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Control of Directed Formations with a Leader-First Follower Structure

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***Abstract*— This paper deals with the directed distance-based control of autonomous formations. Unlike most conventional approaches, a graphical model of the formation with a minimally persistent property is used. Any minimally persistent formation can, through agents additions, be re-organized as one with a leader and a first follower. This paper focuses on leader-first-follower formations that may contain cycles. We show how to establish equations of motion, and how to construct a decentralized control law that will recover shape in the presence of small deviations from the nominal. We also list a number of signification open problems in the general area.**

***Index Terms*— Multiagent System, Directed Formations, Distributed Control, Graph Theory**

I. Introduction

In this paper, we are concerned with agents that move in a formation: specifically the formation at one instant of time must be congruent to the formation at another instant of time, or equivalently, inter-agent distances are preserved over all time. In this paper, like many predecessors, we consider control of formation shape based on inter-agent distance preservation. What distinguishes this work however from most but not all work to this point is that we assign the task of controlling the distance between two agents to a set-point value to only one of the two agents, hence the task is a *directed* one. We model the formation as a directed graph, with agents serving as nodes. A directed edge exists from i to j if agent i is responsible for maintaining its distance from j. A comprehensive review of directed formation control can be found in, for example [1]–[3]. Here we simply note that a key challenge arises when the the underlying directed graph has cycles. Indeed our main contribution is to provide control laws that stabilize despite these cycles.

It is now well understood that the concept of *graph rigidity* [4] could helpfully underpin much of the control law development. In the undirected graph case where two agents work together to maintain the correct separation between them, a distributed control law that stabilizes a formation exists only if the underlying graph is rigid, a point specially emphasized in the contributions of Olfati-Saber and colleagues, e.g. [5]. In the directed graph case, rigidity is not enough. One needs a further concept, termed *persistence*, see [1]. This includes rigidity, but overlays this with a further condition that rules out certain information-flow or sensing patterns, (corresponding to particular choices of one member of an agent pair to control the distance between that pair) that are otherwise consistent with the rigidity property. In a persistent graph, it remains possible to have cycles.

In this paper, we shall confine attention to the control of minimally persistent formations with a Leader-First follower structure. Specifically, such a structure with n agents involves a rigid graph with $2n-3$, well chosen edges; where all but 2 agents have precisely two outgoing edges each; one agent, known as the *leader*, has no outgoing edge; and the remaining agent, known as the *first follower*, has its solitary outgoing edge to the leader. Note that we use *first* to distinguish it from other follower agents, and there exists at most one first follower in any persistent directed formations. Some authors have used a similar term *leader-follower* to refer to directed formations. In the sequel we refer to the leader as having two degrees of freedom, as it is not responsible for meeting any distance constraints, and is as such free to move as it pleases. The first follower on the other hand has one degree of freedom, and the remaining agents have zero degrees of freedom each.

Obviously, understanding the control of such a structure is a precursor to being able to control any persistent formation, whether or not it is minimally persistent. Our prime interest is in the following problem. Suppose that all agents in a formation are correctly positioned prior to $t=0$. Just before $t=0$, they undergo small displacements from their initial positions. After $t=0$, those with degree(s) of freedom are not allowed to exercise that degree(s) of freedom for the purpose of restoring the shape (they may exercise it in connection with securing gross motion of the formation, say towards a target); those required to maintain distances are required to adjust their position in order to restore any incorrect distances to the correct value. In so doing, they can use only relative position information between themselves and the agents from which they are required to maintain their distance. The whole process has to occur so that the closed loop is stable.

This is a form of zero-input stability. This stability then underpins the preservation of a formation shape when those agents with positive degrees of freedom actually move, so that the whole formation moves while maintaining its shape. In this paper, we do not work through the details of establishing formation shape stability when motion occurs (and for this purpose, one must recognize at the outset that unless the motions executed by the agents with positive degree of freedom are in some way regular, e.g. constant direction

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and speed are maintained, one must always expect some distortion of the formation shape away from the nominally correct shape). The analysis in the paper is also a small signal analysis, i.e. we postulate the applicability of linearised models.

We first set up the continuous time equations of motion for various types of agents, characterized in terms of the number of constraints they each maintain. The stability results are provided in Section III, and presented by way of verifying a principal minor condition. The structure of eigenvalues, with an example, is examined in Section IV. The paper ends with some concluding remarks in Section V. Note that the proofs in Section III and Section IV are omitted due to space limitations, and can be found from a preprint [3].

II. Equation of motion

In this section, we set up the equations related to the control of a minimally persistent formation in the plane. Suppose that the formation is correctly located before time 0, with agent j at position p_{0j}. At time 0, we find that all agents are displaced from their correct positions. Agent j is displaced by δp_j (which is a 2-vector). We separate gross motion of the formation (translation and rotation) from correction of its shape. For shape correction: the leader will not move; the first follower will correct his/her distance from the leader, but otherwise not move; the other followers will seek to correct their distances.

If the formation is acyclic, shape control is conceptually easy. The first follower corrects its position. Then those agents which see the first follower and leader adjust their positions. Then more agents correct their positions, and so on. In discrete time, evidently the formation shape will be corrected in a number of time steps equal to the longest directed path in the underlying formation graph. In continous time, the evident triangular coupling indicates that a stability problem will not arise.

For a graph containing cycles, the problem is that when one agent corrects its position with respect to its neighbours, (i.e. the agents from which it is required to maintain its distance) and the agents of which it is a neighbour correct their positions and so on, the agents in the cycle end up "chasing their own tail", and instability is a possibility. Our focus in this paper is dealing with this problem.

A discrete time version of the adjustment process is that at any time k, all agents, having identified the errors in the distances to their neighbours between time $k-1$ and k, adjust their positions instantaneously in the expectation that their neighbours will not move. But in general their neighbours also move at time k, so a further adjustment is needed at time $k+1$. And the process continues. We shall in the following discuss this problem, but in continuous time. We seek an adjustment rule that ensures convergence, and each agent can only use relative position information to its neighbours.

We can regard any motion of the leader and any motion of the first follower preserving the distance to the leader as effectively an external input to the formation - mediated through the positive degree of freedom vertices. The rest of the formation has the task of keeping up, i.e. maintaining the shape of the formation by maintaining the formation's directed distance constraints. In the next subsections we shall consider the motion of individual vertices, and then combine these motions into a single equation. Our analysis will be restricted to small displacements, which allows linearization.

A. Motion of Ordinary Followers

Suppose agent j has to maintain its distance from agents k and m. It looks at these agents and, noting their present positions (displaced from the nominal by δp_k and δp_m), figures where it should have to move in order to restore the distances to the correct value. This position will be displaced from the nominal because agents k and m are displaced from the nominal. Identify this target position [1] as $p_{0j}+\delta p_j^*$, and note that δp_j^* is a function of δp_k and δp_m. Agent j then moves to reduce the distance from this target position, *assuming p_k and p_m do not move*. With A_j 2×2, it uses the law

$$\dot{\delta p_j} = A_j(\delta p_j^*(\delta p_k, \delta p_m) - \delta p_j) \tag{II.1}$$

Now we need to properly evaluate $\delta p_j^*(\delta p_k, \delta p_m)$ as a linear expression in δp_k and δp_m.

In order to maintain the distance constraints, we have the following:

$$||p_{0j}-p_{0m}||^2 = ||[p_{0j}+\delta p_j^*]-[p_{0m}+\delta p_m]||^2 \tag{II.2}$$

$$||p_{0j}-p_{0k}||^2 = ||[p_{0j}+\delta p_j^*]-[p_{0k}+\delta p_k]||^2 \tag{II.3}$$

which leads to (on neglecting second order terms)

$$[p_{0j}-p_{0m}]^T[\delta p_j^*-\delta p_m]=0$$
$$[p_{0j}-p_{0k}]^T[\delta p_j^*-\delta p_k]=0$$

Collecting these and solving yields

$$\begin{aligned}\delta p_j^* &= \begin{bmatrix}(p_{0j}-p_{0m})^T\\(p_{0j}-p_{0k})^T\end{bmatrix}^{-1}\\ &\times \begin{bmatrix}(p_{0j}-p_{0m})^T & 0_{1\times 2}\\ 0_{1\times 2} & (p_{0j}-p_{0k})^T\end{bmatrix}\begin{bmatrix}\delta p_m\\ \delta p_k\end{bmatrix}\end{aligned} \tag{II.4}$$

Using (II.1) there follows

$$\dot{\delta p_j} = -A_j\begin{bmatrix}(p_{0j}-p_{0m})^T\\(p_{0j}-p_{0k})^T\end{bmatrix}^{-1} R_{(j,mk)}\begin{bmatrix}\delta p_j\\ \delta p_m\\ \delta p_k\end{bmatrix} \tag{II.5}$$

where

$$R_{(j,mk)} = \begin{bmatrix}(p_{0m}-p_{0j})^T & (p_{0j}-p_{0m})^T & 0_{1\times 2}\\ (p_{0k}-p_{0j})^T & 0_{1\times 2} & (p_{0j}-p_{0k})^T\end{bmatrix}$$

and $A_j\begin{bmatrix}(p_{0j}-p_{0m})^T\\(p_{0j}-p_{0k})^T\end{bmatrix}^{-1}$ is an adjustable matrix.

We remark that $R_{(j,mk)}$ is a submatrix of the *rigidity matrix* [6] of the entire formation, obtained by deleting all

[1] Strictly speaking, there are 2 points at the correct distance from k and m. But generically, only one is close to p_j.

but the rows corresponding to the outgoing edge from agent j, and all but the columns corresponding to the vertices or agents on which the edges are incident, viz. j, k and m.

We remark further that, under the assumption of non-collinearity of agents j, k and m the inverse of the 2×2 matrix multiplying A_j exists. Non-collinearity is reasonable to assume for a generic formation.

B. Motion of First Follower and the Leader

We impose two conditions on the motion of first follower: (a) it needs to correct its distance from the leader; (b) it never moves along a line orthogonal to the line joining itself to the leader. With p_j and p_k denoting the coordinates of the first follower and leader, we obtain from these two conditions that the target position $p_j + \delta p_j^*$ must satisfy:

$$[p_{0j} - p_{0k}]^T [\delta p_j^* - \delta p_k] = 0 \tag{II.6}$$

and

$$[-y_{0j} + y_{0k} \quad x_{0j} - x_{0k}] \ [\delta p_j^* - \delta p_k] = 0 \tag{II.7}$$

There results:

$$\begin{aligned} \delta \dot{p}_j &= -A_j \begin{bmatrix} x_{0j} - x_{0k} & y_{0j} - y_{0k} \\ -y_{0j} + y_{0k} & x_{0j} - x_{0k} \end{bmatrix}^{-1} \\ &\times \begin{bmatrix} (p_{0j} - p_{0k})^T & -(p_{0j} - p_{0k})^T \\ 0_{2\times1} & 0_{2\times1} \end{bmatrix} \begin{bmatrix} \delta p_j \\ \delta p_k \end{bmatrix} \end{aligned} \tag{II.8}$$

where $\begin{bmatrix} (p_{0j} - p_{0k})^T & -(p_{0j} - p_{0k})^T \\ 0_{2\times1} & 0_{2\times1} \end{bmatrix}$ is a submatrix of the rigidity matrix (the row corresponding to the first follower's outgoing edge and the columns to the first follower and the leader) together with a zero row, and $A_j \begin{bmatrix} x_{0j} - x_{0k} & y_{0j} - y_{0k} \\ -y_{0j} + y_{0k} & x_{0j} - x_{0k} \end{bmatrix}^{-1}$ is again an adjustable (2×2) matrix.

Lastly, for the purpose of controlling formation shape, we assume that the leader (agent k say) does not move:

$$\delta \dot{p}_k = 0 \tag{II.9}$$

C. Putting the equations together

Number the agents from 1 to n, with the first follower and leader as agents $(n-1)$ and n. Number the edges so that edges $(2j-1)$ and $2j$ are out-edges of vertex j, for $j = 1, 2, \ldots (n-2)$. Edge $2n-3$ goes from the first follower to the leader. Putting the various equations together, we get:

$$\frac{d}{dt} \begin{bmatrix} \delta p_1 \\ \delta p_2 \\ \vdots \\ \delta p_{n-1} \\ \delta p_n \end{bmatrix} = \Gamma \begin{bmatrix} R \\ 0_{3\times2n} \end{bmatrix} \begin{bmatrix} \delta p_1 \\ \delta p_2 \\ \vdots \\ \delta p_{n-1} \\ \delta p_n \end{bmatrix} \tag{II.10}$$

where R is the $(2n-3) \times 2n$ rigidity matrix, and Γ is an adjustable block 2×2 matrix. As is well known (see the appendix) for a minimally persistent graph this matrix has a generic rank of $2n-3$.

The three eigenvalues at the origin correspond to there being no motion of the leader, and motion of the first follower restricted to occurring on the line joining the first follower to the leader. Choose coordinates so that the y-coordinate axis is on a line perpendicular to that joining the first follower to the leader. Then we can drop some terms and letting coordinates of p_i be ξ_{2i-1}, ξ_{2i}, we arrive at

$$\frac{d}{dt} \begin{bmatrix} \delta \xi_1 \\ \delta \xi_2 \\ \vdots \\ \delta \xi_{2n-3} \end{bmatrix} = \hat{\Gamma} \hat{R} \begin{bmatrix} \delta \xi_1 \\ \delta \xi_2 \\ \vdots \\ \delta \xi_{2n-3} \end{bmatrix} + \textit{driving term} \tag{II.11}$$

In (II.11), the driving term is a constant, and comes from the non-zero but constant values of the leader's displacement of δp_n and the y-coordinates of δp_{n-1}. Further, $\hat{R}$ is simply R with last three columns deleted, the matrix $\hat{\Gamma}$ is block diagonal, with $(n-2)$ blocks of size 2×2 and one of size 1×1. The 2×2 blocks are actually of the form

$$-A_j \begin{bmatrix} (p_{0j} - p_{0m})^T \\ (p_{0j} - p_{0m})^T \end{bmatrix}^{-1}$$

Since the A_j are adjustable, we have in effect complete freedom over the entries of $\hat{\Gamma}$. We seek to use the freedom to ensure stability. Knowing $\hat{\Gamma}$, we can of course infer A_j which defines the control law for adjusting each agents.

The crucial point is that $\dot{p}_i$ can only depend on p_i, p_j and p_k where agents j and k are those from which agent i must maintain its distance. This forces $\hat{\Gamma}$ to have the block diagonal structure, but does not constrain the individual blocks to be, for example, individually diagonal or multiples of the identity. For this problem, $\hat{\Gamma}$ in fact serves as the controller. Of course, one could contemplate replacing $\hat{\Gamma}$ by some dynamics, in which $\dot{p}_i$ was determined by dynamic processing of p_i, p_j and p_k, but this is beyond the scope of the paper.

D. Choosing the Block Diagonal Control Multiplier

Thus the underlying dynamic equation is one of the form

$$\dot{x} = \Lambda A x. \tag{II.12}$$

Here Λ is a block diagonal matrix. Its entries correspond to gains associated with the control used by each agent. If each agent were to apply the same gain to the two distance constraints we would have Λ of the form $\lambda_1 I_2 \oplus \lambda_2 I_2 \oplus \ldots$. For the moment in this section, we will assume that Λ is diagonal and all the diagonal elements of Λ can be independently chosen. The key result, providing a *constructive procedure*, is as follows.

Theorem 1: Suppose A is an $m \times m$ nonsingular matrix with every leading principal minor nonzero. Then there exists a diagonal Λ such that $Re\,[\lambda_i(\Lambda A)] < 0, \forall i$.

Remark 1: To apply this theorem, we must show that the matrix $\hat{R}$ defined in the previous subsection (corresponding to A in the theorem), has all principal minors nonzero. This is nontrivial and will be done in Section III. Of course, $\hat{\Gamma}$ will correspond to Λ and we achieve the A_j from $\hat{\Gamma}$.

Remark 2: Theorem 1 is sufficient but not necessary. One can also choose Λ with the desired properties, if all leading principal minors are nonzero, after symmetrically reordering the rows and columns of A. Necessary conditions include: nonsingularity of A and the property that not all its principal minors be zero.

Remark 3: If A is complex and Λ is allowed to be complex, then the eigenvalues of ΛA can be arbitrarily chosen by the choice of Λ under the same hypothesis as Theorem 1 [7]. One can find examples with real A and Λ constrained to be real such that arbitrary eigenvalues (with complex values in complex conjugate pairs) are not attainable. Thus the stabilizability result is nontrivially different to the eigenvalue positioning result available for complex A and Λ.

E. *The true nonlinear system*

Section II-C, studies (II.10), equivalently (II.11), which is a forced system with constant forcing function. It is only an approximation of the true nonlinear system, as the matrices $\hat{R}$ and $\hat{\Gamma}$ are defined using values at the *original* equilibrium point, i.e. before any perturbation occurs at $t = 0$, in contrast to being defined using instantaneous values along the trajectory, or values at the equilibrium point applying as $t \to \infty$. The latter equilibrium point differs from the original equilibrium point because of the perturbation occurring at $t = 0$ to ξ_{2n-2}, ξ_{2n-1}, ξ_{2n}.

Thus we should choose the gains in order to stabilize the linear system obtained by linearizing around the *new* equilibrium point of the nonlinear system; however, for small (as opposed to infinitesimal) movements, the difference between stabilizing the two linearized systems is almost immaterial. And the exponential stability of the linear system obtained by linearizing round the new equilibrium point assures that the nonlinear system will also have this property for a certain set of initial conditions.

III. The Principal Minor Condition

The key technical condition required for stabilizability is that a certain matrix have all its leading principal minors non-zero. This section addresses this issue. Recall that with $V = \{1, \cdots, n\}$, the directed graph $G = (V, E)$ has a leader-first-follower structure with n and $n-1$ being the leader and the first follower respectively. Suppose the rigidity matrix R is such that its last row corresponds to the only outgoing edge that the first follower has, i.e. the outgoing edge from $n-1$ to n. The main result of this section is as follows:

Theorem 2: Consider an n-node minimally persistent graph $G = (V, E)$ with $V = \{1, \cdots, n\}$, and n and $n-1$ the leader and the first follower respectively. Suppose $\hat{R}$ is the $(2n-3) \times (2n-3)$ submatrix of the rigidity matrix R of G, obtained by removing the last three columns of R and obeying the row and column ordering noted above. Then there exists an ordering of the first $n-2$ vertices of G and an ordering of the pair of outgoing edges for each of these vertices such that the leading principal minors of the associated $\hat{R}$ are generically nonzero.

The proof hinges on a number of lemmas. We state them without proof to provide some indication of the proof of Theorem 2.

Lemma 1: Under the hypothesis of Theorem 2, $\hat{R}$ defined above, is generically nonsingular.

Let the set V' comprise all the ordinary follower nodes, and consider any $V_0 \subset V'$. Then we define $R(V_0)$ as the principal submatrix of $\hat{R}$ obtained by retaining the columns corresponding to the elements of V_0. Call $G_0 = (V_0, E_0)$ the subgraph of G induced by V_0 and conforming to the row and column ordering noted before. Note $R(V_0)$ is not the rigidity matrix of the induced subgraph G_0, as it may contain edge information regarding certain edges of G which are not in G_0.

Then we have the following two lemmas.

Lemma 2: Under the hypothesis of Theorem 2, $R(V')$ is generically nonsingular.

Lemma 3: Consider a minimally persistent graph with a Leader-First Follower structure $G = (V, E)$, and any induced subgraph $G_1 = (V_1, E_1)$, such that V_1 contains neither the leader nor the first follower. Call $V_2 \subset V - V_1$ the set of vertices in $V - V_1$ that have incoming edges from V_1 in G. Suppose $|V_2| \geq 2$. Define E_o as the set of outgoing edges of nodes of V_1 in G. Construct a new graph $\bar{G} = (\bar{V}, \bar{E})$ with the following properties. (a) $\bar{V} = V_1 \bigcup V_2$. (b) Let $G_2 = (V_2, E_2)$ be any minimally persistent graph with a Leader-First Follower structure, with vertex set V_2, and with edge set E_2 that is not required to be related in any way to the edges in G. Choose $\bar{E} = E_2 \bigcup E_o$ and observe $E_1 \subset E_o$.

Then $\bar{G}$ is a minimally persistent graph with a Leader-First Follower structure, and the leader and first follower belong to V_2.

Lemma 2 can be applied to the graph $\bar{G} = (\bar{V}, \bar{E})$. Thus the internal structure of $\bar{G}$ implies that $R(\bar{V}')$ is block triangular, with top left corner $R(V_1)$. A subtle argument below shows that $|V_2| \geq 2$ necessarily holds. This yields the following theorem.

Theorem 3: Under the hypothesis of Theorem 2, $R(V_1)$ is generically nonsingular for every $V_1 \subset V'$.

Proof: From Lemma 2 the result holds when $V_1 = V'$. Thus suppose $V_1 \neq V'$. Then we can argue that there are at least three outgoing edges from V_1 to $V - V_1$. Indeed by Laman's theorem $|E_1| \leq 2|V_1| - 3$. Further as V_1 does not contain either the leader or the first follower, every node in V_1 has exactly 2 outgoing edges in G. Thus there must be at least three outgoing edges from V_1 to $V - V_1$. Adopt now the notation of Lemma 3. Clearly $|V_2| \neq 0$. Suppose now, to obtain a contradiction, that $|V_2| = 1$. Then in the subgraph induced by $V_1 \bigcup V_2$ there are at least $2|V_1|$ edges, while $|V_1 \bigcup V_2| = |V_1| + 1$. As $2|V_1| > 2(|V_1| + 1) - 3$, Laman's theorem, (which states that no m-vertex induced subgraph of a rigid graph can have more than $2m-3$ edges), is violated. Thus $|V_2| \geq 2$ and the conditions of Lemma 3 apply.

Using the notation and construction used in Lemma 3, call $\hat{\bar{R}}$ and $\hat{R}_2$ the matrices obtained by removing from the rigidity matrices of $\bar{G}$ and G_2 respectively, the two columns corresponding to the common leader and one column corresponding to the common follower. By Lemma 3, both $\bar{G}$ and G_2 are minimally persistent and so by Lemma 1, $\hat{\bar{R}}$ and $\hat{R}_2$ are both generically nonsingular. Then the fact that no node in V_2 has an outgoing edge to the nodes of V_1 in $\bar{G}$, and that all the outgoing edges of V_1 in G are retained in $\bar{G}$, ensures that with $\times$ a *don't care* block,

$$\hat{\bar{R}} = \begin{bmatrix} R(V_1) & \times \\ 0 & \hat{R}_2 \end{bmatrix}.$$

Thus the result follows.

■

Remark 3.1: Because of Theorem 3, we can only conclude at this point that every *even order* principal (rather than every leading principal) minor is nonzero.

We will now show that there is an ordering of vertices possible, and an ordering of the two outgoing rows associated with each vertex such that after this reordering all leading principal minors of $\hat{R}$ are generically nonzero. We begin with the vertex reordering hypothesised below.

Lemma 4: Under the hypothesis of Theorem 2, there exists a sequence of nodes $i_1, \cdots, i_{n-2}$ all in V', that has the following property: For all $1 < j \leq n-2$, i_j has at most one outgoing edge in the subgraph of G induced by $\{i_1, ..., i_j\}$.

Label without loss of generality $i_k = k$. Rows and columns $2j-1$ and $2j$ of $R(V')$ are associated with vertex j. In the ordering of rows of $R(V')$ select the penultimate row of $R(\{1, ..j\})$ to be the outgoing edge of j that is not in the subgraph of G induced by $\{1, ..., j\}$. Consider the $(2j-1)$-th leading principal minor, i.e. under the relabeling above

$$\det\left(\begin{bmatrix} R(\{1,...,j-1\}) & \times \\ 0 & x_j - x_l \end{bmatrix}\right),$$

where $\times$ is a *don't care* vector and l is the node not in $\{1, ...j-1\}$ to which j's second outgoing edge goes. Then as by Theorem 3 $R(\{1, ...j-1\})$ is nonsingular, this determinant is nonzero. Thus after the vertex reordering and ordering of outgoing edges at each vertex, all leading principal minors of $\hat{R}$ are generically non-zero.

IV. Analysis of Eigenstructure

We now show that if in (II.1) and (II.8) $A_j = I$, while the stability condition will always hold for acyclic graphs, for graphs with cycles there may be node coordinates that result in its violation.

When all A_j in (II.1) and (II.8) are identity matrices, denote the corresponding values of $\hat{\Gamma}$ by $-\hat{R}_e$. (Note that the associated Γ is a block sum of diagonal matrices determined by relative position of neighbouring agents. Most of this matrix can be organized as the negative of an outgoing or exiting rigidity matrix, denoted $-R_e$, and $-\hat{R}_e$ denotes the matrix resulting when the last rows and columns are removed.)

A. Structure of $\hat{R}_e^{-1}\hat{R}$

For any set $V_j = \{1, \cdots, j\} \subset V' = \{1, \cdots, n-2\}$, recall the definition of $R(V_j)$ in Section III and the ordering enforced on the first $n-2$ rows of $\hat{R}$ presented in section III. If vertex i in the underlying graph has outgoing edge to vertices k and m, define the two matrices

$$B_j = \begin{bmatrix} x_j - x_k & y_j - y_k \\ x_j - x_m & y_j - y_m \end{bmatrix}, \quad \text{(IV.13)}$$

$$S(V_j) = \left(\bigoplus_{i=1}^{j} B_i\right)^{-1} R(V_j). \quad \text{(IV.14)}$$

Observe that, with $\times$ denoting a *don't care* block element,

$$\hat{R}_e^{-1}\hat{R} = \begin{bmatrix} S(V') & \times \\ 0 & 1 \end{bmatrix}. \quad \text{(IV.15)}$$

Thus we have the following obvious fact:

Fact 1: At least one eigenvalue of $-\hat{R}_e^{-1}\hat{R}$ is -1.

Thus the interesting eigenvalues are those of $S(V')$. Further $S(V')$ has the structure below. Define $e_1 = [1, 0]'$ and $e_2 = [0, 1]'$. Partition $S(V_j)$ into 2×2 blocks; then each matrix on the block diagonal is I_2. Call $-X_{lr}$ the lr-th off-diagonal block element of $S(V_j)$. Then X_{lr} is non-zero iff there is an outgoing edge from l to r in the graph induced by V_j. There are thus at most two off-diagonal non-zero block elements in each block row. If l has an outgoing edge to a node r in the subgraph induced by V_j then if this edge information were in the $(2l-1)$-th row of R, $X_{lr} = B_l^{-1} e_1 e_1' B_l$. If l has a second outgoing edge to a node s in the subgraph induced by V_j and as this edge information must then be encoded in the $2l$-th row of R,

$$X_{ls} = I - X_{lr} = B_l^{-1} e_2 e_2' B_l. \quad \text{(IV.16)}$$

We will call X_{lr} the *edge weight* of the outgoing edge from l to r. Each edge weight has rank 1, and trace 1.

B. Acyclic graphs and graphs with nonoverlapping cycles

We first present a result that applies to more general graphs.

Theorem 4: Suppose q vertices in the graph induced by $V' = \{1, 2, 3, \cdots, n-2\}$ defined above have no incoming edges and m vertices have only one incoming edge each. Then there are at least $2q + m$ eigenvalues of $S(V')$ that are 1.

We next turn to graphs that are acyclic.

Theorem 5: Suppose the graph induced by V' defined above is acyclic. Then all eigenvalues of $S(V')$ are 1.

Evidently for acyclic graphs all eigenvalues of $-\hat{R}_e^{-1}\hat{R}$ are -1, and stability is guaranteed.

In the sequel we call a graph $G'' = (V'', E'')$ a *pure cycle* if with $V'' = \{1, \cdots, k\}$, $E'' = \{\{1, 2\}, \{2, 3\}, \cdots \{k-1, k\}, \{k, 1\}\}$, where $\{i, j\}$ denotes an edge from i to j. If G'' is a subgraph of G then we define its *cycle weight* to be the rank-1 matrix $X_{12}X_{23}\cdots X_{k-1,k}X_{k1}$.

We call the graph induced by V' as being *one with non-overlapping cycles* (for example, Fig. 1) if $V' = \bigcup_{i=1}^{r} V^i$,

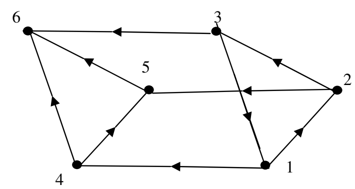

Fig. 1. An example of a minimally persistent formation with Leader-First Follower structure and two non-overlapping cycles

where the graph induced by each V^i is either acyclic or a pure cycle, at least one such graph is a pure cycle, and no node of V^j has an outgoing edge to any node in $\bigcup_{i=1}^{j-1} V^i$.

Then it is clear that for a graph with non-overlapping cycles, under a symmetric permutation of rows and columns, $S(V')$ has a block triangular structure, with $S(V^i)$ the diagonal blocks. Thus the set of eigenvalues of $S(V')$ is simply the union of the set of eigenvalues of these $S(V^i)$. Because of Theorem 5 we just focus on one such V^i for which the induced subgraph is a pure cycle. Then we have the following result:

Theorem 6: Suppose the subgraph induced by $V'' = \{1, \cdots, k\} \subset V'$ is a pure cycle. Define α to be the trace of the cycle weight. Then k eigenvalues of $S(V'')$ are at 1, and the remaining k are

$$1 - \alpha^{1/k} e^{j2\pi l/k} \quad l \in \{0, \cdots, k-1\}. \tag{IV.17}$$

We show by example in the next subsection that for suitably selected agent coordinates, (IV.17) may have negative real parts, i.e. instability ensues.

C. An example

Using the results of the previous subsection we provide an example that for suitably placed vertex positions leads to an unstable $-\hat{R}_e^{-1}\hat{R}$. Consider the graph with non-overlapping cycles in Fig. 1.

Then $-\hat{R}_e^{-1}\hat{R}$ has 6 eigenvalues at -1 and the remaining three are at (IV.17) with $k = 3$. Choose the six node x-coordinates to be [0.4103, 0.8936, 0.0579, 0.3529, 0.8132, 0.0099] and the six y-coordinates to be [0.4565, 0.0185, 0.8214, 0.4683, 0.6154, 0.7919]. Then $\alpha = 1.1407$ and (IV.17) for $l = 0$ is real and negative, implying instability.

On the other hand with

$$\hat{\Gamma} = \begin{bmatrix} 4.8330 & -4.3800 \\ 0.5740 & -0.1180 \end{bmatrix} \oplus \begin{bmatrix} 8.3570 & -8.0290 \\ -0.8040 & 5.9690 \end{bmatrix} \oplus \begin{bmatrix} -0.3524 & 0.3649 \\ 0.0480 & 0.0295 \end{bmatrix} \oplus I_3,$$

$\hat{\Gamma}\hat{R}$ has the eigenvalues [-0.0515 - 0.0473i, -0.0515 + 0.0473i, -0.4298, -1.0000, -1.0000, -1.0000, -3.2276, -4.7944, -10.1634].

V. Concluding Remarks

In this paper, we analyzed the control of minimally persistent formations with a Leader-First Follower structure. The result in [6] allows transformation to this desired structure with simple steps. The results obtained in these two papers could in principle be extended in many ways.

First, the methods of this paper only demonstrate stabilizability. The stability theorem we are relying on, involving multiplying a matrix with nonzero leading principal minors by a diagonal matrix to make it stable, is almost certainly novel; however, it does not address the achieving of control objectives beyond stability.

In fact there is a broader list of issues that need to be addressed in the future, and we record some: (a) The control laws of this paper should really be regarded as nonlinear laws, with the rigidity matrix varying in the course of the motion. We have assumed small motions in order to justify an analysis using a linearised system. An immediate task would be to demonstrate stability of the nonlinear algorithm, and at the same time or separately, construct a framework that embraces all minimally persistent formations, and not just ones of the Leader-First Follower type. (b) The control laws are based on the existence of persistence property in the formation, however, we note that this property alone does not preclude discontinuous flexibility and flip ambiguity. Further requirements are necessary at formation design stage to avoid situations, for example two symmetric(mirror) positions of same agent being too closer, when the control laws may fail. (c) It would be worthwhile extending the results to persistent formations which are not minimal. (d) More sophisticated control laws than the memoryless ones of this paper and its companion could be considered, even just proportional plus integral. (e) More sophisticated agent models, e.g. with mass, or wheel robots, could be considered. (f) Three dimensional problems could be studied.

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