ROBUST LYAPUNOV STABILITY RESULTS AND ADAPTIVE SYSTEMS

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Abstract

Lyapunov stability results are summarized which are robust in character. These results are applied to examine the behaviour of adaptive identification and control algorithms applied to non-stationary plants where certain persistency of excitation conditions are satisfied.

1. Introduction

Most hard results in adaptive identification and control presuppose a linear plant, of known finite order, and with time-invariant parameters. A number also assume that no noise is present. There is obviously a need to establish that these results extend in some way to plants where these idealizations are not exactly met, but may be approximately met. It seems particularly important to consider plants with time-varying parameters, since such plants often provide the rationale for the additional complexity of adaptive methods.

The purpose of this paper is first to state modest developments to those results of stability theory which are robust, i.e., results which can be mildly modified when idealizations made in deriving the original results are only approximately true. Second, we illustrate application of these ideas to time-varying plants, focusing on an adaptive control problem.

The key to obtaining a robust stability result is to have uniform asymptotic stability or exponential asymptotic stability. (These are equivalent in the linear case, while the second implies, but is not implied by, the first in the nonlinear case). In Section 2, we describe how systems with such stability properties can be perturbed with the retention of some form of stability; typically, the perturbation involves the introduction of an input, and bounded-input, bounded-state stability is established. In Section 3, we discuss the application of such ideas to adaptive control. Of the various conclusions drawn, the most general is that the adaptive algorithm described will behave robustly, given satisfaction of a persistency of excitation condition on the reference trajectory for the adaptive control problem, since it is just such a condition which guarantees exponential stability of the idealized algorithm.

The calculations are done principally to show robustness of the adaptive control algorithm in the face of plant parameter variation. We must stress however, that the same methods with remarkably little change allow examination of the effects of measurement noise, plant nonlinearity, and undermodelling of the plant order. They also can be applied to equation and output error identification. All one has to do is to obtain true state-variable equations for an idealized plant, assume excitation conditions which guarantee uniform or exponential stability, and finally, show that the nonideal character of the plant corresponds to one of the standard variations to the equations considered by the robust Lyapunov stability results.

2. Robust Lyapunov Stability Results

In this section, our aim is to delineate a set of robust stability results i.e., stability results which remain valid in the face of variations in the underlying system. All results are for a discrete time. We assume known the definitions for a difference equation of uniform stability, uniform asymptotic stability and global uniform asymptotic stability, see e.g. [1-3]. We also recall that the solution sequence x_k of a difference equation is termed exponentially stable if there exists $K > 0$ and $\alpha \in (0,1)$ such that for all $k \geq k_0 \geq 0$ and all x_{k_0},

$$||x_k|| \leq K\alpha^{k-k_0}||x_{k_0}|| \quad (2.1)$$

For our purposes, it also proves helpful to define a refinement of the concept of exponential stability: we say that a difference equation exhibits exponential stability for all initial conditions in a ball $B_R \triangleq \{||x|| < R\}$ if (2.1) holds whenever $||x_{k_0}|| < R$; of course, $K > 0$ and $\alpha \in (0,1)$. This definition allows us to capture the idea of a system in which trajectories for any initial condition decay exponentially fast to zero, but the rate of decay depends on the initial conditions. Then (2.1) might not hold for all x_{k_0}, since a single α (or for that matter, a single K) could not be found; nevertheless, the system could be exponentially stable for all initial conditions inside an arbitrarily large but finite ball.

Crucial results for adaptive system theory include the following (proofs are omitted due to shortage of space)

Theorem 2.1: Consider the system

$$x_{k+1} = \hat{f}[x_k, k, u_k] \quad (2.2)$$

$$f[x_k,k] = \hat{f}[x_k,k,0] \qquad (2.3)$$

Suppose that with $u_k \equiv 0$, (2.2) is uniformly asymptotically stable for initial conditions in the ball B_R, and the linearization around the zero trajectory is exponentially stable. Suppose further that

$$||\hat{f}(x_k,k,u_k) - f(x_k,k)|| \le M||u_k|| \qquad (2.4)$$

in $B_R \times [0,\infty) \times B_U$, for some constant M. Then (2.2) has the following BIBS property. Take arbitrary but fixed $R_1 \in (0,R)$ and arbitrary $\varepsilon > 0$. Then there exists $N(R_1,\varepsilon)$ such that $x_{k_0} \in \bar{B}_{R_1}$ and $||u_k|| < N(R_1,\varepsilon)$ for all k imply $x_k \in B_{R'+\varepsilon}$ for all k (where $R' = \sup_k\{||x_k|| : ||x_{k_0}|| < R, k \ge k_0 \ge 0\}$).

Further, let x_k^0 denote the solution to (2.2) with $u_k \equiv 0$ and initial condition x_{k_0}; then with $x_{k_0} \in B_{R_1}$ and $||u_k|| < N(R_1,\varepsilon)$, there exists N' such that

$$\sup_{k\ge k_0} ||x_k - x_k^0|| < N' \sup_{k\ge k_0} ||u_k|| \qquad (2.5)$$

and N" such that

$$\lim_{k\to\infty} \sup ||x_k|| < N'' \sup_{k\ge k_0} ||u_k|| \qquad (2.6)$$

The key idea above is that BIBS behaviour results if the control is small enough, even if the initial state is NOT necessarily small, under two main provisos (apart from smoothness). First, the unforced system must be uniformly asymptotically stable (as for the total stability result) and second, the linearization around the zero trajectory alone must be exponentially stable. A secondary aspect of the theorem is that the bound on the control sequence necessary to achieve BIBS behaviour depends on the ball in which initial states may lie; thus for a bigger initial state ball, the control bound will usually be smaller.

Systems With Slowly Varying Parameters

The other results we obtain deal with the following issue. Suppose that a parameter appears in a system equation, and that for all fixed values of the parameter, the system demonstrates exponential stability. Can one consider that if the parameter is varying - at least slowly - then exponential stability is retained? For linear systems, all sorts of results are available, see e.g. [4]. For nonlinear, time-variant, continuous-time systems which are globally exponentially stable, and where the rate of convergence is independent of the parameter, the answer is yes, [4], provided certain smoothness conditions hold. Here we state results for time-varying, discrete-time systems which are exponentially stable for initial conditions in a ball B_R.

Theorem 2.2: Consider the system

$$x_{k+1} = f(x_k, k, p_k) \qquad (2.7)$$

Suppose that: (i) if p_k is constant taking any value in a set P, there exists $\alpha \in (0,1)$ $K > 1$, and R (independent of the particular constant p_k) such that

$$||x_k|| \le K\alpha^{k-k_0}||x_{k_0}|| \qquad (2.8)$$

for all $k \ge k_0 \ge 0$, $x_{k_0} \in B_R$ (this implies $f(0,k,p) = 0$ for all k^{k_0} and all $p \in P$.

Define M by $K\alpha^M < 1/2$ and define a region s in the following way:

$$s = \{x | \exists\ k_0,\ i \le M \text{ and}$$

$$x_{k_0} \in B_R,\ x_{k_0+1},\ \ldots,\ x_{k_0+i} = x \text{ and}$$

$$p_{k_0},\ldots,p_{k_0+i-1} \in P \text{ with}$$

$$x_{j+1} = f(x_j,i,p_j),\ j = k_0,\ldots,k_0+i\}$$

(Thus s is the set of x reachable in up to M steps from an initial $x_{k_0} \in B_R$, with an arbitrary sequence of p_j.)

Suppose further that (ii) there exists δ_1 such that for all $x, y \in s$, $p \in P$, and k,

$$||f(x,k,p) - f(y,k,p)|| \le \delta_1||x-y|| \qquad (2.9)$$

(iii) there exists δ_2 such that for all $x \in s$, p_1 and $p_2 \in P$, and k,

$$||f(x,k,p_1) - f(x,k,p_2)|| \le \delta_2||p_1-p_2||\ ||x|| \qquad (2.10)$$

Then if $p_k \in P$ for all k and

$$||p_{k+1} - p_k|| \le \Delta \qquad (2.11)$$

where*

$$\Delta < [\delta_1^{M-1}\delta_2(M-1)M]^{-1} \qquad (2.12)$$

the system (2.7) is exponentially stable for all $x_{k_0} \in B_R$ and all k_0.

To obtain the result on adaptive control of time-varying plants, we actually need a more involved result than that of the theorem. Consider the scheme of Figure 1 in which $\{v_k\}$ is a time function, fixed once and for all. Suppose $\{v_k\}$ has the property that for any fixed value of p in a set P, the x_k trajectories decay exponentially fast [with a uniform bound and rate of decay, as in (2.8). Now suppose that p is made slowly varying. Were u_k (see the Figure) to stay the same, then Theorem 2.2 would allow us to conclude immediately that exponential stability would be retained for sufficiently slow p variation. But with p changing and v_k fixed, u_k is affected by the change of p. The import of Theorem 2.3 is that in this more complicated situation, exponential stability will nevertheless be retained for slow enough p variation.

Theorem 2.3: Consider the coupled systems

* The bound on Δ is not the best possible, but suffices to prove the theorem.

$$\alpha_{k+1} = A(p_k)\alpha_k + B(p_k)v_k \tag{2.13a}$$

$$u_k = C(p_k)\alpha_k + D(p_k)v_k \tag{2.13b}$$

and

$$x_{k+1} = f(x_k,\ k,\ p_k,\ u_k) \tag{2.14}$$

with v_k a prespecified sequence. Suppose that for all $p \varepsilon P$, $A(p)$ has all its eigenvalues inside $|z| < 1$, so that if p_k varies slowly enough (2.13) is BIBS and BIBO stable and accordingly, since $\{v_k\}$ is prespecified, there exists δ_3 such that for all k $||\alpha_k|| \le \delta_3$ if

$$||p_{k+1} - p_k|| \le \Delta_2 \tag{2.15}$$

for all k. Suppose further that if p_k is constant, taking any value in the set p, k_0 is arbitrary and α_{k_0} is arbitrary save that $||\alpha_{k_0}|| \le \delta_3$, there exists $\alpha \varepsilon (0,1)$ $K > 1$ and R such that

$$||x_k|| \le K\alpha^{k-k_0}||x_{k_0}|| \tag{2.16}$$

for all $k \ge k_0 \ge 0$, $x_{k_0} \varepsilon B_R$. Define M as in Theorem 2.2 and define a region s in the following way. s is the set of x reachable in up to M steps from an initial $x_{k_0} \varepsilon B_R$ with k_0 arbitrary, with an arbitrary sequence p_j save that (2.15) holds, and with arbitrary α_{k_0} satisfying $||\alpha_{k_0}|| \le \delta_3$. Let s_u be the equivalent set of u. Suppose further that there exists δ_1 such that for all $x, y \varepsilon s$, $u_k \varepsilon s_u$, $p \varepsilon P$ and k,

$$||f(x,k,p,u_k) - f(y,k,p,u_k)|| \le \delta_1||x-y||$$

and there exists δ_2 such that for all $x \varepsilon s$, $u_k \varepsilon s_u$, p_1 and $p_2 \varepsilon P$ and k,

$$||f(x,k,p_1,u_k) - f(x,k,p_2,u_k)|| \le \delta_2||p_1 - p_2||\ ||x||$$

and there exists δ_4 such that for all $x \varepsilon s$, all $p \varepsilon P$, all k, and all $u_k^1, u_k^2 \varepsilon s_u$,

$$||f(x,k,p,u_k^1) - f(x,k,p,u_k^2)|| \le \delta_4||u_k^1 - u_k^2||$$

Then if for all k, $p_k \varepsilon P$ and

$$||p_{k+1} - p_k|| \le \Delta$$

where Δ depends on Δ_2, δ_1, δ_2, δ_4, M, and bounding and decay constants associated with (2.13), the system (2.14) is exponentially stable.

3. Adaptive Control

To illustrate the application of the ideas of Section 2 we shall work out in detail the result for plant parameter variation in the case of an adaptive control problem requiring the following of a reference trajectory. The basic conclusion should be clear; if the adaptive algorithm is exponentially convergent with no plant parameter variation, then there should be a bounded error when plant parameter variation is present. We shall work in terms of one of the Goodwin-Caines-Ramadge algorithms, [5]; the technique should however be transferable without conceptual difficulty to other algorithms.

The first, and by far the major part, of the task, is to set up a true state-variable description of the adaptive control algorithm, which does not seem to have been done before. The second part is to appeal to the robustness results of Section 2 to extend convergence results for time-invariant plants to time-varying plants; the application of these results of course presupposes the availability of state-variable equations.

Review of the Goodwin-Caines-Ramadge Algorithm'

Suppose the plant has a delay of d units with equation

$$y_t = -a_1 y_{t-1} \dots - a_n y_{t-n} + b_d u_{t-d} + \dots + b_m u_{t-m} \tag{3.1}$$

For the moment, assume the plant is time-invariant. Now by a standard device, we can find α_i, β_j such that

$$y_t = -\alpha_d y_{t-d} - \dots - \alpha_{n+d-1} y_{t-(n+d-1)} + \beta_d u_{t-d} \dots + \beta_{m+d-1} u_{t-(m+d-1)} \tag{3.2}$$

with $\theta_0 = [-\alpha_d \dots -\alpha_{n+d-1}\ \beta_d \dots \beta_{m+d-1}]'$ and $\phi_t = [y_t \dots y_{t-n+1}\ u_t \dots u_{t-m+1}]'$, one has $y_{t+d} = \phi_t'\theta_0$. Also, let

$$\hat{\theta}_t = [-\hat{\alpha}_{d,t} \dots -\hat{\alpha}_{n+d-1,t}\ \hat{\beta}_{d,t} \dots \hat{\beta}_{m+d-1,t}] \tag{3.3}$$

denote an estimate of θ_0, available at time t just after the reception of y_t and just before the generation of u_t. Suppose that in this estimate, $\hat{\beta}_{d,t}$ is guaranteed nonzero. There are two key steps in the algorithm. The first is to use the equation

$$y^*_{t+d} = \phi_t'\ \hat{\theta}_t \tag{3.4}$$

as a definition for u_t.

The second step of the algorithm is to update the estimate of θ_t upon reception of y_{t+1}. One sets

$$\hat{\theta}_{t+1} = \hat{\theta}_t + \frac{\nu_t \phi_{t-d+1}}{1 + ||\phi_{t-d+1}||^2}[y_{t+1} - \phi'_{t-d+1}\hat{\theta}_t] \tag{3.5}$$

where ν_t is chosen according to certain rules which ensure that $\hat{\beta}_{d,t+1} \neq 0$ [5]. The rules make y_t a discontinuous function of ϕ_{t-d+1}, $\hat{\theta}_t$ and $y_{t+1} - \phi'_{t-d+1}\hat{\theta}_t = \phi'_{t-d+1}(\theta_0 - \hat{\theta}_t)$; we shall assume the definition is adjusted to make ν_t a continuously differentiable function of its arguments. With $\tilde{\theta}_t = \hat{\theta}_t - \hat{\theta}_0$, this means that

$$\tilde{\theta}_{t+1} = \tilde{\theta}_t - \frac{\nu_t(\tilde{\theta}_t,\theta_0,\phi_{t-d+1})\phi_{t-d+1}}{1+||\phi_{t-d+1}||^2}\phi'_{t-d+1}\tilde{\theta}_t \qquad (3.6)$$

with ν_t smooth; also the rules for ν_t ensure that for some $\varepsilon > 0$, $\varepsilon < \nu_t < 1 - \varepsilon$.

Obtaining the State-Variable Equation

Equation (3.6) is part of the state variable equation set. Because ϕ_{t-d+1} involves u_{t-d+1}, which depends on earlier values of $\hat{\theta}_{t-d+1}$ and earlier u_i than u_{t-d+1} (which in their turn depend on earlier $\hat{\theta}_i$), (3.6) above is not acceptable as full state-variable description of the system. (Nor of course is it a linear equation.)

Let $\bar{\phi}_t(\theta_0)$ be the vector ϕ_t which would be obtained if we were to have $\hat{\theta}_t = \theta_0$ for all t, i.e. if no adaption were necessary, and set

$$e_t = \phi_t - \bar{\phi}_t(\theta_0) \qquad (3.7)$$

Obviously, one of the adaptive control goals is to have $\phi_t - \bar{\phi}_t \to 0$ implying that the inputs and outputs of the adapting plant should approach those of a completely adapted plant.

Then one can show that

$$e_t = \bar{A}(\theta_0)e_{t-1} + i_{n+1}[f(\tilde{\theta}_t+\theta_0) - f(\theta_0)][\bar{\phi}_{t-1}(\theta_0) + e_{t-1}] + i_{n+1}[g(\tilde{\theta}_t+\theta_0) - g(\theta_0)]y^*_{t+d} \qquad (3.8)$$

and

$$\bar{\phi}_t = \bar{A}(\theta_0)\bar{\phi}_{t-1} + i_{n+1}\frac{1}{\beta_d}y^*_{t+d} \qquad (3.9)$$

In these two equations, $\bar{A}(\theta_0)$ is a constant matrix, i_{n+1} denotes a unit vector with 1 in position $n+1$, and f and g are certain continuously differentiable functions of their arguments, so long as $\hat{\beta}_{t,d} \neq 0$. With θ_0 fixed, $\{y^*_{t+d}\}$ prespecified and an arbitrary but fixed initial condition on $\bar{\phi}_t$, $\{\bar{\phi}_t\}$ is a fixed sequence.

The relevant equations we need are (5.13), and (5.11) rewritten as

$$\tilde{\theta}_{t+1} = \tilde{\theta}_t - \frac{\nu_t(\tilde{\theta}_t,\theta_0,\bar{\phi}_{t-d+1})(\bar{\phi}_{t-d+1}+e_{t-d+1})(\bar{\phi}_{t-d+1}+e_{t-d+1})'}{1+||\bar{\phi}_{t-d+1}+e_{t-d+1}||^2}\tilde{\theta}_t \qquad (3.10)$$

Together, (3.8) and (3.10) specify, for some m,n

$$e_t = m(\tilde{\theta}_t, e_{t-1}, t)$$

$$\tilde{\theta}_{t+1} = n(\tilde{\theta}_t, e_{t-d+1}, t)$$

From these, we can get state-variable equations with state vector $[e'_{t-d+2}\ e'_{t-d+3} \cdots e'_t\ \tilde{\theta}'_{t+1}]'$:

$$\begin{bmatrix} e_{t-d+2} \\ e_{t-d+3} \\ \vdots \\ e_{t-1} \\ e_t \\ \tilde{\theta}_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{t-d+1} \\ e_{t-d+2} \\ \vdots \\ e_{t-2} \\ e_{t-1} \\ \tilde{\theta}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \\ 0 \\ m(\tilde{\theta}_t, e_{t-1}, t) \\ n(\tilde{\theta}_t, e_{t-d+1}, t) \end{bmatrix} \qquad (3.11)$$

In summary, (3.11) provides the state-variable equations for the adaptive system with time-invariant plant.

Remarks Concerning the State-Variable Equations

We note the following points:

1. In [6], exponential convergence of $\tilde{\theta}_t$ and e_t to zero has been established for initial conditions in an arbitrarily large but finite ball under the assumptions

y^*_t is persistently exciting (as defined in [6]) (3.12)

The plant (3.1) is minimum phase (3.13)

There is no common zero of $z^n + a_1 z^{n-1} + \ldots + a_n$

$z^m + b_1 z^{m-1} + \ldots + b_m$ (3.14)

The calculation in [6] was for $d=1$, but the basic result extends easily.

2. The right hand sides of (3.8) and (3.10) have continuous derivatives with respect to any variable appearing of arbitrary order, provided $\alpha(.,.,.)$ is smooth.

3. The linearizations of (3.8) and (3.10) around the zero trajectory are

$$e_t = \bar{A}(\theta_0)e_{t-1} + i_{n+1}i'_{n+1}L(\tilde{\theta}_t)\bar{\phi}_{t-1}(\theta_0) + i_{n+1}m(\tilde{\theta}_t) \qquad (3.15)$$

and

$$\tilde{\theta}_{t+1} = \tilde{\theta}_t - \frac{\nu_t(0,\theta_0,\bar{\phi}_{t-d+1})\bar{\phi}_{t-d+1}\bar{\phi}'_{t-d+1}}{1+||\bar{\phi}_{t-d+1}|^2}\tilde{\theta}_t \qquad (3.16)$$

where $L(\tilde{\theta}_t)$ is a matrix with entries linear in the entries of $\tilde{\theta}_t$ and $m(\tilde{\theta}_t)$ is a linear function of $\tilde{\theta}_t$. These equations are triangularly coupled. Equation (3.16) is exponentially stable given (3.13), (3.14) and (3.12), i.e. a persistently exciting condition on y^*_t, [6], since as may be shown

$$\bar{\phi}_t = [y^*_t \cdots y^*_{t-n+1}\ u^*_t \cdots u^*_{t-m+1}]' \qquad (3.17)$$

where $\{u^*_t\}$ is the plant input sequence required to produce $\{y^*_t\}$ as the output sequence. Equation (3.15) is then exponentially stable so long as $|zI-\bar{A}|$ has all zeros in $|z| < 1$. Since as may be shown

$$|zI-\bar{A}| = z^{n+d+1}(z^{m-1} + \frac{b_{d+1}}{b_d} z^{m-2} + \ldots + \frac{b_m}{b_d}) \quad (3.18)$$

this is a consequence of the minimum phase character of the plant.

Equations Given Variation of the Plant Parameters

Suppose now that the plant parameters are time-varying with (3.1) replaced by

$$y_t = -a_{1,t}y_{t-1} - a_{2,t}y_{t-2} - \ldots - a_{n,t}y_{t-n} + b_{d,t}u_{t-d} + \ldots + b_{m,t}u_{t-m} \quad (3.19)$$

with a similar replacement for (3.2). Then with

$$\theta_{0,t} = [-\alpha_{d,t} \ldots -\alpha_{n+d-1,t}\ \beta_{d,t} \ldots \beta_{m+d-1,t}]' \quad (3.20)$$

we obtain easily the following variations to (3.8) and (3.10)

$$e_t = \bar{A}(\theta_{0,t})e_{t-1} + i_{n+1}[f(\tilde{\theta}_t + \theta_{0,t}) - f(\theta_{0,t})][\hat{\phi}_{t-1} + e_{t-1}] + i_{n+1}[g(\tilde{\theta}_t + \theta_{0,t}) - g(\theta_{0,t})]y^*_{t+d} \quad (3.21)$$

$$\tilde{\theta}_{t+1} = \tilde{\theta}_t - \frac{\nu_t(\tilde{\theta}_t,\theta_{0,t},\hat{\phi}_{t-d+1}+e_{t-d+1})(\hat{\phi}_{t-d+1}+e_{t-d+1})(\hat{\phi}_{t-d}+e_{t-d+1})'}{1+||\hat{\phi}_{t-d+1}+e_{t-d+1}||^2}\tilde{\theta}_t + (\theta_{0,t} - \theta_{0,t+1}) \quad (3.22)$$

with an obvious consequential variation to (3.11). The quantity $\hat{\phi}_{t-d+1}$ is not the same as $\bar{\phi}_{t-d+1}$; the latter quantity depends on y^*_{t+d} and the particular plant parameters -- see (3.9). We have in fact

$$\hat{\phi}_t = \bar{A}(\theta_{0,t})\hat{\phi}_{t-1} + i_{n+1}\frac{1}{\beta_{d,t}}y^*_{t+d} \quad (3.23)$$

The changes from the pair (3.8) and (3.10) exhibited in (3.21) and (3.22) are of three kinds: (i) $\bar{\phi}_{t-d+1}$ is replaced by $\hat{\phi}_{t-d+1}$, (ii) certain formerly time-invariant parameters have become time-varying and (iii) an additive input term, viz, $\theta_{0,t} - \theta_{0,t+1}$, appears.

Retention of Stability Under Slow Parameter Variation

Consider first the same equation as (3.21) through (3.23) save that the driving term $\theta_{0,t} - \theta_{0,t+1}$ in (3.22) is absent.* Suppose further that $\theta_{0,t}$ varies in such a way that the "frozen" plant is always minimum phase, remains of degree n and $\theta_{0,t}$ varies in a compact set P. Suppose further that y^*_t is persistently exciting. Finally, suppose that initial conditions lie in an arbitrarily large but fixed ball B_R. Evidently, for all frozen parameter values, exponential stability of e_t and $\tilde{\theta}_t$ is obtained, and because P is compact, a single bounding constant and exponent can be achieved in describing the exponential stability. Thus the most important condition in Theorem 2.3, (2.16), is satisfied. The remaining conditions follow easily from the smoothness properties and bounded nature of $\{y^*_k\}$ and B_R. Then Theorem 2.3 implies that the resulting set of equations exhibits exponential stability of e_t and $\tilde{\theta}_t$ provided that $\theta_{0,t}$ varies sufficiently slowly. Theorem 2.1 then implies that when the driving term $\theta_{0,t} - \theta_{0,t+1}$ is included in (3.22), BIBS behaviour will result if $||\theta_{0,t} - \theta_{0,t+1}||$ is sufficiently small, i.e. again if the rate of plant parameter variation is sufficiently small.

4. Conclusions

We have highlighted a number of robust Lyapunov stability results. The usual starting point is an assumption of uniform or exponential asymptotic stability for initial conditions inside a ball, and the usual conclusion is a BIBS result. Next, having pointed out that a number of idealized adaptive algorithms are exponentially stable given some form of persistency of excitation condition, we have shown the capability of such algorithms to perform robustly in the case of variations from the idealizations. In some detail, we have looked at time-variation in the plant parameter, especially for adaptive control.

An important idea is that, in general, the less well-initialized an adaptive scheme is, the less tolerance there will be of variations from the idealizations.

* In order that the ideas of Section 2 be applied, strictly, these equations should be expanded to obtain a state-variable set of equations in the same manner as (3.11) represent an expanded version of (3.18) and (3.16). One can of course imagine this is done, with but trivial differences to the above argument.

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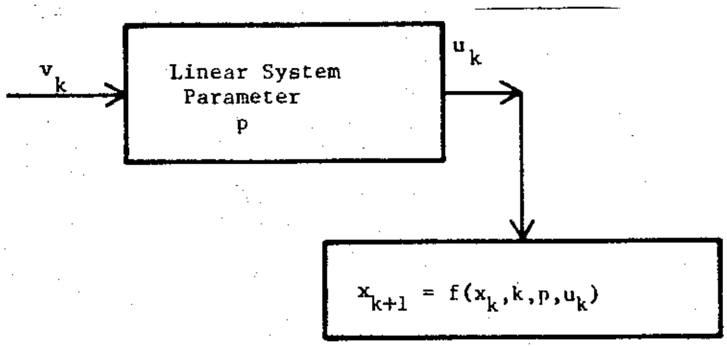

Figure 1: Time-invariant parameter plus exciting signal forcing exponential stability.