GENERIC POLE ASSIGNMENT: PRELIMINARY RESULTS*

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Introduction

It has long been of interest to find explicit conditions for a real, canonical linear system $(C_{pxn}, A_{nxn}, B_{nxm})$ to be completely assignable; i.e., to have the property that for each real, monic, nth degree polynomial

$$\alpha(\lambda) = \lambda^n + \sum_{i=1}^{n} a_i \lambda^i \quad (1)$$

there exists a real, constant output-feedback matrix F_{mxp} for which

$$\det(\lambda I - A - BFC) = \alpha(\lambda) \quad (2)$$

It has also been of interest to identify those systems which are 'generically assignable': (C,A,B) is generically assignable if the set of all coefficient vectors $(a_1, a_2, \ldots, a_n)'$ associated with (1) for which there exist real F satisfying (2) is open and dense in $\mathbb{R}^n$.

Apart from the cases when either C has independent columns or B has independent rows (which can be dealt with using state-feedback theory), little is known about either complete or generic assignability. Perhaps the sharpest result to date, due to Kimura [1] and others, asserts that for 'almost every' linear system, generic pole assignment is possible provided $n \leq m + p - 1$. Willems and Hesselink [2] show that for almost every system with $m = p = 2$ and $n = 4$, generic pole assignment is not possible, even though the number of free parameters in F equals n. Using a version of the implicit function theorem, Hermann and Martin prove that for almost every linear system whose first n Markov matrices, $CB, CAB, \ldots, CA^{n-1}B$ are linearly independent, generic pole assignment is possible provided F is allowed to be complex-valued. Brockett and Byrnes [4] take advantage of certain classical ideas based on elimination theory to develop a formula which gives values of m,n and p, for which almost every linear system is generically assignable - but their development is not constructive.

The purpose of this paper is to develop a useful formulation of the assignment problem for $p = 2$ (§1), and to provide constructive solutions for some special cases. For $m = 2$ and $n = 4$ we characterize the classes of generically assignable and completely assignable systems (§2). In §3 we show that for $m = 3$ and $n = 6$, almost every linear system is generically assignable; we do this by reducing the assignment problem to the problem of finding a real root of a real 5th degree polynomial in one variable. That such a polynomial should exist is agreement with the Brockett-Byrnes formula derived in [4].

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Notation: In the sequel $\mathbb{R}$ denotes the real field and $\mathbb{R}^m$ is real m-space. If x and y are in $\mathbb{R}^m$, we write $x \wedge y$ for their exterior product; we often represent this product by the vector

$$\begin{bmatrix} d_{12} \\ d_{13} \\ \vdots \\ d_{1m} \\ d_{23} \\ \vdots \\ d_{2m} \\ \vdots \\ d_{m-1,m} \end{bmatrix}_{m^* x1}$$

where

$$m^* = \frac{m(m-1)}{2}$$

and d_{ij} is the determinant of the 2x2 matrix consisting of rows i and j of $[x,y]_{mx2}$. In this framework we note that if H_{rxm} is a matrix, then $Hx \wedge Hy = H^*(x \wedge y)$ where

$$H^* = [h_1 \wedge h_2, h_1 \wedge h_3 \ldots, h_1 \wedge h_m, h_2 \wedge h_3 \ldots, h_{m-1} \wedge h_m]_{r^* x m^*}$$

and h_i is the ith column of H.

I. Formulation

We begin with the observation that

$$\begin{aligned} \det(\lambda I - A - BFC) &= \alpha_0(\lambda) \det(I - BFC(\lambda I - A)^{-1}) \\ &= \alpha_0(\lambda) \det(I - \frac{H(\lambda) F}{\alpha_0(\lambda)}) \end{aligned} \quad (3)$$

where $\alpha_0(\lambda) = \det(\lambda I-A)$, and $H_{2xm}(\lambda)$ is the polynomial matrix $\alpha_0(\lambda)C(\lambda I-A)^{-1}B$. If we write f_i, ($i = 1,2$) for the ith column of F_{mx2} and e_i($i = 1,2$) for the ith unit vector in $\mathbb{R}^2$, then (3) can be expanded further by noting that

$$\det(I - \frac{H}{\alpha_0} F) = (e_1 - \frac{H}{\alpha_0} f_1)\wedge(e_2 - \frac{H}{\alpha_0} f_2)$$

$$= e_1 \wedge e_2 - \frac{H}{\alpha_0} f_1 \wedge e_2 - e_1 \wedge \frac{H}{\alpha_0} f_2 - \frac{H}{\alpha_0} f_1 \wedge \frac{H}{\alpha_0} f_2$$

$$= 1 - \frac{1}{\alpha_0} (h_1 f_1 + h_2 f_2 + h_3(f_1 \wedge f_2)) \qquad (4)$$

where h_i($i = 1,2$) is the ith row of H, $\bar{h}_i$($i = 1,2,\ldots,m$) is the ith column of H and h_3 is the polynomial matrix

$$h_3 = -\frac{1}{\alpha_0} [\bar{h}_1 \wedge \bar{h}_2, \bar{h}_1 \wedge \bar{h}_3, \ldots, \bar{h}_1 \wedge \bar{h}_m, \bar{h}_2 \wedge \bar{h}_3, \ldots$$
$$\ldots, \bar{h}_2 \wedge \bar{h}_m, \ldots, \bar{h}_{m-1} \wedge \bar{h}_m]^{\dagger}_{1xm^*}$$

Hence if we define the linear transformation

$L: \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{m^*} \rightarrow$ space of all polynomials of degree < n,

so that

$$L(f_1, f_2, f_3) = -(h_1 f_1 + h_2 f_2 + h_3 f_3) \qquad (5)$$

then we can combine (4) with (3) to obtain

Lemma 1:

$$\text{c.p.}(A + BFC) = \alpha_0 + L(f_1, f_2, f_1 \wedge f_2) \qquad (6)$$

In view of Lemma 1 we see that if $\alpha(\lambda)$ is a real, monic, nth degree polynomial to be assigned with $F = [f_1, f_2]$, then F must be chosen to satisfy

$$L(f_1, f_2, f_1 \wedge f_2) = \alpha(\lambda) - \alpha_0(\lambda) \qquad (7)$$

It is easy to see that this single nonlinear equation in f_1 and f_2 is equivalent to the simultaneous equations

$$L(f_1, f_2, f_3) = \alpha(\lambda) - \alpha_0(\lambda) \qquad (8a)$$

$$f_3 = f_1 \wedge f_2, \qquad (8b)$$

Since $L(f_1, f_2, f_3)$ is linear in f_1, f_2 and f_3 and $L(f_1, f_2, f_1 \wedge f_2)$ is continuous in f_1 and f_2, (C,A,B) cannot be completely assignable or even generically assignable unless L is an epimorphism. Thus if $\underline{L}_{nx(2m+m^*)}$ is a matrix representation of L, then we can state

Proposition 1: A necessary condition for (C,A,B) to be either generically or completely assignable is that

$$\text{rank } \underline{L} = n \qquad (9)$$

†The elements of $\alpha_0 h_3$ are in fact nothing more than a lexigraphically ordered set of independent 2x2 minors of H.

Remark: It is worth noting that (9) is weaker than the requirement that the first n Markov matrices CB, CAB, ..., $CA^{n-1}B$ be linearly independent.

To proceed with our formulation, let us observe that if (9) holds, then dim (ker L) = n^* where

$$n^* = 2m + m^* - n \qquad (10)$$

Thus if $(f_1^{\ i}, f_2^{\ i}, f_3^{\ i})$, $i = 1,2,\ldots,n^*$ is a basis for ker L, then a typical element in the kernel is of the form (M_1x, M_2x, M_3x) where

$$\left.\begin{aligned} M_i &= [f_i^{\ 1}, f_i^{\ 2}, \ldots, f_i^{\ n^*}]_{mxn^*} \quad (i = 1,2) \\ M_3 &= [f_3^{\ 1}, f_3^{\ 2}, \ldots, f_3^{\ n^*}]_{m^*xn^*} \end{aligned}\right\} \qquad (11)$$

and $x \in \mathbb{R}^{n^*}$.

Remark 1: Note that the columns of

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix},$$

which are actually a representation of the basis of ker L, are linearly independent. Use will be made of this fact in the sequel.

To continue, observe that any solution (f_1, f_2, f_3) to (8a) for any $\alpha(\lambda)$, must be of the form

$$(f_1, f_2, f_3) = (f_{10} + M_1x, f_{20} + M_2x, f_{30} + M_3x)$$

where $x \in \mathbb{R}^{n^*}$ and (f_{10}, f_{20}, f_{30}) is a particular solution to (8a). Thus to solve our problem (i.e., both (8a) and (8b)), we must find, if possible, $x \in \mathbb{R}^{n^*}$ for which

$$(f_{10} + M_1x) \wedge (f_{20} + M_2x) = f_{30} + M_3x \qquad (12)$$

Assuming (9) holds, we know that for any $\alpha(\lambda)$, (8a) must have a particular solution (f_{10}, f_{20}, f_{30}). Conversely for any choice of (f_{10}, f_{20}, f_{30}), (8a) uniquely defines a polynomial $\alpha(\lambda) = \alpha_0(\lambda) + L(f_{10}, f_{20}, f_{30})$. From this it follows that for complete assignability (12) must be solvable for all possible $(f_{10}, f_{20}, f_{30}) \in \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{m^*}$ while for generic solvability (12) must be solvable for all (f_{10}, f_{20}, f_{30}) in some open dense subset of $\mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{m^*}$. For the remainder of this paper then, we assume (9) holds, and study when (12) might have one or the other of these properties.

Remark:

One simple condition which insures that (12) will always have a solution x no matter what f_{10}, f_{20}, f_{30} are, is that

$$\text{rank} \begin{bmatrix} M_1 \\ M_3 \end{bmatrix}_{(m+m^*)xn^*} = m + m^* \qquad (13)$$

For if this is so then the linear equations $M_1 x = -f_{10}, M_3 x = -f_{30}$ will always be solvable and any such solution will automatically satisfy (12). Now (13) will be true (at least generically) if $m + m^* \geq n^*$; but using the definition of n^* in (10) we see that this is equivalent to $m \geq n$, which of course is a case which could be handled using state-feedback theory.

II. Case $m = 2$, $n = 4$

In this case $m^* = n^* = 1$, and x is a scalar. Direct expansion of (12) thus yields

$$(M_1 \wedge M_2)x^2 + (f_{10} \wedge M_2 + M_1 \wedge f_{20} - M_3)x + (f_{10} \wedge f_{20} - f_{30}) = 0 \quad (14)$$

which is a quadratic equation in x. For (14) to have a real solution x for either all or almost all f_{10}, f_{20}, f_{30}, it is clearly necessary that

$$M_1 \wedge M_2 = 0 \quad (15)$$

Now if (15) is true, the set of f_{10}, f_{20}, f_{30} for which (14) does not have a solution is given by all f_{10}, f_{20}, f_{30} satisfying

$$f_{10} \wedge M_2 + M_1 \wedge f_{20} - M_3 = 0 \quad (16a)$$

$$f_{10} \wedge f_{20} - f_{30} \neq 0 \quad (16b)$$

Since by Remark 1, M_1, M_2 and M_3 cannot all be zero, the set of (f_{10}, f_{20}, f_{30}) satisfying (16a) must be either empty or a proper variety in $\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}$; therefore the set of (f_{10}, f_{20}, f_{30}) satisfying (16) must be the compliment of an open-dense set in $\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}$. In other words, if (15) holds, (14) is solvable for almost all (f_{10}, f_{20}, f_{30}), so (15) together with (9) are necessary and sufficient for generic pole assignment.

For complete pole assignment, (14) must be solvable for all (f_{10}, f_{20}, f_{30}) so the set of such elements satisfying (16) must be empty. In view of Remark 1, this will be true just in case $M_1 = M_2 = 0$, which of course implies (15). We summarize:

__Proposition 2__: __Let__ $m = p = 2$ __and__ $n = 4$. __Then__ (C,A,B) __is generically assignable if and only if__

$$\text{rank } \underline{L} = 4 \quad (17)$$

and

$$M_1 \wedge M_2 = 0 \quad (18)$$

(C,A,B) __is completely assignable if and only if__ (17) __is true and, in addition__

$$M_1 = M_2 = 0 \quad (19)$$

It is not difficult to see that (18) will almost certainly fail to hold if (C,A,B) is chosen at random. From this it follows that generic pole assignment is not possible for 'almost every' system with $m = p = 2$ and $n = 4$, as was observed previously in [2].

It is possible to interpret the conditions of Proposition 2 in more familiar terms. For this, first note from (5) that $h_1 f_1 + h_2 f_2 = \text{trace}(HF)$ and $h_3 = -(1/\alpha_0)\det H$. Since the transfer matrix $T(\lambda)$ of (C,A,B) equals H/α_0, we can write

$$L(f_1, f_2, f_3) = \alpha_0(f_3 \det T - \text{trace}(TF))$$

where $F = [f_1, f_2]$. Condition (17) is thus equivalent to the requirement that the __numerator polynomials__ of T (i.e., $h_{11}, h_{12}, h_{21}, h_{22}$) together with the __transmission polynomial__ $h_t = \alpha_0 \det T$, span the linear space of real polynomials of degree less than 4. Assuming that this is so a basis for the kernel of L can be represented by a single pair (G,g) where g is a real scalar and G is a real 2x2 matrix satisfying

$$(G,g) \neq 0 \quad (20a)$$

$$\text{trace}(TG) = g \det T \quad (20b)$$

In this framework (19) is equivalent to $G = 0$, which in turn is equivalent to $\det T = 0$. We arrive at the following result, obtained previously in [2].

__Corollary 1__: (C,A,B) __is completely assignable if and only if the transfer matrix__ T __of__ (C,A,B) __is singular, and the numerator polynomials of__ T __span the linear space of all real polynomials of degree less than__ 4.

Since complete assignability implies generic assignability and since the spanning property of $\{h_{11}, h_{12}, h_{21}, h_{22}, h_t\}$ is necessary in either case, the spanning property of $\{h_{11}, h_{12}, h_{21}, h_{22}\}$ in Corollary 1 is necessary and sufficient for generic assignability in the case when $T(\lambda)$ is singular.

Suppose $T(\lambda)$ is nonsingular; then (20) implies $G \neq 0$. For if $G = 0$, then (20b) would imply $g = 0$ contradicting (20a). Condition (18) is equivalent to $\det G = 0$, which in turn is equivalent to rank $G = 1$. Thus (18) is equivalent to the existence of __nonzero__ vectors g_1 and g_2 such that $\text{trace}(Tg_1 g_2') = g \det T$; but $\text{trace}(Tg_1 g_2') = g_2' T g_1$. Therefore we can state

__Corollary 2__: __If the transfer matrix__ T __of__ (C,A,B) __is nonsingular, then generic pole assignment is possible if and only if there exist real nonzero vectors__ g_1 __and__ g_2 __and a real scalar__ g __such that__

$$g_2' T g_1 = g \det T \quad (21)$$

__and, in addition, the numerator polynomials of__ T __together with the transmission polynomial of__ T __span the linear space of all real polynomials of degree less than__ 4.

It is easy to see that if at least one of the entries in T is zero, then nonzero g_1 and g_2 necessarily exist for which (22) holds with $g = 0$. It is easy to show that if all entries of T are nonzero, and if (21) holds with nonzero g_1 and g_2, then it is possible to construct an 'initializing' output-feedback matrix F_0, so that after suitable input and output coordinate transformations, the transfer matrix T_{F_0} of the resulting transformed, closed-loop system has at least one entry equal to zero. In other words, the first condition of Corollary 2 is equivalent to the existence of input-coordinate, output-coordinate and output-feedback transformations which when applied to (C,A,B) result in a new system whose transfer matrix has at least one zero entry.

III. $m = 3$, $n = 6$

In this case $m^* = n^* = 3$, so $x \in \mathbb{R}^3$ and M_1, M_2 and M_3 are 3x3 matrices. We make

Assumption 1: There exists a real scalar μ such that $M_1 + \mu M_2$ is nonsingular and

$$M \equiv [M_1 + \mu M_2]^{-1} M_2$$

has at least two distinct eigenvalues.

In view of Remark 1, it is easy to see that Assumption 1 is 'generic' in the sense that 'almost every' system with $p = 2$, $m = 3$ and $n = 6$ has the required property.

Our approach will be to first make a judicious change of variables in (12), then expand (12), and finally to reduce what results to a 5th degree polynomial equation in one variable. Since M is 3x3, it must have at least one real eigenvalue. If the remaining two eigenvalues are complex, let w denote the real eigenvalue. If all eigenvalues are real, let w denote any one, distinct from the remaining two. In either case let g be an eigenvector for w and let G_{3x2} be a real matrix whose columns span the 2-dimensional M-invariant subspace associated with the two eigenvalues distinct from w. Then

$$MG = GW \quad (22a)$$

$$Mg = gw \quad (22b)$$

$$\det[G,g] \neq 0 \quad (22c)$$

$$\det[wI-W] \neq 0 \quad (22d)$$

where W is a matrix representation of the restriction of M to column span G. Next define

$$T = [t_1, t_2] = [M_1 + \mu M_2]G \quad (23a)$$

$$t_3 = [M_1 + \mu M_2]g$$

Remark 2: Using (22c) and the nonsingularity of $[M_1 + \mu M_2]$ it is easy to see that t_1, t_2 and t_3 are linearly independent.

From (21),(22a) and (23a) it follows that

$$M_2 G = TW \quad (24)$$

Similarly from (21),(22b) and (23b), we obtain

$$M_2 g = t_3 w \quad (25)$$

Thus if we introduce the new variables

$$\begin{bmatrix} y \\ z \end{bmatrix} = [G,g]^{-1} x$$

where $y \in \mathbb{R}^2$, then

$$x = Gy + gz$$

$$[M_1 + \mu M_2]x = Ty + t_3 z \quad (26a)$$

$$M_2 x = TWy + t_3 wz \quad (26b)$$

With reference to (12) we can write

$$(f_{10} + M_1 x)\wedge(f_{20} + M_2 x) = (f_{10} + \mu f_{20} + (M_1 + \mu M_2)x)\wedge(f_{20} + M_2 x)$$

$$= (f_{10} + \mu f_{20} + Ty + t_3 z)\wedge(f_{20} + TWy + t_3 wz)$$

Thus in our new variables, (12) becomes

$$(f_{10} + \mu f_{20} + Ty + t_3 z)\wedge(f_{20} + TWy + t_3 wz) = f_{30} + M_3 Gy + M_3 gz$$

Expanding and collecting terms we obtain

$$(t_1 \wedge t_2)(y \wedge Wy) + (L_1 + L_2 z)y = L_3 z + d \quad (27)$$

where

$$L_1 y = Ty \wedge f_{20} + (f_{10} + \mu f_{20})\wedge(TWy) - M_3 Gy \quad (28a)$$

$$L_2 y = Ty \wedge (t_3 w) + t_3 \wedge (TWy)$$

$$= [t_1 \wedge t_3, t_2 \wedge t_3][wI-W]y \quad (28b)$$

$$L_3 = M_3 g - (f_{10} + \mu f_{20})\wedge(t_3 w) - t_3 \wedge f_{20} \quad (28c)$$

$$d = f_{30} - (f_{10} + \mu f_{20})\wedge f_{20} \quad (28d)$$

To proceed we need

Lemma 2: $t_1 \wedge t_3, t_2 \wedge t_3$ and $t_1 \wedge t_2$ are linearly independent.

Proof: Suppose

$$a_1 t_1 \wedge t_2 + a_2 t_1 \wedge t_3 + a_3 t_2 \wedge t_3 = 0,$$

To prove $a_1 = 0$, write

$$a_1 t_1 \wedge t_2 \wedge t_3 + a_2 t_1 \wedge t_3 \wedge t_3 + a_3 t_2 \wedge t_3 \wedge t_3 = 0$$

But $t_1 \wedge t_3 \wedge t_3 = t_2 \wedge t_3 \wedge t_3 = 0$ and since by Remark 2, $t_1 \wedge t_2 \wedge t_3 \neq 0$, it follows that $a_1 = 0$. By similar reasoning, $a_2 = a_3 = 0$. □

Since $[wI-W]$ is nonsingular {cf.(22d)}, it follows from Lemma 2 and (28b) that $[t_1 \wedge t_2, L_2]$ is nonsingular as well. Define

$$\begin{bmatrix} q \\ P \end{bmatrix} \equiv [t_1 \wedge t_2, L_2]^{-1}$$

where q is a row vector. Then

$$\left.\begin{aligned} q(t_1 \wedge t_2) &= 1 \\ qL_2 &= 0 \end{aligned}\right\} \quad (29)$$

and

$$\left.\begin{aligned} P(t_1 \wedge t_2) &= 0 \\ PL_2 &= I \end{aligned}\right\} \quad (30)$$

Application of P to (27) yields

$$(PL_1 + Iz)y = PL_3 z + Pd$$

or

$$y = (PL_1 + Iz)^{-1}(PL_3 z + Pd)$$

Thus

$$y = \frac{\bar{y}}{\beta(z)} \quad (31)$$

where $\beta(z) = \det(PL_1 + Iz)$ and $\bar{y}$ is a vector of polynomials in z, each polynomial having degree no greater than 2.

Application of q to (27) and using (29) yields

$$y \wedge Wy + qL_1 y = qL_3 z + qd$$

From (31) we obtain,

$$\frac{1}{\beta^2}(\bar{y} \wedge W\bar{y}) + \frac{1}{\beta} qL_1 \bar{y} = qL_3 z + qd$$

or

$$\bar{y} \wedge W\bar{y} + \beta qL_1 \bar{y} = \beta^2 qL_3 z + \beta^2 qd \quad (32)$$

Now $\bar{y} \wedge W\bar{y}$, $\beta qL_1 \bar{y}$ and $\beta^2 qd$ all must be polynomials in z of degree not exceeding 4, while $\beta^2 qL_2 z$ will be of degree 5 if

$$qL_3 \neq 0 \quad (33)$$

In other words, if (33) holds then (32) is a polynomial equation in z of degree 5.

From (28c)

$$qL_3 = qM_3 g - q((f_{10} + \mu f_{20}) \wedge (t_3 w) + t_3 \wedge f_{20}). \quad (34)$$

Since by Remark 2, $t_3 \neq 0$, (33) must necessarily hold for almost all (f_{10}, f_{20}, f_{30}). We conclude:

<u>Proposition 3</u>: <u>For the case</u> $p = 2$, $n = 6$, $m = 3$, <u>almost every linear system is generically assignable</u>.

Concluding Remarks

It is possible to use the approach outlined in this paper to solve the assignment problem for other special cases. Results along these lines, which encompass those of Kimura and others (for p = 2) will be presented in another paper.

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