THE INVERSE PROBLEM OF OPTIMAL CONTROL

by

B. D. O. Anderson*

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*Stanford Electronics Laboratories
Stanford, California

CONTENTS

ABSTRACT

The relation is explored between linear feedback laws chosen to reduce the sensitivity of a linear time-invariant system to plant parameter variations, and linear feedback laws derived on an optimal control basis from quadratic loss functions. A general equivalence between the two types of design is established for multiple-input, multiple-output systems. The sensitivity improvement can be calculated for optimally design systems, and a quadratic loss function can be found for a system designed to reduce the sensitivity to plant parameter variations.

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I. INTRODUCTION

The paper considers a specific inverse problem of optimal control: Given a multivariable finite dimensional dynamical system with a linear feedback law, when is such a system optimal, and what is the corresponding loss function?

As pointed out in [1], which actually considers many of the questions we raise here, a sufficient condition for obtaining a linear feedback law is that the loss function be a quadratic function of the states and controls. If a stochastic environment is assumed, [2] demonstrates that quadratic loss functions are necessary to produce linear feedback laws. Consequently, in this paper we take a more restricted viewpoint than the title of the paper might imply, to ask when is a linear feedback system optimal, and what is the corresponding quadratic loss function.

The single variable case has been considered in [1], and our conclusions are surprisingly similar to those of this reference. We establish that a necessary and sufficient condition for optimality is, roughly, that feedback should render the system less sensitive to plant parameter variations. Necessity has actually been established in an earlier paper [3], for the more general case of a multivariable time-varying system. The sufficiency is considerably more difficult to prove, and a proof for time-varying systems does not seem to be in sight. It would appear however that, from the practical point of view, necessity is more useful.

The paper draws heavily on modern control theory to establish the results, particularly material which may be found in [4] and [5]. In contrast however to [1], we do not require the use of certain canonical forms for completely controllable systems. Two other features of the paper are its reliance upon spectral factorization results of Youla [6] and a system theory characterization of positive real matrices [7].

The plan of the paper is as follows. Section 2 is concerned with the formulation of plant equations, and the presentation in review fashion of background control theory material. We also mention here the mathematical characterization of a certain physical property of a system, namely that its sensitivity to plant parameter variation should be reduced by the application of feedback. Section 3 serves as a review of the optimal control problem and contains some elementary extensions of results in [1]. The main material of the paper is to be found in Section 4. Principal results are Theorem 5, showing that optimality implies improvement in respect of sensitivity, and Theorem 8, showing that improvement in respect of sensitivity to plant parameter variations implies optimality. This section also explains how the loss function corresponding to an optimal control law can be computed.

II. PLANT DESCRIPTION AND PRELIMINARIES

We consider a linear, time-invariant, finite-dimensional plant, that is, a plant which may be described by the following state equations:

$$\dot{x} = Fx + Gu \tag{1a}$$

$$y = H'x \tag{1b}$$

Here x is an n-vector, the state

u is a p-vector, the input

y is an m-vector, the output

F, G, H are constant matrices, of appropriate dimension

and the superscript prime denotes matrix transposition

A diagrammatic representation is shown in Fig. 1. The transfer function matrix $W(s)$ relating the Laplace transform of u, $U(s)$, to the Laplace transform of y, $Y(s)$, through

$$Y(s) = W(s)\, U(s) \tag{2a}$$

is given by

$$W(s) = H'(sI - F)^{-1}G \tag{2b}$$

Any matrix triple $\{F,G,H\}$ satisfying (2b) is termed a realization of $W(s)$. If F has the least possible dimension, $\{F,G,H\}$ is termed a minimal realization.

The introduction of state variable feedback means that we force u to be composed of linear combinations of the state variables, that is, in the absence of external input,

$$u = -K'x \tag{3}$$

where K is a constant $n \times p$ matrix. This is illustrated in Fig. 2.

There exist transfer function matrices relating the vector variables at point 1 in the figure to those at point 3 before and after the "closing of the loop", that is, before and after the variable at 3 is connected to the summing point.

It is not hard to establish that these matrices are

$$T_o(s) = K'\Phi(s)G \qquad \text{(open loop)} \tag{4}$$

$$T_c(s) = K'\Phi_K(s)G \qquad \text{(closed loop)} \tag{5}$$

where

$$\Phi(s) = (sI-F)^{-1} \tag{6}$$

and

$$\Phi_K(s) = (sI-F+GK')^{-1} \tag{7}$$

Observe that the transfer function matrix in (3), $T_o(s)$, is also the transfer function matrix relating the variable at point 2 to that at point 3 after the loop is closed.

The closed loop and open loop functions are related through the <u>return difference matrix</u>, a generalization of the return difference for single variable systems. The return difference matrix is defined as

$$T(s) = I + K'\Phi(s)G \tag{8}$$

Lemma 1. The closed loop responce $T_c(s)$, open-loop response $T_o(s)$, and return difference matrix $T(s)$ are related by

$$T_o(s) = T_c(s)T(s) = T(s)T_c(s) \tag{9}$$

Proof.

$$\begin{aligned} T_o(s) &= K'(sI-F)^{-1}G \\ &= K'(sI-F+GK')^{-1}(sI-F+GK')(sI-F)^{-1}G \\ &= K'(sI-F+GK')^{-1}[I+GK'(sI-F)^{-1}]G \\ &= K'(sI-F+GK')^{-1}G[I+K'(sI-F)^{-1}G] \\ &= T_c(s)T(s) \qquad (9a) \end{aligned}$$

Similarly, it can be shown that

$$T_o(s) = T(s)T_c(s) \tag{9b}$$

The behavior of the return difference matrix yields information about the sensitivity of the system to plant parameter variations. It is possible to compare the closed-loop plant with an open-loop plant of the same transfer function matrix, the transfer function matrix being achieved by using a controller in cascade with the main plant (1). The comparison is summed up by

Lemma 2. The sensitivity of the closed-loop plant with respect to plant parameter variations will be less than that of the equivalent open-loop plant if

$$T'(-j\omega)T(j\omega)-I > 0 \tag{10}$$

on the $j\omega$ axis (the notation "> 0" is shorthand for "is positive definite").

Proof.

See [8] or [9]. These references also contain a full discussion of the significance of (10).

Equation (10) is a generalization of a well-known result for the single input case:

$$|T(j\omega)|^2 > 1 \tag{11}$$

where T is here a function rather than a matrix. Further, the stronger the inequality in (11) is, the less sensitive will be the system to plant parameter variations; so also for the inequality in (10).

We shall have occasion in the sequel to use the concepts of complete controllability and complete observability [4]. We term the matrix pair $[F,G]$ completely controllable if (with n the dimension of F)

$$\text{rank } [G, FG, F^2G, \ldots, F^{n-1}G] = n \tag{12}$$

The pair $[F,H']$ is completely observable if the pair $[F',H]$ is completely controllable, i.e.,

$$\text{rank } [H, F'H, (F')^2H, \ldots, (F')^{n-1}H] = n \tag{13}$$

The physical meaning of these concepts has been discussed now in many places in the literature; see [4] for a good introduction. Lemma 3 meanwhile provides a connection between these concepts and that of a minimal realization:

Lemma 3. Let $\{F,G,H\}$ be a triple and $W(s)$ a matrix of transfer functions satisfying (2b), i.e.,

$$W(s) = H'(sI-F)^{-1}G \tag{2b}$$

Then

(a) $\{F,G,H\}$ is minimal if and only if $\{F,G\}$ is completely controllable and $[F,H']$ is completely observable.

(b) If $\{F,G,H\}$ is minimal, all other minimal realizations are given by $\{T^{-1}FT, T^{-1}G, T'H\}$ for some nonsingular T.

(c) If $\{F,G,H\}$ is a nonminimal realization while $[F,G]$ is completely controllable, there exists a nonsingular T such that

$$T^{-1}FT = \begin{bmatrix} F_1 & 0 \\ F_{12} & F_{22} \end{bmatrix} \tag{14a}$$

$$T^{-1}G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \tag{14b}$$

$$(T)'H = \begin{bmatrix} H_1 \\ 0 \end{bmatrix} \tag{14c}$$

with $\{F_1, G_1, H_1\}$ a minimal realization.

Proof.

See [4]. Note that (c) is a particular case of a more general result discussed in [4]. The form in which it is stated here will be useful for the sequel, however.

To conclude this section we discuss briefly the concept of the <u>degree of a matrix of rational transfer functions</u>. If $W(s)$ is such a matrix, with $W(\infty) = 0$, then the degree is the same thing as the dimension of F in a minimal realization of $W(s)$.

Alternatively, let W have a pole at s_o; we define $\delta[W;s_o]$ as the maximal order that s_o has as a pole of any minor of W, see [5], [10]. Then the degree of W, written $\delta[W]$ is given by

$$\delta[W] = \sum \delta[W;s_o] \tag{15}$$

the summation being over all poles of W, [5], [10].

A related concept is that of the <u>Smith-McMillan Canonical form</u> of W, [5], [10]. This is a decomposition of $W(s)$ as

$$W(s) = A_1(s) \text{ diag} \left[\frac{\epsilon_1}{\psi_1}, \frac{\epsilon_2}{\psi_2}, \ldots, \frac{\epsilon_p}{\psi_p} \right] A_2(s) \tag{16}$$

where, assuming $W(s)$ is square for simplicity, $A_1(s)$ and $A_2(s)$ are polynomial matrices with constant determinant, ϵ_i and ψ_i $(i=1,\ldots,p)$ are polynomials such that for each i (i) ϵ_i divides ϵ_{i+1} (ii) ψ_{i+1} divides ψ_i, (iii) ϵ_i and ψ_i are prime, (iv) ϵ_i and ψ_i have leading coefficient unity, (v) ϵ_i and ψ_i are canonic.

Some properties of the degree concept and the Smith-McMillan canonical form are collected in the next lemma.

<u>Lemma 4</u>. Let $W(s)$ be a matrix of rational transfer functions, for simplicity assumed to be square, and to have $W(\infty)$ finite. Let the Smith-McMillan canonical form of $W(s)$ be as in (16)

(a) If K is a constant matrix, $\delta[W + K;s_o] = \delta[W;s_o]$ for all poles s_o of W.

(b) If $\{F,G,H\}$ is a minimal realization of $W(s) - W(\infty)$, $\det(sI-F) = \psi_1\psi_2 \ldots \psi_p$ to within a constant multiplier.

(c) If s_o is a zero of ψ_i of order ν_i, then $\delta[W;s_o] = \sum_i \nu_i$.
(d) Let $W_1(s)$, $W_2(s)$ be two matrices with the same properties as $W(s)$. Let s_o be a pole of W_2, but not of W_1. Suppose $\det W_1(s_o)$ is nonzero. Then $\delta[W_1W_2;s_o] = \delta[W_2;s_o]$.

Proof.

Parts (a), (b), (c) are well known and can be found in for example [10]. To prove part (d), we note from [10, Theorem 5.26] that the ν_i [see part (c)] for W_2 are identical to the ν_i for W_1W_2. The result follows then from part (c).

III. THE OPTIMIZATION PROBLEM

This section serves as a review of the optimization problem, and states some theorems which are completely straightforward generalizations of those in [1].

The optimization problem may be simply stated as: find u so that the performance index

$$V(x_o,t_o;u) = \int_{t_o}^{\infty} (x'Qx + 2x'Su + u'Ru)dt \tag{15}$$

is minimized (here x_o is the state of the system at time t_o; Q, S and R are constant matrices of appropriate dimension). For the problem to have a well-defined solution, a number of conditions must be imposed; these are on Q, S and R, as well as on the plant. Before stating them, however, we shall indicate a problem simplification allowing replacement of (15) by (21).

One requirement generally imposed is that the integrand in (15) be strictly concave in u [11]. This requires R to be positive definite, implying the existence of a unique positive definite square root $R^{1/2}$. Then we can set

$$\hat{u} = R^{1/2}u \tag{16a}$$

$$\hat{G} = GR^{-1/2} \tag{16b}$$

and observe that from the external point of view the system described by F, $\hat{G}$, H with input $\hat{u}$ is the same as that described by F, G, H with input u. With this replacement (15) becomes (dropping the hat notation)

$$V(x_o,t_o;u) = \int_{t_o}^{\infty} (x'Qx + 2x'Su + u'u)dt$$

$$= \int_{t_o}^{\infty} [x'(Q-SS')x + (u'+x'S)(u+S'x)]dt \tag{17}$$

We now make a second change in the control variable, taking

$$\hat{u} = u + S'x \tag{18a}$$

which requires

$$\hat{F} = F-GS' \tag{18b}$$

and a feedback law

$$\hat{K} = K-S \tag{18c}$$

where we are permitting in (18c) the control u to be derived via possibly both state variable feedback and external control. Equations (18) guarantee that the system described by $\hat{F},\hat{G},\hat{H},\hat{K}$ performs in the same way as that described by F,G,H,K from the external point of view.

Dropping the hat notation again, we observe that the minimization is now of

$$V(x_o,t_o;u) = \int_{t_o}^{\infty} [x'(Q-SS')x + u'u]dt \tag{19}$$

It is known that for a minimum to exist, we require $Q-SS'$ to be non-negative definite [11]. Choosing L such that

$$Q-SS' = LL' \tag{20}$$

we observe finally that we must minimize

$$V(x_o, t_o; u) = \int_{t_o}^{\infty} (x'LL'x + u'u)dt \tag{21}$$

The problem may now be solved as follows, see [1,11]. Let $P(t)$ be the unique symmetric solution of

$$-\dot{P} = PF + F'P - PGG'P + LL' \tag{22a}$$

with

$$P(t_1) = 0 \tag{22b}$$

If the plant, or more accurately the matrix pair $[F,G]$, is completely controllable, and we assume this to be the case, there exists P_∞, a solution to (22a) defined by

$$\lim_{t_1 \to \infty} P(t_1) = 0 \tag{22c}$$

Call this matrix simple P. Define a control law K by

$$K = PG \tag{23}$$

The following theorems follow [1] directly:

Theorem 1. Consider the problem of minimizing the performance index (21) for the plant (1), assumed completely controllable. This minimization is achieved by choosing $u = -K'x$, K being given by (23). The matrix P in (23) is derived from the differential equation and limiting process (22).

We call the control law K stable if the eigenvalues of $F-GK'$ have negative real parts. Equivalently, by (4) and (6), the poles of the closed loop transfer function $T_c(s)$ are in the strict left half plane.

Then [1]

Theorem 2. Under the conditions of Theorem 1, if also $[F,L']$ is completely observable, then P is positive definite and K is a stable law.

It is also true that, irrespective of the complete observability or otherwise of $[F,L']$:

Theorem 3. If F has negative real part eigenvalues, the control law K is stable.

That optimal feedback can automatically guarantee stability, even if the open loop plant is unstable, is an important feature brought out in Theorem 2.

The final theorem of this section, also following [1], is

Theorem 4. Suppose the plant (1) is completely controllable and $[F, L']$ is completely observable. Let K be a fixed control law. Then K is both stable and optimal if and only if there exists a positive definite symmetric P such that

$$PG = K \tag{24}$$

$$-P(F-GK') - (F'-KG')P = LL' + KK' \tag{25}$$

IV. RELATION BETWEEN SENSITIVITY IMPROVEMENT AND OPTIMALITY

The principal results of the paper are to be found in this section. Theorem 5 was originally established in [3] for the case of linear multivariable time-varying control systems. Since time-varying systems include as a special case time-invariant systems, the theorem as now proved is not new. However, [3] gave merely a time-domain description, whereas here we give a frequency-domain description. The argument is similar to that of Kalman [1] for the single variable case.

Theorem 5. Let K be an optimal control law for the completely controllable plant (1) with the performance index(21). Suppose also that $[F, L']$ is completely observable. Then

$$[I+G'\Phi'(-s)K][I+K'\Phi'(s)G] = I + [G'\Phi'(-s)L][L'\Phi(s)G] \tag{26}$$

Proof.

Theorem 4 applies, and by direct calculation using (24),

$$[I+G'\Phi'(-s)K][I+K'\Phi(s)G]$$

$$= I + G'\Phi'(-s)\,[PGG'P+(-sI-F')P+P(sI-F)]\Phi(s)G$$

$$= I + G'\Phi'(-s)[-P(F-GK')-(F'-KG')P-KK']\Phi(s)G$$

again by (24),

$$= I + [G'\Phi'(-s)L][L'\Phi(s)G] \qquad \text{by (25).}$$

Q.E.D.

Since $I + K'\Phi(s)G$ is the return difference matrix (18), we see that

$$T'(j\omega)T(j\omega) - I = [G'\Phi'(-j\omega)L][L'\Phi(j\omega)G] \tag{27}$$
$$\geq 0$$

with equality holding only where $L'\Phi(j\omega)G$ is singular. This could indeed be the case; for example, the original plant might have two input lines coupling in identical fashion, thus implying that G has two equal columns. Accordingly, it is not possible to use Lemma 2 to deduce that improvement in sensitivity to plant parameter variation will occur, but certainly it can be concluded that there is no degradation. Moreover, the conditions under which there is no improvement can be deduced in theory from (27).

The next result is by way of a partial converse to Theorem 5. In contrast to the proof of the result for the single variable case, [1], it is not necessary to appeal to a canonical form for completely controllable systems.

Theorem 6. Let K be a stable control law for the completely controllable plant (1). Suppose that $[F, L']$ is completely observable, and that

$$[I+G'\Phi'(-s)K][I+K'\Phi(s)G] = I + [G'\Phi'(-s)L][L'\Phi(s)G] \tag{28}$$

Then K is optimal for the performance index (21).

Proof.

The hypotheses of the theorem allows us by Theorem 1 to solve the optimal control law problem and deduce the existence of a feedback law K_* yielding the minimum performance index. Moreover, by Theorem 5, K_* must satisfy

$$[I+G'\Phi'(-s)K_*][I+K_*'\Phi(s)G] = I + [G'\Phi'(-s)L][L'\Phi(s)G] \tag{29}$$

From (28) and (29) it then follows that

$$[I+G'\Phi'(-s)K_*][I+K_*'\Phi(s)G] = [I+G'\Phi'(-s)K][I+K'\Phi(s)G] \tag{30}$$

We shall use this equation to conclude the equality of K and K_*. It is necessary to recall the stability of K (following by hypothesis) and of K_* (following by Theorem 2).

From (8a) we have

$$K'\Phi(s)G = K'(sI-F+GK')^{-1}G[I+K'(sI-F)^{-1}G]$$

from which

$$[I+K'(sI-F)^{-1}G]^{-1} = [I-K'(sI-F+GK')^{-1}G] \tag{31}$$

This allows us to write (30) as

$$[I-G'\Phi'_{K_*}(-s)K_*][I-K'_*\Phi_{K_*}(s)G]$$

$$= [I-G'\Phi'_K(-s)K][I-K'\Phi_K(s)G] \tag{32}$$

where the poles of $\Phi_K(s)$ and $\Phi_{K_*}(s)$ are all in the left half plane.

Now observe that if $A(s)$ is set equal to either side of (32), then $A'(-s) = A(s)$; a matrix with this property is termed parahermetian. Moreover, we note that $A(j\omega)$ is nonnegative definite for all ω, being of the form $B'(-j\omega)B(j\omega)$. These two properties (parahermetian and nonnegative definite for $s = j\omega$) allow application of the spectral factorization theory of Youla [6, Theorem 2]. Noting that (32) is of the form

$$A(s) = B_1'(-s)B_1(s) = B_2'(-s)B_2(s) \tag{33}$$

it is possible to conclude using Youla's result that there exist a rational paraunitary matrix $C(s)$ such that

$$B_1(s) = C(s)B_2(s) \tag{34}$$

A paraunitary matrix $C(s)$ is a matrix satisfying

$$C'(-s)C(s) = I \tag{35}$$

and since $C'(-j\omega) = C'^*(j\omega)$ it is evident from (35) that the poles of $C(s)$ cannot be on the $j\omega$-axis (which includes the point $\omega = \infty$). Applying (34) to the case in hand we conclude

$$[I-K_*'\Phi_{K_*}(s)\,G] = C(s)[I-K'\Phi_K(s)G] \tag{36a}$$

and

$$C'(-s)[I-K_*'\Phi_{K_*}(s)G] = [I-K'\Phi_K(s)G] \tag{36b}$$

From (36a) and the stability of K_*, it follows that $C(s)$ has no right half plane poles. From (36b) and the stability of K, it follows that $C(s)$ has no left half plane poles. Since $C(s)$, being paraunitary, also cannot have $j\omega$-axis poles or a pole at infinity, $C(s)$ must be constant and orthogonal. Letting $s = \infty$ in (36a) yields then $C = I$ and

$$I - K_*'\Phi_{K_*}(s)G = I - K'\Phi_K(s)G \tag{37}$$

Applying (31) then yields

$$K_*'\Phi(s)G = K'\Phi(s)G \tag{38a}$$

The completely controllability of $[F,G]$ then ensures that

$$K_* = K \tag{39}$$

This is because an equivalent statement to (38a) is

$$K_*'e^{Ft}G = K'e^{Ft}G \tag{38b}$$

and complete controllability ensures that the columns of $e^{Ft}G$ as t varies include all possible n-vectors.

Q.E.D.

The preceding theorem is merely a partial converse to Theorem 5, depending as it does on the knowledge of the relation (28). A stronger converse is provided by the following theorems 7 and 8, which in essence show that a <u>design where sensitivity to plant parameter variation is reduced is also a design which is optimal with regard to some performance index</u>. Before proving the theorems we need another lemma:

<u>Lemma 5</u>. Let $Z(s)$ be a square matrix of rational functions such that (i) $Z(\infty) = 0$, (ii) $Z(s)$ is analytic for $\text{Re}\ s \geq 0$ and (iii)

$$Z(j\omega) + Z'(-j\omega) > 0 \tag{39}$$

for all values of s on the imaginary axis. Let $\{A,B,C\}$ be a minimal realization for $Z(j\omega)$.

Then there exists a minimal $\{A,B,D\}$ such that

$$\text{(a)}\quad Z(s) + Z'(-s) = B'(-sI-A')^{-1}DD'(sI-A)^{-1}B \tag{40a}$$

(b) $D'(sI-A)^{-1}B$ is square and nonsingular in $\text{Re}\ s \geq 0$

<u>Proof</u>.

See [7, Theorem 1], where the above facts are established during the course of the proof of the theorem.

We remark that this theorem is a statement in state space terms of the spectral factorization theorem of Youla, [6]. If $Q(s)$ is the matrix, unique to within an orthogonal multiplier, such that

$$Z(s) + Z'(-s) = Q'(-s)Q(s) \tag{40b}$$

with Q and its inverse Q^{-1} analytic in $\text{Re}\ s \geq 0$, then $\{A,B,D\}$ is a minimal realization for Q.

We can now state

Theorem 7. Let K be a stable control law for the completely controllable plant described by Eq. (1a). Suppose $[F,K']$ is completely observable, and that

$$[I+G'\Phi'(-j\omega)K][I+K'\Phi(j\omega)G] - I > 0 \tag{41}$$

for all ω. Then there exists a matrix L such that $[F,L']$ is completely observable, and

$$[I+G'\Phi'(-s)K][I+K'\Phi(s)G] - I = [G'\Phi'(-s)L][L'\Phi(s)G] \tag{26}$$

Proof.

Using (31), we see that (41) is equivalent to

$$I - [I-G'\Phi_K'(-j\omega)K][I-K'\Phi_K(j\omega)G] > 0 \tag{42a}$$

or

$$G'\Phi_K'(-j\omega)K + K'\Phi_K(j\omega)G - G'\Phi_K'(-j\omega)KK'\Phi_K(j\omega)G > 0 \tag{42b}$$

Let P be a symmetric solution of the equation

$$PF + F'P = -KK' \tag{43}$$

At least one such solution exists [12, p. 225]. Then

$$-G'\Phi_K'(-s)KK'\Phi_K(s)G = G'(-sI-F')^{-1}[P(F-sI)+(F'+sI)P](sI-F)^{-1}G$$

$$= -G'(-sI-F')^{-1}PG - G'P(sI-F)^{-1}G$$

and so (42) becomes

$$G'\Phi_K'(-j\omega)(K-PG) + (K-PG)'\Phi_K(j\omega)G > 0 \tag{44}$$

To apply Lemma 5 we require $[F-GK',(K-PG)]$ to be completely observable; this may not be the case.* However, we can use Lemma 3 to write (44) as

$$G_1'\Phi_1'(-j\omega)H_1' + H_1'\Phi_1(j\omega)G_1 > 0 \tag{45}$$

where

$$T^{-1}(F-GK')T = \begin{bmatrix} F_1 & 0 \\ F_{12} & F_{22} \end{bmatrix} \tag{40a}$$

*Note that $[F,G]$ and thus $[F-GK',G]$ is completely controllable by hypothesis.

$$T^{-1}G = \begin{bmatrix} G_1 \\ \\ G_2 \end{bmatrix} \tag{46b}$$

$$T'(K-PG) = \begin{bmatrix} H_1 \\ \\ 0 \end{bmatrix} \tag{46c}$$

where $[F_1, H_1']$ is completely observable. Now we can use Lemma 5, noting that Φ_1 has left half plane poles since the same is true of Φ_K, K being a stable control law by assumption. Thus there exists L_1 (corre sponding to D of the lemma) such that

$$G_1\Phi_1'(-s)H_1' + H_1'\Phi_1(j\omega)G_1 = G_1'\Phi_1'(-s)L_1L_1'\Phi_1(s)G_1 \tag{47}$$

Now define L as $(T^{-1})'L_1$, and apply (46) backwards to get

$$G'\Phi_K'(-s)(K-PG) + (K-PG)'\Phi_K(s)G = G'\Phi_K'(-s)LL'\Phi_K(s)G \tag{48}$$

or equivalently, observing the equality of the left sides of (42) and (44):

$$I-[I-G'\Phi_K'(-s)K][I-K'\Phi_K(s)G] = G'\Phi_K'(-s)LL'\Phi_K(s)G \tag{49}$$

An important feature of (49) is that $L'\Phi_K(s)G$ is nonsingular throughout the half plane $\text{Re } s \geq 0$. This follows from (b) of Lemma 4.

Equation (49) can be transformed by multiplying on the right by $[I-K'\Phi_K(s)G]^{-1}$ and on the left by $[I-G'\Phi_K'(-s)G]^{-1}$; the result, using (31) again is

$$[I+G'\Phi'(-s)K][I+K'\Phi(s)G] - I = G'\Phi'(-s)LL'\Phi(s)G \tag{50}$$

with the relation

$$L'\Phi(s)G = L'\Phi_K(s)G[I-K'\Phi_K(s)G]^{-1} \tag{51}$$

being easy to establish.

To conclude the theorem we must now show that $[F,L']$ is completely observable. If this is not the case, then $\{F,G,L\}$ is not minimal; we can suppose then that $\{\overline{F},\overline{G},\overline{L}\}$ is *a minimal realization* for $L'\Phi(s)G$, i.e., we suppose

$$L'\Phi(s)G = \overline{L}'(sI-\overline{F})^{-1}\overline{G} \tag{52}$$

where $\overline{F}$ will have smaller dimension than F. Some of the zeros of $\det\,(sI-F)$ must then not be zeros of $\det\,(sI-\overline{F})$. We consider separately the case of zeros with $\mathrm{Re}\ s \geq 0$, and those with $\mathrm{Re}\ s < 0$, examining the former first.

Suppose $I + K'\Phi(s)G$ has the *Smith-McMillan canonical form*

$$I + K'\Phi(s)G = A_1(s)\ \mathrm{diag}\left[\frac{\epsilon_1}{\psi_1},\ \ldots,\ \frac{\epsilon_p}{\psi_p}\right]A_2(s) \tag{53}$$

with the similar form for $L'\Phi_K(s)G$ being

$$L'\Phi_K(s)G = A_3(s)\ \mathrm{diag}\begin{bmatrix}\frac{\lambda_1}{\mu_1}, & \ldots, & \frac{\lambda_p}{\mu_p}\end{bmatrix} A_4(s) \tag{54}$$

Both of these matrices are nonsingular except at isolated points which are the zeros of $\epsilon_1,\ldots,\epsilon_p$, and $\lambda_1,\ldots,\lambda_p$. These zeros all lie in the left half plane since we have ensured nonsingularity for $\mathrm{Re}\ s \geq 0$ of $L'\Phi_K(s)G$, while $[1+K'\Phi(s)G]^{-1} = [1-K'\Phi_K(s)G]$ which has poles only in $\mathrm{Re}\ s < 0$.

We have from (51), (53), and (54) that

$$\det\,[L'\Phi(s)G] = \det\,[L'\Phi_K(s)G]\ \det\,[I+K'\Phi(s)G]$$

$$= k_1\,\frac{\epsilon_1\ldots\epsilon_p\ \lambda_1\ldots\lambda_p}{\psi_1\ldots\psi_p\ \mu_1\ldots\mu_p}\quad \text{(where } k_1 \text{ is a constant)} \tag{55}$$

We know from the form of the left hand side of (55) that there must be cancellations on the right hand side--at a minimum all the μ_i must cancel with numerator terms because by (53) and lemma 4 $L'\Phi(s)G$ is a polynomial matrix divided by $\psi_1\ldots\psi_p$. But we also note that there can be no cancellation of a zero of ψ_i in $\mathrm{Re}\ s \geq 0$ with a numerator term, because all zeros of the numerator lie in $\mathrm{Re}\ s < 0$.

The Smith-McMillan form of $L'\Phi(s)G$ can be written as

$$\overline{L}'(sI-\overline{F})^{-1}\overline{G} = L'\Phi(s)G = A_5(s)\ \mathrm{diag}\begin{bmatrix}\frac{\alpha_1}{\beta_1}, & \ldots, & \frac{\alpha_p}{\beta_p}\end{bmatrix} A_6(s) \tag{56}$$

and evidently

$$\det\,(L'\Phi(s)G) = k_2\,\frac{\alpha_1\;\ldots\;\alpha_p}{\beta_1\;\ldots\;\beta_p}\quad (k_2 \text{ a constant}) \tag{57}$$

Here $\det\,(sI-\overline{F}) = \beta_1\ldots\beta_p$ (see Lemma 4, part (b)). The two expressions (55) and (57) for $\det\,(L'\Phi(s)G)$ and the remarks following (55) imply that $\det\,(sI-\overline{F})$ and $\det\,(sI-F)$ can only differ by factors which are zero in the left half plane $\mathrm{Re}\ s < 0$.

We now show that this is impossible. Let s_o be a zero of $\det(sI-F)$ in the left half plane, of order ν; suppose it has order $\overline{\nu}$ as a zero of $\det\,(sI-\overline{F})$ with possibly $\overline{\nu} = 0$. Now $[I+G'\Phi(-s)K]$ is nonsingular in the left half plane, since its inverse $[I-G'\Phi_K'(-s)K]$ is analytic there. Consequently

$$\delta[\{I+G'\Phi'(-s)K\}\{I+K'\Phi(s)G\} - I;\ s_o]$$

$$= \delta[K'\Phi(s)G;\ s_o] \quad \text{applying Lemma 4(a),(b),(d)}$$

$= \nu$ applying lemma 4(b),(c), and the complete observability $[F,K']$.

On the other hand, it has been ensured that $L'\Phi(s)G$ is nonsingular in the right half plane, and thus that $G'\Phi'(-s)L$ is nonsingular in the left half plane. Then

$$\delta[G'\Phi'(-s)LL'\Phi(s)G;\ s_o]$$

$$= \delta[L'\Phi(s)G;\ s_o] \quad \text{applying lemma 4(d),}$$

$= \overline{\nu}$ applying lemma 4(b),(c).

But ν and $\bar{\nu}$ must be equal, being degrees of the same quantity. Thus the left half plane zeros of $\det(sI-F)$ are the same as those of $\det(sI-\bar{F})$. Now we have shown that the zeros of $\det(sI-F)$ and $\det(sI-\bar{F})$ are identical; consequently, F and $\bar{F}$ have the same dimension, $\{F,G,L\}$ is minimal, and $[F,L']$ is completely observable.

Q.E.D.

We comment that this theorem implicitly describes a method of computation of the loss function for which a given control law is optimal.

Theorems 6 and 7 can now be combined to yield the following theorem, which essentially states that a stable design where reduction is sensitivity to plant parameter variations has been achieved is also a design which is optimal for some quadratic loss function.

Theorem 8. Let K be a stable control law for the completely controllable plant (1) such that $[F,K']$ is completely observable and

$$[I+G'\Phi'(-j\omega)K][I+K'\Phi(j\omega)G] - I > 0 \tag{41}$$

for all ω. Then K is an optimal law for a performance index of the form

$$V(x_o,t_o;u) = \int_{t_o}^{\infty} (x'LL'x + u'u)dt \tag{21}$$

Proof. The existence of L follows by Theorem 7; K is optimal by Theorem 6.

Q.E.D.

V. CONCLUSIONS

Design of control systems to reduce their sensitivity to plant parameters, either because of variations or the unknown character of the latter, is often an engineering objective. Optimal design can also be an engineering objective. This paper serves to link the concepts involved in the two design techniques, showing that the fulfillment of one objective can often lead to fulfillment of the other. The paper moreover sets out the analysis allowing quantitative study of both optimality and sensitivity reduction simultaneously. Theorem 5 allows quantitative examination of the way in which optimal design produces sensitivity reduction, while Theorem 7 deals with the construction of a quadratic loss function given a return difference matrix satisfying the sensitivity reduction condition.

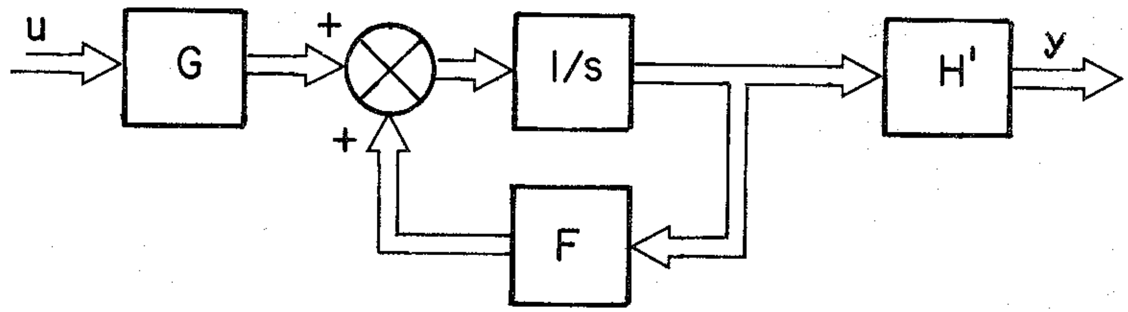

FIG. 1. PLANT WITHOUT FEEDBACK.

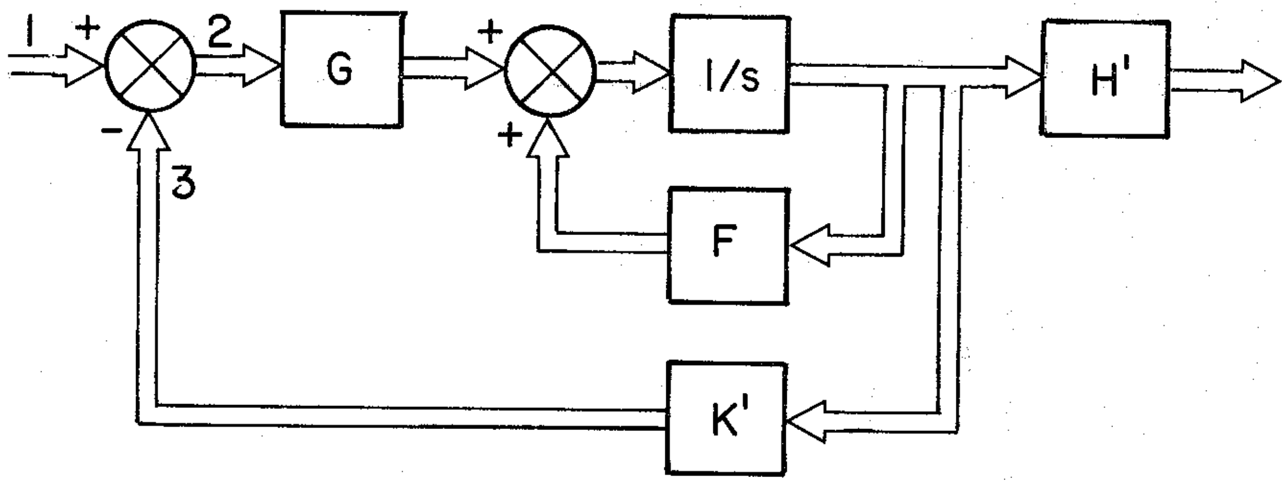

FIG. 2. PLANT WITH STATE VARIABLE FEEDBACK.

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13. ABSTRACT

The relation is explored between linear feedback laws chosen to reduce the sensitivity of a linear time-invariant system to plant parameter variations, and linear feedback laws derived on an optimal control basis from quadratic loss functions. A general equivalence between the two types of design is established for multiple-input, multiple-output systems. The sensitivity improvement can be calculated for optimally design systems, and a quadratic loss function can be found for a system designed to reduce the sensitivity to plant parameter variations.

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