CONNECTING FORWARD AND BACKWARD AUTOREGRESSIVE MODELS

B.D.O. Anderson[1] and A.C. Tsoi*

[1] Department of Systems Engineering, Institute of Advanced Studies, Australian National University, Canberra, ACT 2600, Australia.

* Department of Electrical Engineering, University of Auckland, Auckland, New Zealand.

ABSTRACT

The prime concern of this paper is to describe procedures for computing the polynomial defining a backward autoregressive recursion from the polynomial defining a forward autoregressive recursion without necessarily using a Levinson-Wiggins-Robinson type of algorithm. Two distinct procedures are given, one involving the computation of covariance data, the other not.

1. INTRODUCTION

A well-known problem in statistics or communication engineering is to fit an autoregressive model of prescribed order to a set of given covariance estimates. The vector or multi-channel recursive determination of the matrix coefficients from the set of given covariance matrices is the task of the Levinson-Whittle-Wiggins-Robinson (LWR) algorithm, [1-3]. The LWR algorithm constructs the forward polynomials, the backward polynomials, and the reflection coefficient matrices recursively from the given set of covariance matrices, increasing the degree of the forward and backward polynomial at each step of the iteration. Besides serving as a tool for calculating the reflection coefficients, the backward polynomials also define a collection of backward AR processes of increasing order which fit the prescribed covariance.

The LWR algorithm solves one of several related problems which can be described as follows. With A,B,R_i and K_i denoting the forward polynomial, backward polynomial, covariance and reflection coefficients respectively,

- P1: Given the covariance matrices R_i, find A,B and K_i
- P2a: Given the forward polynomial A, find B,R_i, and K_i
- P2b: Given the backward polynomial B, find A,R_i and K_i
- P3: Given the reflection coefficient matrices K_i, find A,B, and R_i.

Problem P1 is solves by the standard LWR algorithm. In this paper, we will consider the solutions to problems P2-P3. We will consider the problem P2a in a rather more detailed fashion. Problem P2b may be solved by a trivial modification of some steps in the solution of P2a. Problem P3 turns out to have a known solution.

We note a recent solution to P2a [4,22] which first finds the R_i from A, and then uses the standard LWR algorithm. Later in this paper, we draw distinctions between our work and that of [4,22].

Let us now note some other relevant literature. Connections between forward and reverse time modelling of covariances using state variable equations can be found in [5-8]. The ideas of especially [7,8] are of use in this paper in developing a procedure for passing from a forward to a backward polynomial.

A further collection of relevant literature is provided from stability theory. Thus several authors, e.g. [9], have pointed out the relationship between a least squares prediction problem and the problem of checking the unit-circle stability of a prescribed polynomial. The idea of a backward polynomial arises here in disguise, the backward polynomial basically being the forward polynomial with coefficients reversed. Vector results can be found in [10] and [11]. In these papers, the forward/backward polynomial relation is no longer so simple.

2. REVIEW OF FORWARD AND BACKWARD MODELS

Consider the system

$$x_{t+1} = Fx_t + Gu_t \qquad y_t = Hx_t + Ju_t \tag{2.1}$$

where [F,G] is a controllable pair and $E[u_t] = 0$, $E[u_t u_s'] = I\delta_{ts}$*. Suppose also that $|\lambda_i(F)| < 1$ for all i and that $E[x_o u_t'] = 0\ \forall t > 0$. Then (2.1) can be regarded as a forward time model for a certain stationary random process, provided the initial time is in the infinitely remote past. Now with $\Pi = [x_t x_t']$, so that

$$\Pi = F\Pi F' + GG' \tag{2.2}$$

a reverse time model [8] is defined by

$$x_t^b = F_b x_{t+1}^b + G_b u_t^b \qquad y_t^b = H_b x_{t+1}^b + J_b u_t^b \tag{2.3}$$

where

$$F_b = F^{-1}(I - GG'\Pi^{-1}) = \Pi F'\Pi^{-1}, G_b = -F^{-1}G,$$

$$J_b = J - HF^{-1}G,\ H_b = HF^{-1} + (J - HF^{-1}G)G'\Pi^{-1}$$

$$x_t^b = x_t,\ y_t^b = y_t. \tag{2.4}$$

---

* The analysis is virtually the same if with Q_f an arbitrary positive definite matrix $E[u_t u_s'] = Q_f \delta_{t,s}$.

CH1788-9/82/0000-0529$00.75 © 1982 IEEE

Further

$$E[u_t^b] = 0, [E\, u_t^b u_s^{b'}] = (I - G'\Pi^{-1}G)\delta_{t,s} = Q_b\delta_{t,s},$$

$$E[x_T u_t^{b'}] = 0, \ \forall\ t < T \tag{2.5}$$

The above reverse time model is unsuitable when F is singular. To treat F singular, we shall recast the above model into a form which does allow F to be made singular. Simple calculations show that

$$\begin{bmatrix} G_b \\ J_b \end{bmatrix} (I-G'\Pi^{-1}G)[G_b' \ J_b']$$
$$= \begin{bmatrix} \Pi-\Pi F'\Pi^{-1}F\Pi & (\Pi-\Pi F'\Pi^{-1}F\Pi)H'-\Pi F'\Pi^{-1}GJ' \\ \hline H(\Pi-\Pi F'\Pi^{-1}F\Pi) & J(I-G'\Pi^{-1}G)J' \\ -JG'\Pi^{-1}F & -JG'\Pi^{-1}F\Pi H'-H\Pi F'\Pi^{-1}GJ' \\ & +H(\Pi-\Pi F'\Pi^{-1}F)H' \end{bmatrix} \tag{2.6}$$

This has a factorization as

$$\begin{bmatrix} \bar{G}_b \\ \bar{J}_b \end{bmatrix} [\bar{G}_b' \ J_b']$$

and the reverse model is given by

$$F_b = \Pi F'\Pi^{-1}, \qquad \bar{G}_b,\ \bar{J}_b \text{ computed as above}$$

$$H_b = JG'\Pi^{-1} + H\Pi F'\Pi^{-1}, \tag{2.7}$$

with

$$E[\bar{u}_t^b] = 0, \qquad E[\bar{u}_t^b\ \bar{u}_s^{b'}] = I\delta_{t,s}$$

$$E[x_T\ \bar{u}_t^{b'}] = 0, \qquad \forall t < T \tag{2.8}$$

If the transfer function matrices of the forward and backward models (2.1) and (2.3) are defined as

$$W(z) = H(zI - F)^{-1}G + J,\ W_b(z) = H_b z[I - F_b z]^{-1}G_b + J_b \tag{2.9}$$

then, analogously with the continuous-time result, [7], one can argue that the spectrum of y is given by

$$\Phi(z) = W(z)W^*(z) = W_b(z)Q_bW_b^*(z) \tag{2.10}$$

Here, the superscript asterisk denotes transposition and replacement of z by z^{-1}. In matrix fraction terms, suppose that

$$W(z) = A(z)B^{-1}(z) \tag{2.11}$$

Then one can find a $B_b(z)$ such that

$$W_b(z) = A(z)B_b^{-1}(z) \tag{2.12}$$

Of course, trivial variations apply if (2.7) and (2.8) replace (2.4) and (2.5).

Above, we assumed that $|\lambda_i(F)| < 1$ which ensured that y is stationary. If however, we simply assume that no two eigenvalues of F are reciprocal, (2.2) is guaranteed to have a solution, one can define quantities as in (2.4) or as in (2.7), and still conclude (2.10). One no longer has the stochastic interpretation however.

3. APPLICATION OF FORWARD/REVERSE MODELS TO AR PROCESSES

Assume we are given a forward AR model

$$y_t + \sum_{i=1}^{N} A_i y_{t-i} = u_t \tag{3.1}$$

with $E(u_t) = 0, [E\, u_t u_s'] = I\delta_{t,s}$*. For the moment, we assume y_t is stationary. We shall describe how the associated backward AR model can be obtained using the ideas of the previous section. We can provide a state-variable model for y_t of the form (2.1) via

$$F = \begin{bmatrix} 0 & I & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & I \\ -A_N & -A_{N-1} & \dots & -A_1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ I \end{bmatrix} \tag{3.2}$$

$$H = -[A_N\ A_{N-1}\ \dots\ A_1], \quad J = I$$

The state vector x_t is related to y_t by

$$x_t = [y_{t-N}' \dots y_{t-1}']' \tag{3.3}$$

Notice that [F,G] forms a controllable pair. Because of (3.3) it is readily seen that $\Pi = E[x_t x_t']$ defined via (2.2) is Toeplitz.

Now suppose that a reverse time model is constructed, as per (2.4), supposing temporarily that F is nonsingular. Then

$$F^{-1} = \begin{bmatrix} -A_N^{-1}A_{N-1} & -A_N^{-1}A_{N-2} & \dots & -A_N^{-1}A_1 & -A_N^{-1} \\ I & 0 & & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \dots & I & 0 \end{bmatrix}$$

which ensures that for some B_i

$$F_b = F^{-1}(I - GG'\ ^{-1}) = \begin{bmatrix} -B_1 & -B_2 & \dots & -B_{N-1} & -B_N \\ I & 0 & \dots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \dots & I & 0 \end{bmatrix} \tag{3.5}$$

* Again, the analysis would be virtually the same if $E[u_t u_s'] = Q_f\delta_{t,s}$.

In fact,

$$B_i = A_N^{-1}(A_{N-i} - P_{1,N-i+1}) \;,\; i = 1, \dots N-1$$

$$B_N = A_N^{-1}(I - P_{11}) \qquad (3.6)$$

where P_{ij} is the block i-j entry of Π^{-1}. We also obtain using (2.4)

$$H_b = [0 \;\dots\; 0 \; I] \;,\; J_b = 0 \;,\; G_b = [-A_N^{-1} \; 0 \;\dots\; 0]' \qquad (3.7)$$

The backward model has the same state-vector as the forward model.

Hence

$$\sum_{i=0}^{N} B_{N-i}y_{t-i} = -A_N^{-1}u_t^b \qquad (B_o = I) \qquad (3.8a)$$

In case F is singular, a perturbation argument shows that F_b which is given by $\Pi F'\Pi^{-1}$ even if F is singular, still has the block companion form of (3.5). Also, when F is nonsingular,

$$\begin{bmatrix} G_b \\ J_b \end{bmatrix} (I - G'\Pi^{-1}G)\; [G_b' \;\; J_b'] = \left[\begin{array}{cccc|c} A_N^{-1}(I - G'\Pi^{-1}G)(A_N^{-1})' & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 0 \end{array}\right]$$

So the earlier agrument yields

$$\begin{bmatrix} \bar{G}_b \\ \bar{J}_b \end{bmatrix} [\bar{G}_b' \;\; \bar{J}_b'] = \bar{Q}_b \oplus 0$$

allowing us to take

$$\bar{J}_b = 0 \qquad \bar{G}_b = [\bar{Q}_b^{\frac{1}{2}} \;\; 0 \;\dots\; 0]$$

So once again, we have a backward AR model:

$$\sum_{i+0}^{N} B_{N-i}y_{t-i} = \bar{Q}_b^{\frac{1}{2}}\,\bar{u}_t^b \qquad (B_0 = I) \qquad (3.8b)$$

Equations (3.1) and (3.8) exhibit the related forward and backward polynomials

$$A_N(z) = I + A_1 z^{-1} + \dots + A_N z^{-N} \qquad (3.9a)$$

$$B_N(z) = B_N + B_{N-1}z^{-1} + \dots + Iz^{-N} \qquad (3.9b)$$

REMARKS

1. Above, we have given a procedure for computing $B_N(z)$ from $A_N(z)$. One proceeds via $A_N(z) \to \Pi \to \Pi^{-1} \to B_N(z)$. The step $A_N(z) \to \Pi$ is based on solving a Lyapunov equation. For the development of fast procedures, see [4]. If $A_N(z)$ is known to be stable, the Lyapunov equation solution can be obtained via a doubling algorithm [12 see p67]. The step $\Pi \to \Pi^{-1}$ could in principle use a specialized block Toeplitz algorithm; should it use the specialized LWR type algorithm, forward and backward polynomials of increasing orders will be found. However, should other doubling-type algorithms eg [13-15] be used, this will not necessarily be the case. Finally (3.6) yields $B_N(z)$.

2. The procedure will work even if det $A_N(z)$ does not have all its zeros in $|z| < 1$ - so long as no two zeros of det $A_N(z)$ have a product of 1. The matrix $Q_b^{-\frac{1}{2}}$ may no longer be real - but this is irrelevant unless a stochastic process interpretation is desired.

3. From (2.10), we obtain the standard relation see e.g. [16] between the forward and backward polynomials. Evidently, (3.1) and (3.8) yield

$$W(z)=(I+A_1z^{-1}+\dots+A_Nz^{-N})^{-1},\; W_b(z)=(B_N+B_{N-1}z^{-1}+\dots Iz^{-N})^{-1}$$

and so with

$$A_N(z)=I+A_1z^{-1}+\dots+A_Nz^{-N} \qquad B_N(z)=B_N+B_{N-1}z^{-1}+\dots+Iz^{-N}$$

we have

$$A_N{}^*A_N = B_N{}^*\bar{Q}_b^{-1}B_N \qquad (3.10)$$

Notice also that after a simple calculation

$$\det[A_N(z)] = \det z^{-N}B_N(z^{-1}) = \det z^{-N}B_N{}^*(z) \qquad (3.11)$$

4. COMPUTATION OF BACKWARD POLYNOMIAL WITHOUT INTERMEDIATE COVARIANCE CALCULATION

In this section, we shall describe a discrete-time version of a continuous-time procedure essentially described in [10]. The task is as follows. Given a square polynomial in z^{-1}

$$A_N(z) = I + A_1z^{-1} + \dots + A_Nz^{-N} \qquad (4.1)$$

with $|A_N(z)|$ not zero for z_i, z_j such that $z_i\, z_j = 1$, find a polynomial

$$B_N(z) = B_N + B_{N-1}z^{-1} + \dots + B_1z^{-(N-1)} + Iz^{-N} \qquad (4.2)$$

such that*

$$A_N^*\, A_N = B_N^*\, \bar{Q}_b^{-1}\, B_N \qquad (4.3)$$

for some $\bar{Q}_b = \bar{Q}_b'$, and

$$\det A_N(z) = \det z^{-N}\, B_N(z^{-1}) \qquad (4.4)$$

In an extended version of the paper it is proved that there exists a unique C(z), polynomial in z^{-1}, such that

$$A_N^{-1}(z)\,[A_N^*(z)]^{-1} = A_N{}^{-1}(z)\; C(z) + C^*(z)\,[A_N^*(z)]^{-1} \qquad (4.5)$$

$$C(1) = \tfrac{1}{2}[A_N'(1)]^{-1} \qquad (4.6)$$

and $A_N{}^{-1}(z)\; C(z)$ is proper, i.e. is finite when $z^{-1}=0$.

---

* Trivial modification of the succeeding material will allow replacement of the left side of (4.3) by $A_N^*\, Q_f^{-1}A_N$ for some positive definite symmetric Q_f.

Now let

$$A_N^{-1}C = DE^{-1} \qquad (4.7)$$

where D,E are right coprime [17], and polynomial in z^{-1}. Then

$$E^* A_N^{-1} A_N^{-*} E = E^*D + D^*E \qquad (4.8)$$

and, (see the right side), is polynomial in z and z^{-1}. Further, $|E|$ is a divisor of $|A_N|$ by (4.7), the quotient being polynomial in z^{-1}; similarly, $|A_N^*|$, the quotient being polynomial in z; this means that the left side of (4.8) has a determinant of the form 1/(polynomial in z and z^{-1}). Hence the determinant must be constant and det E is a constant multiple of det A_N. The left side of (4.8) is also para-hermitian and positive definite almost everywhere. Consequently, by elementary transformations (here the constancy of the determinant is critical), see [18], we can find a V, polynomial in z^{-1}, such that

$$E^*A_N^{-1}(A_N^{-*})E = V^*V \qquad (4.9)$$

Further V is unimodular, i.e. V^{-1} is polynomial in z^{-1}. Then

$$A_N^*A_N = (EV^{-1})\,(EV^{-1})^*$$

Define $N = EV^{-1}|_{z^{-1}=0}$ and $\bar{Q}_b = NN'$. Because det $A_N(z)$ is a multiple of det E(z) and is nonzero at $z^{-1} = 0$, and because V is unimodular N is nonsingular. Hence with $F(z) = EV^{-1}N^{-1}$

$$A_N^* A_N = F\,\bar{Q}_b F^* \qquad (4.10)$$

and det A_N = det F (That det A_N is a multiple of det F is clear; that the multiple is 1 follows by considering $z^{-1} = 0$). Now let $F(z) = I + F_1z^{-1} + \dots + F_mz^{-m}$ By considering the terms in z^N and higher degree on each side of (4.10), we see that m = N. It is then straightforward to verify that

$$B_N(z) = z^{-N} F^*(z) \qquad (4.11)$$

satisfies (4.2) through (4.4).

Remark. The calculation of C(z) in (4.5) allows another approach to covariance calculation. For if $R_k = E[y_t y'_{t-k}]$, then

$$A_N^{-1}(z)\; C(z) = \sum_{k\geq 0} z^{-k}R_k \qquad (4.12)$$

whence we can easily recursively obtain the R_i if $C(z) = \sum_{i=0}^{m} C_i z^{-i}$ by

$$\begin{aligned} C_o &= R_o \\ C_1 &= A_1R_0+R_1 \\ C_2 &= A_2R_0+A_1R_1+R_2 \\ &\vdots \end{aligned} \qquad (4.13)$$

5. COMPUTATION OF REFLECTION COEFFICIENTS AND COVARIANCE FROM FORWARD AND BACKWARD POLYNOMIALS

It is relatively straightforward to obtain the reflection coefficients knowing the forward and backward polynomials $A_N(z)$ and $B_N(z)$. From the standard LWR equations [16]

$$A_n(z) = z^{-1}A_{n-1}(z) - K_n^{\alpha}B_{n-1}(z). \qquad (5.1)$$

$$B_n(z) = -z^{-1}K_n^{\beta}A_{n-1}(z) + B_{n-1}(z). \qquad (5.2)$$

(where K_n^α and K_n^β are the forward and backward reflection coefficients) we have

$$K_n^\alpha = -\text{coefficient of } z^{-n} \text{ in } A_n(z) \qquad (5.3a)$$

$$K_n^\beta = -B_n(z^{-1})|_{z^{-1}=0} \qquad (5.3b)$$

and it is trivial to verify from (5.1) and (5.2) that

$$A_{n-1}(z) = z[1 - K_n^\alpha K_n^\beta]^{-1}[A_n(z) + K_n^\alpha B_n(z)] \qquad (5.4a)$$

$$B_{n-1}(z) = [I - K_n^\beta K_n^\alpha]^{-1}[K_n^\beta A_n(z) + B_n(z)] \qquad (5.4b)$$

so that (5.4) allows a recursion with decreasing n; (5.3) allows the reflection coefficients to be found. Of course, the inverses in (5.4) must exist; a sufficient condition is that det $A_n(z)$ be stable.*

Now consider the determination of the covariance. We have already indicated two procedures: (a) Find Π using the Lyapunov equation (2.2) (here, B_N is not used); (b) Find C(z) in (4.5) and use the recursion (4.13) (again, B_N is not used). We now indicate a third procedure. One finds Π^{-1} and from it Π, with the key point being that a simple formula exists for Π^{-1} in terms of $A_N(z)$ and $B_N(z)$.

The formula for Π^{-1} comes about in the following way. Define a block matrix R_{N+1} with L-j entry

$$R_{N+1}^{L,j} = E(y_{t+i-1}y'_{t+j-1}) \qquad (5.5)$$

(assuming the y_t process is stationary). Then it is easy to see using the forward recursion for y_t, (3.1), that

$$[A_N\; A_{N-1}\; \dots\; A_1\; I]R_{N+1} = [0\; 0\; \dots\; 0\; I] \qquad (5.6a)$$

$$R_{N+1}[I\; B_1'\; \dots\; B_{N-1}'\; B_N']' = [\bar{Q}_b 0\; \dots\; 0\; 0]' \qquad (5.6b)$$

Observe now that $\Pi = E[x_tx_t']$ is a submatrix of R_{N+1} obtained by deleting the last block row and column. Accordingly, a now standard formula due to Gohberg and Heinig [19-21] yields Π^{-1} in terms of $\bar{Q}_b$ and triangular block Toeplitz matrices formed using the A_i and B_i.

* In fact, what we are describing here is the matrix generalization of the Jury table test: det $A_N(z)$ is stable if and only if the singular values of the K_n^α and K_n^β lie in (-1,1).

Of course, once Π is known, the quantities $E[y_t y_t']$, ... $E[y_t y_{t-n+1}']$ are known, and the remaining covariance coefficient matrices can be found recursively using the Yule-Walker relations. It can also be noted that a formula is available for R_{N+1}^{-1} in terms of the A_i, B_i, see [19-21].

6. CONCLUSION

In this paper, we have described two procedures for passing from a forward polynomial to a backward polynomial associated with AR processes of stationary spectrum. The first procedure involves finding covariance data before finding the backward polynomial, the second does not. We have also described how this second procedure can be used to rapidly construct the covariance data.** When forward and backward polynomials are both known, and reflection coefficients and covariance data are not known, a simple recursion yields the reflection coefficients while the covariance data is accessible via several approaches.

As forshadowed in the introduction, we comment on the task of passing from a backward polynomial to a forward polynomial. This is virtually the same problem as we have solved. Using the relations described in Section 4, see (4.1) through (4.4), it is easy to see that $[A_N, B_N]$ is a forward-backward pair if and only if $[z^{-N} B_N(z^{-1}),\ z^{-N} A_N(z^{-1})]$ is a forward-backward pair. So given B_N, we can find A_N using the forward to backward algorithms of the paper, taking $z^{-N} B_N(z^{-1})$ as the starting point.*

Also, as commented earlier, the problem of finding the forward polynomial, the backward polynomial and the covariance matrices from the reflection coefficient matrices K_i is straightforward. We note that the forward and backward polynomials may be obtained from the reflection coefficient matrices by running the LWR algorithm backwards, [23]. Once the forward-backward pair is obtained, the covariance matrices may be obtained via any of the approaches described in section 4 or 5.

REFERENCES

[1] N. Levinson, "The Wiener RMS (Root-Mean Square) Error Criterion in Filter Design and Prediction", J. Math. and Phys. Vol 25, pp 261-278, 1947.

[2] P. Whittle, "On the Fitting of Multivariable Autoregressions and the Approximate Canonical Factorization of a Spectral Density Matrix", Biometrika, Vol 50, pp 129-134, 1963.

[3] R.A. Wiggins and E.A. Robinson, "Recursive Solution to the Multichannel Filtering Problem", J. Ceophys. Res. Vol 70, pp 1885-1891, 1965.

[4] M. Morf and B. Porat, "Efficient Solution of Lyapunov Equation for Matrix Autoregressive Model and its Applications to the Inverse Levinson Problem", Proc. 1981 Conf. on Decision and Control, San Diego, U.S.A., Dec. 1981, pp 1070-1074.

[5] A. Lindquist and G. Picci, "On the Stochastic Realization Problem" SIAM J. Control and Optimization, Vol 17, pp 361-389, May 1979.

* Strictly, we must be given the value of Q_b as well as B_N.

** The first procedure involves the solution of a Lyapunov equation and hence the methods proposed in [4] and [22] are applicable. The second procedure does not require the solution of a Lyapunov equation and hence it is different from methods proposed in [4] and [22]. In fact the second procedure gives a solution to the problem posed in the conclusions of [4] and [22].

[6] L. Ljung and T. Kailath, "Backwards Markovian Models for Second-Order Stochastic Processes", IEEE Trans. Information Theory, Vol IT-22, pp 488-491, July 1976.

[7] B.D.O. Anderson and T. Kailath, "Forwards, Backwards and Dynamically Reversible Markovian Models of Second-Order Processes", IEEE Trans. Circuits and Systems, Vol CAS-26, pp 956-965, Nov. 1979.

[8] G. Verghese and T. Kailath, "A Further Note on Backwards Markovian Models", IEEE Trans Information Theory, Vol IT-25, pp 121-124, Jan. 1979. Correction to the above paper in ibid, Vol IT-25, pp 501, July 1979.

[9] A.J. Bekhout, "Stability and Least Square Estimation", Automatica Vol 11, pp 637-638, 1975.

[10] B.D.O. Anderson and R.R. Bitmead, "Stability of Matrix Polynomials", Int. J. Control. Vol 26, pp 235-247, 1977.

[11] Y. Monden and S. Arimoto, "Generalized Rouché Theorem and its Application to Multivariate Autoregressions", IEEE Trans Acoustics, Speech and Signal Processing, Vol ASSP-28, No 6, pp 733-738, December 1980.

[12] B.D.O. Anderson and J.B. Moore, "Optimal Filtering" Prentice Hall, Inc; Englewood Cliff, N.J., 1979.

[13] F.G. Gustavson and D.Y.Y. Yun, "Fast Algorithm for rational Hermite Approximation and Solution of Toeplitz Systems", IEEE Trans, Circuits and Systems, Vol CAS-26, No 9, pp 750-755, September 1979.

[14] R.R. Bitmead and B.D.O. Anderson, "Asymptotically Fast Solution of Toeplitz and Related Systems of Linear Equations", Linear Algebra and its Applications, Vol 34, pp 103-116, 1980.

[15] M. Morf, "Doubling Algorithms for Toeplitz and Related Equations", Proc. IEEE Conf. on Acoustics, Speech and Signal Processing, Denver, Co, pp 954-959, 1980.

[16] M. Morf, A. Vieira, D.T.L. Lee and T. Kailath, "Recursive Multi-Channel Maximum Entropy Spectral Estimation", IEEE Trans. Geoscience Electronics, Vol GE-16, pp 85-94, April 1978.

[17] T. Kailath, Linear Systems Prentice Hall Inc., Englewood Cliffs, N.J. 1980.

[18] P.R. Motyka and J.A. Cadzow, "The Factorization of Discrete-Process Spectral Matrices, "IEEE Trans. Aut. Contrl., Vol AC-12, pp 698-706, 1967.

[19] I.C. Gohberg and G. Heinig, "Inversion of Finite Toeplitz Matrices with Entries Being Element from a Non-Commutative Algebra", Rev Roumaine Math. Pures Appl. XIX(5), pp 623-665, 1974.

[20] T. Kailath, A. Vieira and M. Morf, "Inverses of Toeplitz Operators, Innovations, and Orthogonal Polynomials", SIAM Review, Vol 20, pp 106-119, Jan. 1978.

[21] B. Friedlander, M. Morf, T. Kailath and L. Ljung, "New Inversion Formulas for Matrices Classified in Terms of their Distances from Toeplitz Matrices", Linear Algebra and Its Applications, Vol 27, pp 31-60, 1979.

[22] M.T. Hadidi and M. Morf, "Efficient Solution of the Inverse Levinson Problem in the Multichannel Case via the Lyapunov Equation", Proc. 1981 Conf. on Decision and Control, San Diego, U.S.A., pp 1075-1076, Dec. 1981.

[23] M. Morf, A. Vieira, T. Kailath, "Covariance Characterization by Partial Auto Correlation Matrices", Annals of Statistics, Vol 6, No 3, pp 643-648.