ON THE SHAPE OF H_2 PERFORMANCE CRITERIA [1]

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Abstract: An H_2 control performance criterion dependent on a controller parameter vector is analyzed from the point of view of evaluating the domain of attraction of its global minimum when a gradient-based algorithm is used in tuning the parameters. The objective of this analysis is twofold: (i) examine how some design parameters of the criterion can be used to enlarge the domain of attraction; (ii) examine how the minimization of a sequence of intermediate criteria can possibly lead to the global minimum of the original criterion without the danger of entrapment in a local minimum.

Keywords: Optimal control, control system synthesis, performance indices.

1. INTRODUCTION

Many control design methods are based on control performance criteria minimization. The associated optimization problems appear in model-based design as well as in data-based design methods. For model-based design, solutions may be achieved by means of tools such as Riccati equations and Linear Matrix Inequalities (LMI's). In control design methods based on input-output data, where no model is available and a fixed parametrized controller structure is used, iterative optimization of the parameter vector is required, and it has been found that a limiting factor in many cases is the convergence to local, nonglobal, minima of the performance criterion.

In this paper we analyze the properties of an H_2 parameter-dependent control performance criterion to determine under which conditions this criterion is free of local minima in the parameter space. These conditions involve the data set used for the data-based design, the initial condition for its iterative optimization, and the formulation of the performance criterion itself.

In contrast to earlier work, this paper focuses on the analysis of the cost function itself, rather than on the methods used to optimize it. We analyze whether or not the global minimum of the cost function has a large enough domain of attraction (when a gradient descent algorithm is used) and what can be done to increase that domain without compromising the final performance. We favor

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improving our understanding of the underlying problem rather than increasing the sophistication of the solution. In so doing, we also expect our results to be useful for model-based optimization. Our results provide guidelines to choose the performance criterion, design the experiments and initialize the optimization algorithm so that it will converge to the global minimum. To keep the analysis simple, we focus on a noise-free scalar system only; this situation allows one to transform almost any reasonable H_2 control performance criterion into one without local minima by just manipulating some simple design parameters. In the noisy case, though the analytical results to be presented here remain valid, the noise rejection part of the criterion cannot be directly manipulated and the influence of the other design parameters is therefore indirect and more complicated.

The paper is organized as follows. Notations and background are explained in Section 2. Section 3 contains our main results on the properties of the H_2 optimal control criterion. These properties are exploited in Section 4 to propose a scheme that converges to the global minimum of the criterion. Conclusions are drawn in Section 5.

2. PRELIMINARIES

Consider a linear time-invariant discrete-time single-input single-output process

$$y(t) = G(z)u(t), \tag{1}$$

where z is the forward time-shift operator, $G(z)$ is the process transfer function, assumed rational and proper, and $u(t)$ is the control input. This process is controlled by a linear time-invariant controller $C(z,\rho)$ which is assumed to have a parametric structure as specified below.

Assumption 1. Linear parametrization:

$$C(z,\rho) = \rho^T \bar{C}(z) \tag{2}$$

where $\rho \in \mathcal{D}_\rho \subseteq \Re^p$ and $\bar{C}(z)$ is a column vector of known rational functions. ■

It is further assumed that $C(z,\rho)G(z)$ has positive relative degree for all $\rho \in \mathcal{D}_\rho$. Some of the most common controller structures are linearly parametrized, PID with fixed derivative pole being the most popular. Indeed, a PID can be written as in (2) with $\rho^T = \begin{bmatrix} k_p & k_i & k_d \end{bmatrix}$ and

$$\bar{C}^T(z) = \begin{bmatrix} 1 & \frac{z}{z-1} & \frac{z-1}{z} \end{bmatrix}.$$

The control action $u(t)$ can be written as

$$u(t) = C(z,\rho)(r(t) - y(t)) \tag{3}$$

where $r(t)$ is the reference signal, which is assumed to be quasi-stationary, so that its spectrum $\Phi_r(e^{\jmath\omega})$ is well-defined (Ljung (1999)). The system (1)-(3) in closed loop becomes

$$\begin{aligned} y(t,\rho) &= T(z,\rho)r(t) \\ T(z,\rho) &= \frac{C(z,\rho)G(z)}{1+C(z,\rho)G(z)} = C(z,\rho)G(z)S(z,\rho) \\ S(z,\rho) &= \frac{1}{1+C(z,\rho)G(z)} \end{aligned}$$

where we have now made the dependence on the controller parameter ρ explicit in the output signal $y(t,\rho)$.

For later reference, we provide here some additional definitions. We say that the quasi-stationary signal $r(t)$ is persistently exciting of order k (PEk) if its spectrum $\Phi_r(e^{\jmath\omega})$ has at least k nonzero components. We say that a vector field $V(e^{\jmath\omega}) = [v_1(e^{\jmath\omega})\ v_2(e^{\jmath\omega})\ \ldots\ v_k(e^{\jmath\omega})\ ]^T$, where each $v_i(e^{\jmath\omega})$ is a function of the frequency variable ω, has full rank if the functions $v_i(e^{\jmath\omega})$ form a linearly independent (LI) set over the reals, that is, if $\nexists \eta \in \Re^k,\ \eta \neq \mathbf{0} : \eta^T V(e^{\jmath\omega}) \equiv 0$.

We search for the controller parameters that make the output of the system as close as possible to the desired one, that is, we solve an optimization problem $\min_\rho J(\rho)$ where $J(\rho)$ is a control performance criterion. This control design formulation is representative of several well-known control designs. It is closely related to LQR and LQG problems (Anderson and Moore (1971)) and has been extensively studied in the contexts of model reference adaptive control (MRAC) (Åström and Wittenmark (1995)) and generalized predictive control (GPC) (Bitmead et al. (1990)).

We deal primarily with design methods that, contrary to the tradition of GPC or LQR, search for the solution of this optimization problem *without knowledge of the process transfer function* $G(z)$. They do so by means of iterative gradient schemes where the estimation of the gradient is made directly from data: IFT (Hjalmarsson et al. (1998)) and FDT (Kammer et al. (2000)). These methods only guarantee convergence to a local minimum.

In adaptive control and data-based control design a model for the process is not known *a priori*, so neither is the cost function. Only local information about the cost function can be obtained from data collected on the system, so iterative gradient-based methods are used. The iteration is given by

$$\rho_{i+1} = \rho_i - \gamma_i \nabla J(\rho_i) \tag{4}$$

where $\nabla J(\rho) = \frac{\partial J(\rho)}{\partial \rho}$ and $\gamma_i > 0\,\forall i$. A set of initial conditions for which the algorithm converges to the global minimum of $J(\rho)$ is called a *domain of attraction* (DOA) of the algorithm. To define the DOA of the algorithm (4), use is made of the property in the following theorem.

Theorem 2.1. Consider a twice-differentiable function $J(\cdot): \Re^n \to \Re^+$. Assume that this function has an isolated global minimum ρ_* and define the set $\mathcal{B}_\alpha(\rho_*) = \{\rho : (\rho - \rho_*)^T(\rho - \rho_*) < \alpha\}$. If

$$(\rho - \rho_*)^T \nabla J(\rho) > 0 \ \forall \rho \in \mathcal{B}_\alpha(\rho_*), \rho \neq \rho_* \quad (5)$$

then there exists a sequence $\gamma_i,\ i = 1, \ldots, \infty$ such that $\mathcal{B}_\alpha(\rho_*)$ is a DOA of algorithm (4) for $J(\rho)$. ■

Convergence to the global minimum is determined by the nonexistence of local extrema expressed by a condition like (5). We note that condition (5) is quite similar to quasi-convexity of $J(\rho)$, which is a sufficient condition for (5) (Greenberg and Pierskalla (1971)).

3. PROPERTIES OF THE H_2 PERFORMANCE CRITERION

In formulating a performance criterion one tries to express some optimal trade-off between reference tracking and economy of control effort. The reference tracking objective is measured by the size of $T(z,\rho)r(t) - y_d(t)$, where $y_d(t)$ is a desired response, which is often specified as the output of a reference model: $y_d(t) = T_d(z)r(t)$. The control effort is typically measured by the size of the control signal $u(t)$. In an H_2 performance criterion the size of each signal is measured by an H_2 norm. For the control effort, this would thus yield $J_u(\rho) = E[(u(t))^2]$. For reasons that will become apparent later, we shall replace the control criterion by the more general form $J_u(\rho) = E[(u(t) - u_d(t))^2]$ where $u_d(t) = U_d(z)r(t)$ is the desired control effort. The transfer function $U_d(z)$ is to be specified by the designer, just like the reference model $T_d(z)$. We examine a sensible choice of U_d later. The H_2 criterion examined in this paper is then the following weighted sum:

$$J(\rho) = \lambda J_y(\rho) + (1 - \lambda)J_u(\rho). \quad (6)$$

for some scalar $\lambda \in [0, 1]$, and where

$$J_y(\rho) = E[(T(z,\rho)r(t) - y_d(t))^2] \quad (7)$$
$$J_u(\rho) = E[(u(t) - u_d(t))^2]. \quad (8)$$

3.1 Analysis of each term

Let Γ be the set of all control parameter values that render the closed-loop system BIBO-stable, that is, $\Gamma \triangleq \{\rho : T(z,\rho)$ is BIBO-stable$\}$. For $\rho \in \Gamma$ Parseval's theorem yields:

$$J_y(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\ (T(e^{\jmath\omega},\rho) - T_d(e^{\jmath\omega}))\ |^2\ \Phi_r(\omega)d\omega$$
$$J_u(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |C(e^{\jmath\omega},\rho)S(e^{\jmath\omega},\rho) - U_d(e^{\jmath\omega})|^2 \Phi_r(\omega)d\omega \quad (9)$$

3.1.1. Reference tracking: J_y. The ideal performance for J_y would be obtained if ρ was such that $T(z,\rho) = T_d(z)$, in which case $J_y(\rho)$ would vanish. Let us define the ideal controller $C_d(z)$, which would achieve exactly this performance:

$$C_d(z) = \frac{T_d(z)}{G(z)(1 - T_d(z))} \quad (10)$$

Inserting (10) into the criterion J_y, and dropping the dependence on ω for simplicity of notation, the reference tracking criterion can be written as

$$J_y(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\ S(\rho)S_d G\ |^2 |C_d - \bar{C}^T\rho|^2\ \Phi_r d\omega \quad (11)$$

where we have defined $S_d(z) = 1 - T_d(z)$. Note that, apart from the multiplicative term $|\ S(e^{\jmath\omega},\rho)\ |^2$ inside the integral, the cost function in (11) is quadratic in the decision variable ρ.

This similarity to a quadratic criterion is explored in (Campi et al. (2002)), where the approximation $S(z,\rho) = S_d(z)$ is made, thus obtaining a quadratic cost function. However, this approximation implies, in general, that a different minimum is attained. Here we do not make such approximation, but remark that when $S(z,\rho) \approx S_d(z)$ the cost is approximately quadratic in ρ. This causes the H_2 criterion to satisfy (5) in a set in which these functions are close enough to each other, as stated in the following Theorem.

Theorem 3.1. Let $\bar{C}(e^{\jmath\omega})$ be full-rank, $r(t)$ be PEp and assume that $\exists \rho_d \in \Gamma : C(z,\rho_d) = C_d(z)$. Let $\Upsilon \subseteq \Gamma$ be a connected set such that $\rho_d \in \Upsilon$ and, for all $\rho \in \Upsilon$:

$$-\pi/2 < \angle S(e^{\jmath\omega},\rho) - \angle S_d(e^{\jmath\omega}) < \pi/2\ \forall\omega \quad (12)$$

Then $J_y(\rho)$ has no local extrema in Υ. ■

For reasons of space, we only give a sketch of the proof, which will be necessary for further developments. The gradient of $J_y(\rho)$ can be put in the form

$$\nabla J_y(\rho) = M_y(\rho)(\rho - \rho_d)$$

where $M_y(\rho)$ is the following matrix:

$$M_y(\rho) = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \Phi_r\,|\ GS(\rho)\,|^2\ \Re\{S_d^* S(\rho)\bar{C}\bar{C}^*\}d\omega\right)$$

where $\Re\{\}$ denotes the real part and $V^*(e^{\jmath\omega}) \triangleq V^T(e^{-\jmath\omega})$. If the symmetric part of $M_y(\rho)$, say $M_y^s(\rho)$, is positive definite, then

$$\begin{aligned}\nabla J_y(\rho)^T(\rho - \rho_y) &= (\rho - \rho_d)^T M_y(\rho)(\rho - \rho_d) \\ &= (\rho - \rho_d)^T M_y^s(\rho)(\rho - \rho_d) > 0\end{aligned}$$

$M_y^s(\rho)$ is an integral of the Hermitian matrix $\bar{C}(z)\bar{C}^*(z)$ multiplied by scalar factors. So, $M_y^s(\rho)$ is positive definite if the elements of $\bar{C}(z)$ are linearly independent and if all scalar factors are strictly positive, which is guaranteed by the phase condition (12) and the persistence of excitation condition on $r(t)$.

3.1.2. Control effort: J_u The ideal controller for $J_u(\rho)$, for which this criterion would vanish, is defined as $C_u(z) = \frac{U_d(z)}{1-G(z)U_d(z)}$, with the corresponding sensitivity function $S_u(z) = \frac{1}{1+C_u(z)G(z)}$. Inserting this controller in the criterion yields the alternative expression:

$$J_u(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\, S(\rho)S_u\,|^2 |C_u - \bar{C}^T\rho|^2\ \Phi_r d\omega \quad (13)$$

This expression is entirely similar to the expression (11), and when $S(z,\rho) \approx S_u(z)$ the cost $J_u(\rho)$ is approximately quadratic in ρ. This yields the following result, similar to Theorem 3.1.

Theorem 3.2. Let $\bar{C}(e^{\jmath\omega})$ be full-rank and assume that $\exists\rho_u \in \Gamma : C(z,\rho_u) = C_u(z)$. Let $\Upsilon \subseteq \bar{\Gamma}$ be a connected set such that $\rho_u \in \Upsilon$ and, for all $\rho \in \Upsilon$:

$$-\pi/2 < \angle S(e^{\jmath\omega},\rho) - \angle S_u(e^{\jmath\omega}) < \pi/2\ \forall\omega \quad (14)$$

Then $J_u(\rho)$ has no local extrema in Υ. ■

We then have $\nabla J_u(\rho) = M_u(\rho)(\rho - \rho_u)$ with

$$M_u(\rho) = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \Phi_r\,|\,S(\rho)\,|^2\ \Re\{S_u^* S(\rho)\bar{C}\bar{C}^*\} d\omega\right)$$

In the formulation of a control performance criterion, the reference model T_d is a design variable whose choice is essentially determined by the required closed loop performance specifications. In our criterion (9) for $J_u(\rho)$, the transfer function U_d is also a design variable, whose choice is determined by the limitations imposed on the control energy. The following Lemma establishes an interesting connection between the choice of the reference model T_d and a possible choice for U_d.

Lemma 3.1. The controller $C_u(z)$ that realizes $U_d(z)$ is identical to the controller $C_d(z)$ that realizes $T_d(z)$ if the design transfer function $U_d(z)$ is chosen as $U_d(z) = \frac{T_d(z)}{G(z)}$.

Proof: The controller C_u is defined by $\frac{C_u}{1+C_uG} = U_d$. Therefore $\frac{C_uG}{1+C_uG} = U_dG = T_d$. ■

Observations:
(i) We should caution that this choice for U_d requires knowledge of the plant model $G(z)$. In data-based control design, G is either unknown or is only known approximately.
(ii) Even if the design choice $U_d = \frac{T_d}{G}$ is made, this does not necessarily mean that the two terms of the criterion, J_y and J_u, will be minimzed by the same controller parameter ρ because the weighting functions in (11) and (13) are different. However, the two optimal controller parameters will be identical if the ideal controller $C_d = C_u$ is in the controller set, i.e. $\exists\rho_* : C_d(z) = \rho_*^T\bar{C}(z)$.

3.2 Putting the pieces together

The gradient of the criterion (6) can be written as

$$\begin{aligned}\nabla J(\rho) &= \lambda M_y(\rho)(\rho - \rho_d) + (1-\lambda)M_u(\rho)(\rho - \rho_u) \\ &= M(\rho)(\rho - \rho_*) \qquad (15)\end{aligned}$$

where $M(\rho) \triangleq \lambda M_y(\rho) + (1-\lambda)M_u(\rho)$, and the global minimum ρ_* is a linear combination of the global minima of each cost component:

$$\rho_* = \begin{bmatrix}\Theta & I - \Theta\end{bmatrix}\begin{bmatrix}\rho_d \\ \rho_u\end{bmatrix} \quad (16)$$

where $\Theta \triangleq \lambda M^{-1}(\rho)M_y(\rho)$. These definitions make sense only if $M(\rho)$ is nonsingular. A sufficient condition for this is that both $M_y^s(\rho)$ and $M_u^s(\rho)$ are positive definite. Under these conditions the total cost $J(\rho)$ has the same properties as its components and we can state a result which is similar to the ones presented previously.

Theorem 3.3. Let $\bar{C}(e^{\jmath\omega})$ be full-rank and $r(t)$ be PEp. Let $\Upsilon \subseteq \Gamma$ be a connected set such that $\rho_y, \rho_u \in \Upsilon$ and that, for all $\rho \in \Upsilon$,

$$|\ \angle S(e^{\jmath\omega},\rho) - \angle S_d(e^{\jmath\omega})\ | < \frac{\pi}{2}$$
$$|\ \angle S(e^{\jmath\omega},\rho) - \angle S_u(e^{\jmath\omega})\ | < \frac{\pi}{2}$$

Assume further that $\rho_* \in \Upsilon$. Then the H_2 cost $J(\rho)$ in (6) has no local extrema in Υ. ■

4. COST FUNCTION SHAPING

The conditions under which the cost functions J_y and J_u are well behaved are given in terms of the problem data: process model $G(z)$, reference model $T_d(z)$, control signal model $U_d(z)$, reference signal $r(t)$, etc. Some of these data can be manipulated by the designer, and we are interested in manipulating them in order to force these cost functions to be well behaved in the largest possible set or at least in a set which contains the current controller; we call this "cost function shaping". The central idea of cost function shaping is to generate intermediate cost functions which are increasingly close to the desired one, such that for each cost function we obtain convergence to its global minimum and at the end of this process we have achieved the global minimum of the desired cost function. We now examine how each variable can be manipulated to achieve this objective.

4.1 Manipulation of the reference spectrum

We have seen that the SPR (strictly positive real) property of a particular transfer function is sufficient for uniqueness of extrema, hence convergence of gradient-based algorithms, within a given set. For each part of the total criterion, a different transfer function is concerned, as expressed in (12) and (14), unless the special choice of Lemma 3.1 can be made for U_d. The corresponding transfer functions, S_d and S_u, are only partially at the designer's disposal. However, the SPR condition is not a necessary condition; it can be circumvented by a proper manipulation of the reference $r(t)$. We first explore the properties of the sensitivity function that enter the SPR conditions.

Lemma 4.1. Consider the sensitivity function $S(z,\rho) = (1 + \rho^T \bar{C}(z)G(z))^{-1}$. For all $\rho_1, \rho_2 \in \Gamma$:

$$\angle S(1,\rho_1) = \angle S(1,\rho_2) \quad (17)$$

$$\angle S(-1,\rho_1) = \angle S(-1,\rho_2) \quad (18)$$

Proof:

$$\angle S(e^{\jmath\omega},\rho) = \sum_{i=1}^{n} \angle(e^{\jmath\omega} - \mathrm{b}_i) - \sum_{i=1}^{n} \angle(e^{\jmath\omega} - \mathrm{a}_i(\rho))$$

where b_i are the poles of the loop transfer function $\bar{C}(e^{\jmath\omega})G(e^{\jmath\omega})$ and $\mathrm{a}_i(\rho)$ are the closed-loop poles. For $\omega = 0$ we have

$$\angle S(1,\rho) = \sum_{i=1}^{n} \angle(1 - \mathrm{b}_i) - \sum_{i=1}^{n} \angle(1 - \mathrm{a}_i(\rho))$$

But $\sum_{i=1}^{n} \angle(1 - \mathrm{b}_i)$ does not depend on ρ and

$$\sum_{i=1}^{n} \angle(1 - \mathrm{a}_i(\rho)) = 0 \ \forall i, \rho \in \Gamma$$

because $\mathrm{a}_i(\rho)$ belong to the unit disc for all $\rho \in \Gamma$. The same argument is valid for $\omega = \pi$. ■

From the property above and the continuity of $S(e^{\jmath\omega},\rho)$ we can also conclude that the phase difference between two sensitivity functions is small for frequencies close to $\omega = 0$ and $\omega = \pi$.

Lemma 4.2. For all $\rho \in \Gamma$, $\exists \omega_l, \omega_h$ such that:

$$|\angle S(e^{\jmath\omega},\rho) - \angle S_d(e^{\jmath\omega})| < \frac{\pi}{2} \ \forall \omega \le \omega_l$$

$$|\angle S(e^{\jmath\omega},\rho) - \angle S_d(e^{\jmath\omega})| < \frac{\pi}{2} \ \forall \omega \ge \omega_h \quad ■$$

Thus, if only those frequencies for which the phase difference is small are weighted by the reference spectrum, the integral will still be strictly positive as desired, even if the phase difference exceeds $\pi/2$ in some frequency ranges. So, if the reference spectrum is concentrated at the borders of the frequency spectrum, then the minimum is unique in Γ, as formalized in the following theorem.

Theorem 4.1. Let $\bar{C}(e^{\jmath\omega})$ be full-rank and let $r(t)$ be PEp. Then there exist ω_l, ω_h, with $0 < \omega_l < \omega_h < \pi$, such that $\Phi_r = 0 \ \forall \omega \in (\omega_l, \omega_h)$ implies that ρ_d is the unique extremum of $J_y(\rho)$ in Γ. ■

4.2 Manipulation of T_d

We have seen that choosing an initial controller ρ_0 such that $\frac{S_d(z)}{S(z,\rho_0)}$ is SPR should be appropriate to obtain convergence to the global optimum. It is often the case that the controller to be tuned is in operation, that it is stabilizing the plant, but achieving poor performance. Since "one bird in the hand is worth two in the bush", it may not be a good idea to abruptly change the controller parameters to another value at which not even stability is totally assured. So, instead of bringing ρ_0 close to ρ_d, let us consider temporarily changing $T_d(z)$ so as to bring ρ_d close to ρ_0 in the initial stages of our controller tuning strategy.

Starting from an initial controller which delivers a given performance, say $T_0(z)$, which we consider poor, we choose a first *cautious* reference model $T_d^1(z)$, close to $T_0(z)$, aiming at a modest performance improvement. Once the global optimum of this new criterion (ρ_*^1) has been achieved, we can pick a second, more ambitious, reference model $T_d^2(z)$, and optimize it starting from ρ_*^1 as initial controller. This argument can be used successively, with several intermediate reference models, until the desired reference model $T_d(z)$ is achieved. This concept of cautious control is a familiar one in data-based control design (Kammer et al. (2000), Hjalmarsson et al. (1998)) and in iterative identification and control design (see e.g. Lee et al. (1993), Zang et al. (1995)).

4.3 Manipulation of U_d

From our analysis above, we see that relaxing the tracking part of the criterion, $J_y(\rho)$, by iterative manipulation of T_d has the property of enhancing the probability of converging to the global minimum of the criterion. The same can be done with the control criterion J_u, although it is not usual. Regarding the control effort objective, the traditional $u_d(t) \equiv 0$ requirement does not even make sense by itself, since the optimal solution for J_u then consists in turning off the controller. These considerations led us to the proposal of the new H_2 criterion (6), with $T_d(z)$ and $U_d(z)$ as designer choices.

4.4 Case study

Consider the problem of adjusting the gain ρ of a PI controller $C(z, \rho) = \rho \frac{z-0.3}{z-1}$. The process transfer function is $G(z) = \frac{1}{z-0.8}$ but this information is not available to the designer. This system must track a reference consisting of a square-wave with unitary amplitude and period $T = 30$ s. The desired tracking performance is specified by the reference model $T_d(z) = 2\frac{z-0.3}{z^2+0.2z+0.2}$. The H_2 cost $J_1(\rho) = 0.7E[(y(t) - y_d(t))^2] + 0.3E[u^2(t)]$ is plotted as the continuous line in Figure 1. It presents a local maximum around $\rho = 0.35$, so the DOA is limited to $\rho > 0.35$.

If we allow the criterion to search for the control action $U_d(z) = 0.5\frac{(z-0.3)(z-0.8)}{z^2-1.3z+0.65}$, instead of $U_d(z) = 0$, the cost becomes $J_2(\rho) = 0.7E[(y(t) - y_d(t))^2] + 0.3E[(u(t) - u_d(t))^2]$ given in the dashed line, which is quasi-convex. Notice that the global minima are very close for both cases, so the achieved performance in both cases is almost the same. However, should we try to minimize J_1 starting from a low gain controller we will probably get stuck at the local minimum at $\rho \approx 0.3$. On the other hand, if we insist that $J_1(\rho)$ is what we want to minimize, then we can just use the minimization of $J_2(\rho)$ as an intermediate task to this end.

Alternatively, it is possible to manipulate the spectrum of the reference so that it excites only those frequencies at which the phase difference (12) is small, that is at very low and/or very high frequencies. In the present case, where the reference is a sequence of steps, taking less data points to calculate the cost will make the reference appear as a higher frequency signal, since in practical computations, expectation is approximated by a sum:

$$J_1(\rho) \cong \hat{J}_1(\rho, N) = \frac{1}{N}\sum_{t=1}^{N}[0.7(y(t) - y_d(t))^2 + 0.3(u(t))^2]$$

If only a small number of data (N) is used, then only the transient part of the response is taken into account, so it is as if a high-frequency signal had been applied. The result of taking only 20 data points is shown as the line with circles in Figure 1. The approximate cost with 20 data points $\hat{J}_1(\rho, 20)$ is quasi-convex and presents the same minimum as the exact cost $J_1(\rho)$.

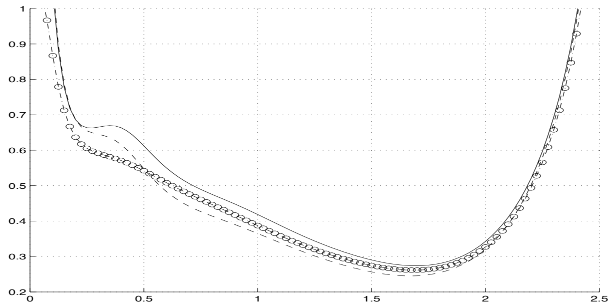

Fig. 1. $J_1(\rho)$ (full), $J_2(\rho)$ (dashed), and $\hat{J}_1(\rho, 20)$ (circles).

5. CONCLUSION

We have shown how the domain of attraction of the global minimum of a simple H_2 control performance criterion can be enlarged by either manipulating some design variables of the criterion, or by replacing the criterion by a succession of logically chosen intermediate criteria that have larger domains of attraction.

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