeed, the relationship between the analysis conducted in this paper (assuming efficient unbiased estimators) and its applicability for biased estimators is yet to be completely understood. However, the goal of many localization algorithms, e.g. see [1], [2], is to obtain unbiased estimates (despite the fact that biased estimators have the ability to outperform the Cramer-Rao bound in terms of the mean square error achieved). Indeed, many localization algorithms attempt to remove the bias via additional processing [2]. Hence, the results obtained in this paper are still of practical significance. We also discuss the practicality of these results further in Section 5.

Maximizing the determinant is equivalent to minimizing the uncertainty volume in an unbiased and efficient estimate of the parameter $\mathbf{x}$. Hence we want to find the form of the determinant of (7). Clearly (7) can be written as

$$\mathcal{I}\left(\mathbf{x}\right) = \frac{1}{\sigma_t^2}\begin{bmatrix} \sum(\frac{1}{2} - \frac{cos(2\phi_i)}{2}) & \sum\frac{\sin(2\phi_i)}{2} & \sum\sin(\phi_i) \\ \sum\frac{\sin(2\phi_i)}{2} & \sum(\frac{1}{2} + \frac{\cos(2\phi_i)}{2}) & \sum\cos(\phi_i) \\ \sum\sin(\phi_i) & \sum\cos(\phi_i) & N \end{bmatrix}$$

where the sum $\sum$ is taken over the index $i = \{1, \ldots, N\}$, i.e. $\sum = \sum_{i=1}^{N}$. Taking the determinant of the 3×3 matrix and rearranging yields

$$\begin{aligned} \det\left(\mathcal{I}(\mathbf{x})\right) = \tfrac{1}{\sigma_t^6}\Big[& \tfrac{N^3}{4} - \tfrac{N}{4}\left(\textstyle\sum\cos(2\phi_i)\right)^2 \\ & -\tfrac{N}{2}\left(\left(\textstyle\sum\cos(\phi_i)\right)^2 + \left(\textstyle\sum\sin(\phi_i)\right)^2\right) \\ & - \textstyle\sum\sin(2\phi_i)\left(\tfrac{N}{4}\sum\sin(2\phi_i) - \sum\cos(\phi_i)\sum\sin(\phi_i)\right) \\ & - \tfrac{1}{2}\textstyle\sum\cos(2\phi_i)\left(\left(\sum\sin(\phi_i)\right)^2 - \left(\sum\cos(\phi_i)\right)^2\right)\Big] \end{aligned} \tag{8}$$

In the next section we characterize those sensor placements which maximize the determinant and minimize the bound on the variance (or mean-square-error) of an unbiased estimator.

4. The Geometry of Time-of-Arrival Based Localization

In this section we consider the relative sensor-target geometry for the problem of time-of-arrival based localization.

A. On the Optimal Localization Geometry for N Sensors

In this subsection we examine the optimal relative sensor-target geometry for time-of-arrival based localization with an arbitrary number $N \geq 3$ of sensors. An unbiased and efficient estimate of $\mathbf{x}$ (or more likely $\mathbf{p}$) will achieve the smallest mean-square-error (or variance) when the sensor-target geometry obeys the configuration derived in this subsection. The following is the main result of this paper.

Theorem 1: The Fisher information determinant (8) is upper-bounded by $\frac{N^3}{4\sigma_t^6}$ which is achieved if and only if

$$\begin{array}{ccc} \sum_{i=1}^{N}\sin(\phi_i(\mathbf{x})) = 0 & \text{and} & \sum_{i=1}^{N}\sin(2\phi_i(\mathbf{x})) = 0 \\ \sum_{i=1}^{N}\cos(\phi_i(\mathbf{x})) = 0 & \text{and} & \sum_{i=1}^{N}\cos(2\phi_i(\mathbf{x})) = 0 \end{array} \tag{9}$$

are simultaneously satisfied with $N \geq 3$.

In order to prove Theorem 1 we need the following lemma.

Lemma 1: Let α, β and γ be three quantities bounded in magnitude by N. Then $\alpha\beta\gamma < \frac{N}{4}\alpha^2 + \frac{N}{2}\beta^2 + \frac{N}{2}\gamma^2$ unless $\alpha = \beta = \gamma = 0$.

Proof: We provide the following constructive proof:

$$\begin{aligned}\alpha\beta\gamma &= \frac{1}{4}\alpha\beta\gamma + \frac{1}{2}\alpha\beta\gamma + \frac{1}{2}\alpha\beta\gamma \\ &\leq \frac{N}{4}|\alpha\beta| + \frac{N}{4}|\alpha\gamma| + \frac{N}{2}|\beta\gamma| \\ &\leq \frac{N}{8}\left(\alpha^2+\beta^2\right) + \frac{N}{8}\left(\alpha^2+\gamma^2\right) + \frac{N}{4}\left(\beta^2+\gamma^2\right) \\ &= \frac{N}{4}\alpha^2 + \frac{3N}{8}\beta^2 + \frac{3N}{8}\gamma^2 \\ &\leq \frac{N}{4}\alpha^2 + \frac{N}{2}\beta^2 + \frac{N}{2}\gamma^2\end{aligned}$$

Notice that the last inequality is strict unless $\beta = \gamma = 0$. Then $\alpha\beta\gamma < \frac{N}{4}\alpha^2$ unless $\alpha = 0$. ∎

Now we proceed to prove Theorem 1.

Proof: [of Theorem 1] The determinant upper-bound of $\frac{N^3}{4\sigma_t^6}$ can be verified by showing that for all $N \geq 3$ and ϕ_i,

$$\begin{gathered}\tfrac{N}{4}\left(\sum\cos(2\phi_i)\right)^2 + \tfrac{N}{2}\left(\sum\cos(\phi_i)\right)^2 + \tfrac{N}{2}\left(\sum\sin(\phi_i)\right)^2 \\ +\tfrac{N}{4}\left(\sum\sin(2\phi_i)\right)^2 + \tfrac{1}{2}\sum\cos(2\phi_i)\left(\sum\sin(\phi_i)\right)^2 \\ \geq \sum\sin(2\phi_i)\sum\cos(\phi_i)\sum\sin(\phi_i) \\ +\tfrac{1}{2}\sum\cos(2\phi_i)\left(\sum\cos(\phi_i)\right)^2\end{gathered}$$

holds. To do this we firstly point out that

$$\frac{N}{2}\left(\sum\cos(\phi_i)\right)^2 \geq \frac{1}{2}\sum\cos(2\phi_i)\left(\sum\cos(\phi_i)\right)^2$$

holds for all ϕ_i since $\sum\cos(2\phi_i) \leq N$. Moreover, it is straightforward to show that

$$\begin{gathered}\tfrac{N}{2}\left(\sum\cos(\phi_i)\right)^2 + \tfrac{N}{2}\left(\sum\sin(\phi_i)\right)^2 + \tfrac{N}{4}\left(\sum\sin(2\phi_i)\right)^2 \\ \geq \sum\sin(2\phi_i)\sum\cos(\phi_i)\sum\sin(\phi_i)\end{gathered}$$

using Lemma 1 with $\alpha = \sum\sin(2\phi_i)$, $\beta = \sum\sin(\phi_i)$ and $\gamma = \sum\cos(\phi_i)$. Therefore we conclude the upper-bound of the determinant (8) is given by $\frac{N^3}{4\sigma_t^6}$ and that the upper-bound is achieved if and only if the conditions (9) are satisfied. ∎

The bearing values which satisfy the conditions (9) given in Theorem 1 lead directly to the required sensor-target angular relationships which minimize the Cramer-Rao bound. Such sensor configurations are referred to as optimal sensor configurations. Later in the paper we discuss the practicality of assuming unbiased and efficient localization. In the following proposition we provide an intuitively appealing example of a sensor configuration which minimizes the Cramer-Rao bound.

Proposition 1: One particular optimal (in the mean-square-error sense) sensor-target configuration for unbiased and efficient estimation of the target location $\mathbf{p}$ *(or more generally the event location* $\mathbf{x}$*) occurs when* $N \geq 3$ *and*

$$\vartheta_{ij} = \vartheta_{ji} = \frac{2}{N}\pi \tag{10}$$

for all $i, j \in \{1, \ldots, N \geq 3\}$ *with* $j - i = 1$ *and where* $\vartheta_{ij} = \vartheta_{ji} \in [0, \pi)$ *with* $j - i = 1$ *is the angle subtended at the target by the two adjacent sensors* i *and* j.

Proof: We need to relate the conditions given in Theorem 1 to the angles $\vartheta_{ij} = \vartheta_{ji} \in [0, \pi)$ and the condition (10) given in the proposition. With no loss of generality let $\phi_1 = 0$ and $\phi_j \geq \phi_i$ when $j > i$ such that the condition (10) of Proposition 1 implies $\phi_j = \frac{2}{N}\pi + \phi_i$ for all $j \in \{2, \ldots, N\}$ where $j - i = 1$. Now if N is odd and $M = (N+1)/2$ then $\sin(\phi_k) = -\sin(\phi_l)$ where $k \in \{2, \ldots, M\}$ and $l \in \{M+1, \ldots, N\}$ and $\sum_{i=2}^{N}\cos(\phi_i(\mathbf{x})) = -1$ which implies $\sum_{i=1}^{N}\sin(\phi_i(\mathbf{x})) = 0$ and $\sum_{i=1}^{N}\cos(\phi_i(\mathbf{x})) = 0$. If N is even and $M = N/2$ then $\sin(\phi_k) = -\sin(\phi_l)$ and $\cos(\phi_k) = -\cos(\phi_l)$ where $k \in \{1, \ldots, M\}$ and $l \in \{M+1, \ldots, N\}$ which similarly implies $\sum_{i=1}^{N}\sin(\phi_i(\mathbf{x})) = 0$ and $\sum_{i=1}^{N}\cos(\phi_i(\mathbf{x})) = 0$. Similar reasoning will show that $\phi_j = \frac{2}{N}\pi + \phi_i$ for all $j \in \{2, \ldots, N\}$ satisfies $\sum_{i=1}^{N}\sin(2\phi_i(\mathbf{x})) = 0$ and $\sum_{i=1}^{N}\cos(2\phi_i(\mathbf{x})) = 0$. Therefore, we have proved the sufficiency of the condition given in the proposition. ∎

Note that the optimal geometry conceptually requires all of the sensors to have equi-angular spacing around the target and is independent of the sensor-target ranges.

Note that the condition (10) of an equi-angular sensor configuration surrounding the target is an intuitively pleasing optimal configuration which minimizes the Cramer-Rao lower bound. However, it is not unique and for $N > 5$ there exists multiple optimal sensor configurations which satisfy the required conditions (9) given in Theorem 1.

One would also expect that if the variance of the noise associated with the measurement of t_i was dependent on i, being perhaps a function of the sensor-target range then the optimal sensor placement would be dependent on the ranges, e.g. see part 1 [4]. In the remainder of this section we examine some illustrative examples.

B. On the Geometry for Three Sensors and One Target

In this section we examine the practically important case involving $N = 3$ sensors and we graphically illustrate Proposition 1. With $N = 3$ sensors and three time-of-arrival measurements (or two time-difference measurements) it is possible that an ambiguity in the estimate of the target location $\mathbf{p}$ exists since two hyperbola branches can intersect in more than one location. In the case of a localization ambiguity it might still be possible to localize given additional (*a priori*) knowledge of a region containing the target's position. In any case, here we are primarily concerned with characterizing the sensor placement for the $N = 3$ sensors which results in the minimization of the Cramer-Rao bound.

From Proposition 1 we know that the optimal geometry is obtained when the three sensors uniformly surround the target and is invariant to the target ranges r_i, $\forall i \in \{1, 2, 3\}$. An example of the optimal sensor geometry for time-of-arrival based localization with three sensors is given in Figure 1.

From Figure 1 we observe an arbitrary optimal placement of $N = 3$ sensors around a single target. Note that the angle of separation between *adjacent* sensors is equal for all adjacent sensor pairs and the sensor placement is independent of the sensor-target ranges.

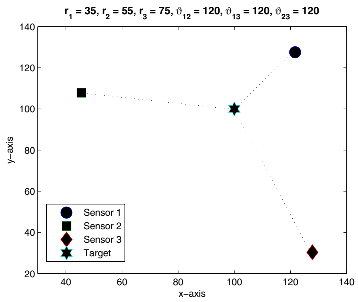

Fig. 1: An example of optimal geometric sensor placements for time-of-arrival based localization with three sensors. The particular target ranges and the angles $\vartheta_{ij} = \vartheta_{ji} \in [0, \pi)$ subtended at the target by the two sensors i and j (in degrees) are given in the figure title.

In order to evaluate the localization geometry in general scenarios we can explore a number of graphical examples. Firstly, we plot the value of the determinant with $\sigma_t^2 = 1$ directly over the possible values of $A, B \in [0, 2\pi)$ where $A = \phi_3(\mathbf{p}) - \phi_1(\mathbf{p})$ and $B = \phi_2(\mathbf{p}) - \phi_1(\mathbf{p})$. The surface and contour plots are given in Figure 2.

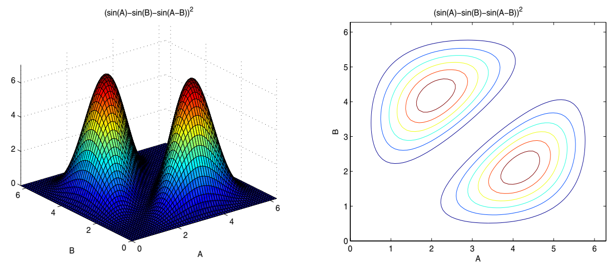

Fig. 2: The value of the Fisher information determinant for three sensors measuring the time-of-signal-arrival.

We can observe in Figure 2 that for the bearing differences $A, B \in [0, 2\pi)$, the value of the determinant is maximum when $A = \frac{4}{3}\pi$ and $B = \frac{2}{3}\pi$ or when $A = \frac{2}{3}\pi$ and $B = \frac{4}{3}\pi$. When $\phi_3 \geq \phi_2 \geq \phi_1$ as assumed then the only maximum consistent with the restriction is $A = \frac{4}{3}\pi$ and $B = \frac{2}{3}\pi$.

Now consider an where the sensors are arranged to form a unit equilateral triangle. Let the sensor coordinates be given by $\mathbf{s}_1 = [-1/2 \ 0]^T$, $\mathbf{s}_2 = [1/2 \ 0]^T$ and $\mathbf{s}_3 = [0 \ \sqrt{3}/2]^T$. We then plot the surface of the determinant for target coordinates obeying $x_p \in [-1, 1]$ and $y_p \in [-1/2, 1]$. The determinant surface is given in Figure 3 along with the contour plot.

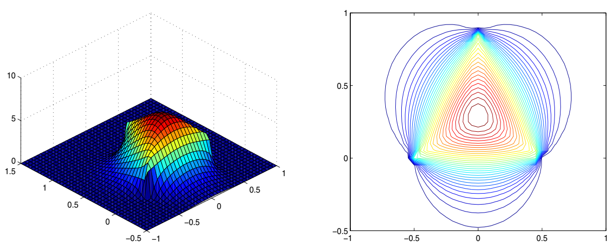

Fig. 3: The value of the Fisher information determinant for three sensors arranged in an equilateral triangle measuring the time-of-signal-arrival.

From Figure 3 we see the determinant is maximized when the target is at the center of the triangle. Indeed, if the target is anywhere within the triangle, then the geometry is well-suited to obtaining a bound suggesting the possibility of accurate localization and conversely if the target moves outside the triangle then this accuracy depreciates.

C. On the Geometry for Four Sensors and One Target

In this subsection we explore, via illustration, the geometric characteristics with $N = 4$ sensors and one target. Firstly, we will plot the contours of the Fisher information determinant for fixed values $\phi_1 = 0$ and $\phi_2 \in \{\frac{\pi}{12}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{3\pi}{2}\}$ with $\phi_3, \phi_4 \in [0, 2\pi)$. The plots are given in Figure 4.

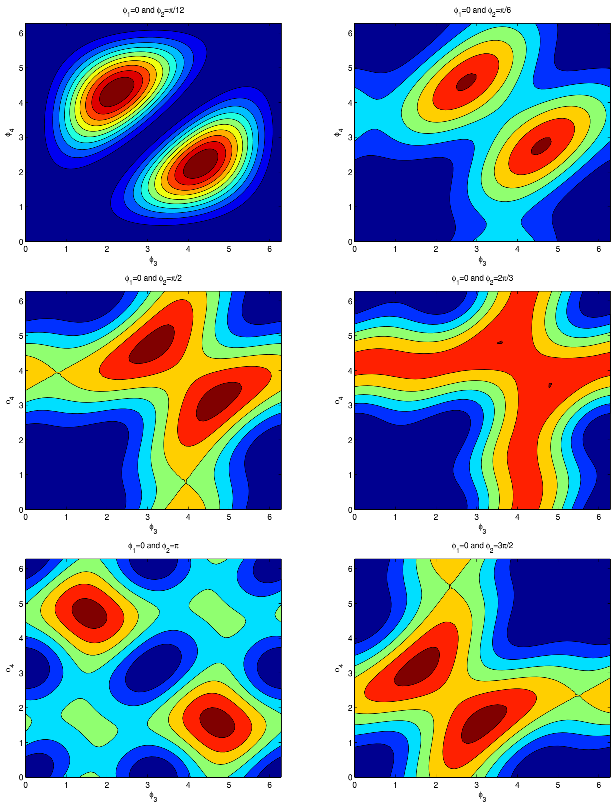

Fig. 4: The contours of the Fisher information determinant for four sensors and various ϕ_i values.

From Figure 4 we can observe the optimal geometries in terms of $\phi_3, \phi_4 \in [0, 2\pi)$ for the given fixed values of $\phi_1 = 0$ and $\phi_2 \in \{\frac{\pi}{12}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{3\pi}{2}\}$. The optimal geometry occurs when $\phi_1 = 0$, $\phi_2 = \frac{\pi}{2}$, $\phi_3 = \pi$ and $\phi_4 = \frac{3\pi}{2}$ as expected. That is, when the sensors are equally spaced around the target (regardless of the target ranges).

Now consider an arrangement of four sensors such that $\mathbf{s}_1 = [-1/2 \ 1/2]^T$, $\mathbf{s}_2 = [1/2 \ 1/2]^T$, $\mathbf{s}_3 = [1/2 \ -1/2]^T$ and $\mathbf{s}_4 = [-1/2 \ -1/2]^T$. The sensors are arranged to form a unit square centered at the origin. We plot the value of the Fisher information determinant for target coordinates obeying $x_p \in [-5/4, 5/4]$ and $y_p \in [-5/4, 5/4]$. The determinant surface is given in Figure 5 along with the associated contour plot.

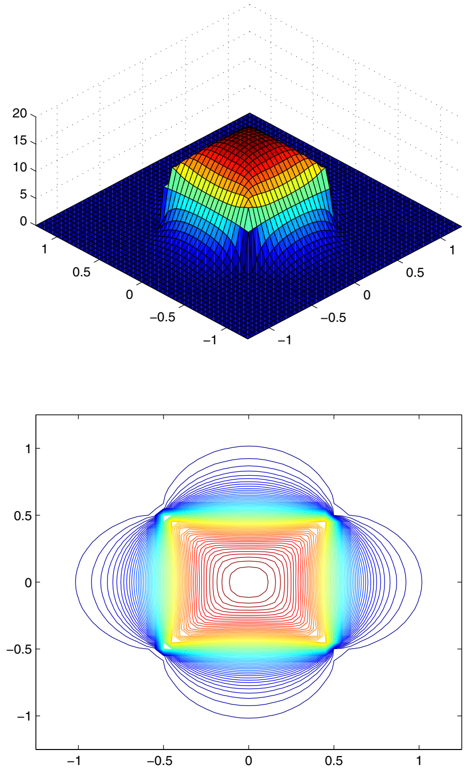

Fig. 5: The value of the Fisher information determinant for four sensors arranged in a unit square measuring the time-of-signal-arrival.

From Figure 5 we note that the optimal geometry is clearly achieved when the target is located at the origin (or at the center of the unit square) as expected.

5. Discussion

The results in this paper assume an unbiased and efficient estimator is used to estimate the target location. However, the estimation technique used in practice is likely to be biased [1], [2]. For example, even the well-known maximum likelihood localization techniques are only asymptotically unbiased and efficient, i.e. require the number of sensors to approach infinity. However, many localization algorithms are actually designed with unbiasedness in mind and with a goal of achieving the Cramer-Rao lower bound [7]. The Cramer-Rao bound for unbiased estimators (i.e. used in this paper) is therefore an interesting benchmark with which intuitively pleasing results on sensor placement have been derived. However, these results can only be considered a guide for practical sensor placement with an accuracy dependent on the bias and efficiency characteristics of the particular estimator employed.

It is well known that the variance (or mean-square-error) of an estimate can actually be made smaller at the expense of increasing the bias [8]. The work of [9], [10] explores the concept of bias-variance trade offs in estimation. In [6], [10] a *biased* Cramer-Rao inequality and in [9], [10] a *uniform* Cramer-Rao inequality are developed and can be used to study this so-called bias-variance trade off. In practice, we are often limited by the choice of estimation algorithm with the maximum likelihood algorithm being statistically optimal and asymptotically unbiased and efficient. Given a specific estimator (or possibly a class of estimator) then the results of [6], [9], [10], [8] can be used to extend the results given here to practical estimation algorithms such as maximum likelihood. Moreover, the results of [8] might potentially be useful in designing localization algorithms and relative sensor placements schemes which permit estimation with a variance below the unbiased Cramer-Rao bound. These problems are yet to be rigorously addressed within the localization literature.

Finally, we remark that in this paper we considered only the single-target scenario. However, the concepts proposed can be extended to multiple-target localization. In general, an optimal sensor placement scheme for multiple target localization will result in a sub-optimal placement in terms of each individual target (or all but one of the targets).

6. Conclusion

We have given a direct and rigorous characterization of the relative sensor-target geometry for time-of-arrival based localization in terms of the potential localization performance of unbiased and efficient estimators. We have shown that an equi-angular surrounding of the target by an arbitrary number of sensors is an optimal sensor placement. However, this sensor configuration is not unique and we provide a necessary and sufficient condition on the sensor-target bearings for a particular sensor configuration to minimize the Cramer-Rao bound. Any sensor configuration which minimizes the Cramer-Rao bound is independent of the sensor-target ranges.

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