DIGITAL LATTICE AND LADDER BLOCK STRUCTURES

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Abstract

Several techniques are available for implementing a scalar digital transfer function using lattice and ladder structures. The lattice and ladder structure proposed by Gray and Markel have the advantage of being internally scaled and usually lead to low-noise realizations.

There are some advantages in implementing a scalar digital filter in block form. There is a technique available for obtaining a lattice and ladder structure for implementing a block digital filter. In this paper new algorithms to realize block digital filters in the form of cascaded lattice or ladder two-pairs are proposed. Block lattice and ladder structures similar to Gray and Markel scalar structures are also developed. Also a technique for obtaining block lattice and ladder structures with internal scaling is developed.

I. Introduction

Block implementation of a scalar digital filter has some advantages such as lesser number of computations, the possibility of using FFT techniques for intermediate computations, implementation by array processors, reduced roundoff noise and high speed computation. A scalar digital filter can be implemented in block form in many ways. Block state structures have been studied by Gnanasekaran and Mitra [1], Barnes and Shinnaka [2], Ananthakrishna and Mitra [3]. In this paper, some techniques for realizing a block digital filter in block lattice and ladder form are presented.

Lattice and ladder realization of scalar digital filters have been proposed by Mitra, Kamat and Huey [4] and Gray and Markel [5]. Gray and Markel's structure has the advantage of being internally scaled. Gnanasekaran has proposed a technique for realizing the block lattice and ladder realization of a digital filter [6]. However, in his realization, the design equations become very complicated when the order of the filter is large.

In this paper new algorithms to realize block digital filters in the form of cascaded lattice or ladder two-pairs are proposed. Block lattice and ladder structures similar to Gray and Markel scalar structure are also developed. Finally, a technique for obtaining block lattice and ladder structures with internal scaling is developed.

The general block equation for an N-th order scalar digital transfer function

\[ h(z) = \left( \sum_{i=0}^{N} a_i z^{-i} \right) \left( \sum_{i=0}^{N} b_i z^{-i} \right) \]

is given by [3]:

\[ \begin{array}{c} Y_k \ Y_{k+1} \ldots Y_{k+L-1} \\ X_k \ X_{k+1} \ldots X_{k+L-1} \end{array} = \begin{bmatrix} a_0 & \cdots & a_{L-1} \\ \vdots & \ddots & \vdots \\ a_{L-1} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} b_0 & \cdots & b_{L-1} \\ \vdots & \ddots & \vdots \\ b_{L-1} & \cdots & b_0 \end{bmatrix} \begin{array}{c} Y_k \ Y_{k+1} \ldots Y_{k+L-1} \\ X_k \ X_{k+1} \ldots X_{k+L-1} \end{array} \]

and, \( A \) and \( B \) are \( L \times L \) matrices given by

\[ A_k = \begin{bmatrix} a_{KL} & \cdots & a_{KL+L-1} \\ \vdots & \ddots & \vdots \\ a_{KL+L-1} & \cdots & a_{KL} \end{bmatrix}, \quad B_k = \begin{bmatrix} b_{KL} & \cdots & b_{KL+L-1} \\ \vdots & \ddots & \vdots \\ b_{KL+L-1} & \cdots & b_{KL} \end{bmatrix} \]

where \( a_i = b_i = 0 \) for \( i < 0 \) and for \( i > N \). \( p \) is defined to be the block order of the transfer function \( H \) such that \( Y = HX \), where \( Y \) and \( X \) are the \( Z \)-transforms of the output and input blocks, respectively.

The basic building block in the realization scheme is the block digital two pair with \( L \times 1 \) input vectors \( X_1 \) and \( X_2 \), and \( L \times 1 \) output vectors \( Y_1 \) and \( Y_2 \). The inputs and the outputs are related by the chain matrix:

\[ \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} \]

or by the transfer matrix:

\[ \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ Y_2 \end{bmatrix} \]

The transfer parameters are related to the chain parameters by

\[ T_{11} = CA^{-1}, \quad T_{12} = (D-CA^{-1}B), \quad T_{21} = A^{-1}, \quad T_{22} = -A^{-1}B \]
The block digital filter (1) will be realized by cascading \( p \) such blocks.

II. Cascaded Block Lattice and Ladder Structure

The main idea in the realization scheme is to realize the \( p \)-th order black transfer function \( G(z) \) such that

\[
Y = G_p(z)X = Y(p) = G_p(z)X(p)
\]

as a constrained two-pair where the constraining transfer function \( G_{p-1}(z) \) such that

\[
y(p-1) = G_{p-1}(z)x(p-1)
\]

is of lower block order than \( G_p(z) \). \( G_{p-1}(z) \) can then be realized as another constrained two-pair where the constraining transfer function \( G_{p-2}(z) \) is of lower block order than \( G_{p-1}(z) \). This process is continued until the constraining transfer function reduces to a constant matrix. In equation (4) \( Z = z \).

Let \( x(i) \) and \( y(i-1) \) be the input vectors and \( y(i) \) and \( x(i-1) \) be the output vectors of \( i \)-th stage. The block filter will be realized by cascading \( p \) stages. From (2) and (4) the transfer function \( G_1(z) \) such that \( Y(1) = G_1(z)X(1) \) is obtained as

\[
G_1(z) = (C_1 + D_1G_0^{-1})(A_1 + B_1G_0^{-1})^{-1}
\]

where \( A_1, B_1, C_1 \) and \( D_1 \) are the chain parameters of the \( i \)-th stage and \( G_0^{-1} \) is the constraining transfer function such that \( Y(0) = G_0^{-1}(Z)X(0) \).

From (5) we can obtain \( G_{p-1}(z) \) as

\[
G_{p-1}(z) = (D_1^{-1} - G_1 B_1)^{-1}(G_1 A_1^{-1} - C_1)
\]

The general procedure for synthesizing a block \( p \)-th order transfer function

\[
G_p(z) = \left( \sum_{i=0}^{p-1} A_i p^{-1} Z^{-i} \right) \left( \sum_{i=0}^{p-1} B_i p^{-1} Z^{-i} \right)^{-1}
\]

is given below. Without loss of generality \( B_0,p \) can be assumed to be an identity matrix.

Let the chain parameters for the \( p \)-th stage be chosen to be

\[
A_p = I, \quad B_p = B_p Z^{-1}, \quad C_p = A_0,p, \quad D_p = A_p P Z^{-1}
\]

Let the constraining transfer function \( G_{p-1} \) be given by

\[
G_{p-1}(z) = \left( \sum_{i=0}^{p-1} A_i p^{-1} Z^{-i} \right) \left( \sum_{i=0}^{p-1} B_i p^{-1} Z^{-i} \right)^{-1}
\]

where \( B_0,p-1 \) is assumed to be an identity matrix. From (3), (8) and (9) we have

\[
G_p = (C + D G_{p-1})(A + B G_{p-1})^{-1} = (A_0,p + A_p p Z^{-1} G_{p-1})(1 + B_p p Z^{-1} G_{p-1})^{-1}
\]

From (7) and (10) by comparing the coefficients of the powers of \( Z^{-1} \) we get

\[
A_0,p B_k,p-1 + A_p p A_k-1,p-1 = A_k,p
\]

\[
B_k,p-1 + B_p p A_k-1,p-1 = B_k,p
\]

where \( k=1, \ldots, (p-1) \) and

\[
A_p-1,p-1 = I
\]

Solving equation (11) for \( k=1, \ldots, (p-1) \) we get

\[
A_{k-1,p-1} = (A_p p^{-1} B_k p^{-1} A_0,p B_k p^{-1})^{-1}(A_k,p A_0,p B_k,p)
\]

\[
B_{k-1,p-1} = B_k p^{-1} B_p p A_{k-1,p-1}
\]

and this defines \( G_{p-1}(Z) \) in terms of \( G_p(Z) \) provided the inverse exists. This process is repeated for \( n=p, (p-1), \ldots, 1 \) giving the equations

\[
A_n = I, B_n Z^{-1}, C_n = A_0,n, D_n = A_n Z^{-1}
\]

\[
A_{k-1,n-1} = (A_n n^{-1} A_0,n B_n,n^{-1})^{-1}(A_k,n A_0,n B_k,n)
\]

\[
B_{k-1,n-1} = B_k n^{-1} B_n,n^{-1} B_{n-1,k-1,n-1}
\]

\[
A_{n-1,n-1} = I
\]

where \( k=1, \ldots, (n-1) \). The \( i \)-th stage will then be given by

\[
\begin{bmatrix} x(i) \\ y(i) \end{bmatrix} = \begin{bmatrix} I & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} x(i-1) \\ y(i-1) \end{bmatrix}
\]

where \( B_i = Z B_i, D_i = Z D_i \) are constant matrices. This gives us the lattice structure. Using the transfer parameters we can get the ladder structure.

In this section we have presented a procedure for synthesizing any given \( p \)-th order block transfer function as a cascaded lattice or ladder structure. This lattice structure can be obtained provided the inverses in equation (13) exist.

III. A Structure Similar to Gray and Markel Structure

The transfer function (7) can also be synthesized in a way similar to the Gray and Markel procedure for synthesizing a scalar digital filter as a cascaded lattice and ladder structure [5]. In this section this procedure will be outlined for a block transfer function
\[ G_p(Z) = \left( \sum_{i=0}^{p} A_i z^{-i} \right) \left( \sum_{i=0}^{p} B_i z^{-i} \right)^{-1} \]

In [5] Gray and Markel have outlined a procedure for synthesizing a scalar digital transfer function as a cascaded lattice and ladder structure. They have proposed two procedures, one of which yields two multipliers per stage and the other one, a single multiplier per stage. These structures have the advantage of an internal scaling property.

We wish to design a block digital filter that will implement the transfer function (7), which is given above for convenience. The block lattice and ladder filter will be formulated as follows.

Let \( A_p(Z) \) and \( B_p(Z) \) denote the matrix polynomials

\[ A_p(Z) = \sum_{n=0}^{p} A_{n,p} z^{-n} \tag{14} \]
\[ B_p(Z) = \sum_{n=0}^{p} B_{n,p} z^{-n} \tag{15} \]

where \( B_{0,p} = 1 \). From \( B_p(Z) \) we can define a block \( p \)-th order matrix polynomial \( P_p(Z) \) as

\[ P_p(Z) = Z^{-p}B_{p}(Z^{-1}) = \sum_{n=0}^{p} B_{n,p} z^{-n} \]
\[ = \sum_{n=0}^{p} B_{n,p} z^{-n} \tag{16} \]

Let \( K = B_{p,p} \); then \( B_p(Z) - K P_p(Z) \) is of block order \( (p-1)^2 \).

Defining \( B_{p-1}(Z) \), a matrix polynomial of block order \( (p-1) \), by the equation

\[ (1-K^2)B_{p-1} = B_p - K P_p \tag{17} \]

where for simplicity the \( Z \)-dependence of the variables is not explicitly shown and \( P_{p-1}(Z) \) by

\[ P_{p-1}(Z) = Z^{-(p-1)}B_{p-1}(Z^{-1}) \tag{18} \]

we get

\[ (1-K^2)P_{p-1} = B_{p-1} - K P_{p-1} \tag{19} \]

Solving the simultaneous matrix equations (17) and (19) gives us,

\[ B_p = B_{p-1} + K Z^{-1} P_{p-1} \]
\[ P_p = K B_{p-1} + Z^{-1} P_{p-1} \tag{20} \]

provided \( (1-K^2) \) is invertible.

Also from (17) it can be easily seen that \( B_{0,p} = I \).

Hence, the same process can be repeated to get \( B_{p-1} \) and \( P_{p-1} \) from \( B_{p-2} \) and \( P_{p-2} \) and so on.

From (7), (14) and (15) we have

\[ G_p(Z) = A_p(Z)[B_p(Z)]^{-1} \tag{22} \]

as the transfer function to be implemented. Using (14) and (16) and letting

\[ H_p = A_p P_p \]

\[ [A_p(Z) - H_p P_p] \]

will be a matrix polynomial of block order \( (p-1) \). This will be defined to be the polynomial \( A_{p-1}(Z) \). This process can also be repeated. Hence, we have

\[ A_{p-1} = A_p - H_p P_p \tag{23} \]

Therefore

\[ A_p = A_{p-1} + H_p P_p \]

\[ = A_{p-2} + H_{p-1} P_{p-1} + H_p P_p \]
\[ = \sum_{i=0}^{p} H_i P_i \tag{24} \]

where \( H_0 = 0, 0 = A_0(Z) \) and \( P_0 = I \).

The transfer function \( G_p(Z) \) will now be given by

\[ G_p(Z) = \sum_{i=0}^{p} H_i P_i \tag{25} \]

and the output block \( Y \) is given in terms of input block \( X \) as

\[ Y = \sum_{i=0}^{p} H_i P_i B_i^{-1} X \tag{26} \]

Hence the output is obtained as a weighted sum of vectors \( P_i B_i^{-1} X, \) \( i=1,2,\ldots,p \).

The tap parameters \( H_i \) and \( K_i \) can be obtained recursively from \( A_p(Z) \) and \( B_p(Z) \) using the following equations for \( m>p, p-1,\ldots, 1 \):

\[ P_m(Z) = Z^{-p}B_m(Z^{-1}) \tag{27} \]
\[ K_m = B_m, m \tag{28} \]
\[ B_m(Z) = (I-K_m^{-2})^{-1}[B_m(Z) - K_m P_m(Z)] \tag{29} \]
\[ H_m = A_m, m \tag{30} \]
\[ A_m(Z) = A_m(Z) - H_m P_m(Z) \tag{31} \]

provided \( (I-K_m^{-2})^{-1} \) exists. From (27)-(29), it can easily be shown that

\[ \begin{bmatrix} B_m \\ P_m \end{bmatrix} = \begin{bmatrix} I & K_m \\ K_m & I \end{bmatrix} \begin{bmatrix} B_{m-1} \\ P_{m-1} \end{bmatrix} \tag{32} \]

Hence

\[ \begin{bmatrix} B_m B_p^{-1} X \\ P_m B_p^{-1} \end{bmatrix} = \begin{bmatrix} I & K_m \\ K_m & I \end{bmatrix} \begin{bmatrix} B_{m-1} B_p^{-1} X \\ B_{m-1} P_{m-1} B_p^{-1} X \end{bmatrix} \tag{33} \]
Once $P^{-1}_m X$ is obtained for $m=1,\ldots,p$, the output $Y$ can be obtained as a weighted sum of these vectors. From (33) it clearly can be seen that the structure is obtained as a cascaded lattice or ladder.

The conditions on $B_p(Z)$ which will guarantee that $(I-X_m Z)$ is invertible for all $m$ (thus permitting execution of the algorithm) have yet to be established. Preliminary indications are that stability of $G_p(Z)$ will suffice.

This formulation of the block lattice and ladder does not have the advantage of internal scaling property, the scalar formulation has. But this formulation does reduce to Gray and Markel's form with the advantage of internal scaling when the block length is unity.

IV. Block Lattice and Ladder Structure with Internal Scaling

Our aim is to generate block lattice structures which exhibit an internal scaling property as in the scalar structure proposed by Gray and Markel. One way of getting the block lattice structure with scaling is to obtain block versions of each of the polynomials obtained in the scalar case, which will be the same as those in the previous section for a block length of unity.

The block transfer function we would like to implement is given by

$$G_p(Z) = \left( \sum_{i=0}^p A_i Z^{-i} \right) \left( \sum_{i=0}^p B_i Z^{-i} \right)^{-1}$$

where $A_i$ and $B_i$ are of the form of $A_i$ and $B_i$ of (1). For simplicity we assume that $pL=N$. If this is not the case zero coefficients can be added to the scalar transfer function.

Let

$$A_p(Z) = \sum_{i=0}^p A_i Z^{-i}$$
$$B_p(Z) = \sum_{i=0}^p B_i Z^{-i}$$

If the corresponding scalar transfer function is

$$h_N(z) = \left( \sum_{i=0}^N a_i z^{-i} \right) \left( \sum_{i=0}^N b_i z^{-i} \right)^{-1}$$

where $b_{0,N}$ is assumed to be unity, then $B_p(Z)$ is given by (36) where

$$B_p(Z) = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} B_{1,p} = \begin{bmatrix} b_{1L,N} & \cdots & b_{1L+1,N} \\ \vdots & \ddots & \vdots \\ b_{1L+1,N} & \cdots & b_{1L+1,N} \end{bmatrix}$$

for $i=1,\ldots,p$.

Let $a_N(z)$ and $b_N(z)$ be the numerator and denominator polynomials of $b_N(z)$. Then $B_p(Z)$ is the block version of the polynomial $b_N(z)$.

If $P_p(Z)$ is obtained from $B_p(Z)$ as

$$P_p(Z) = Z P_p t(Z^{-1})$$
then $P_p(Z)$ will be the block version of $P_N(Z)$ obtained from $b_N(z)$ by the relation

$$P_N(z) = z^{-N} b_N(z^{-1})$$

Defining $\gamma_1 = (P_0, P_1, \ldots, P_p)$, $\theta_1 = \frac{1}{1-\gamma_1}$ and

$$\begin{bmatrix} B_p(1) \\ P_p(1) \end{bmatrix} = \begin{bmatrix} \alpha_i I & \beta_i I \\ \beta_i J & \alpha_i J \end{bmatrix} \begin{bmatrix} B_p \\ P_p \end{bmatrix}$$

where $J = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ and $J^L = Z I$

we find that $B_p(1)$ and $P_p(1)$ will be the block versions of the scalar polynomials $b_{N-L}(z)$ and $P_{N-L}(z)$ obtained using the scalar Gray and Markel lattice formulation. Repeating this process $L$ times we get

$$\begin{bmatrix} B_p(L) \\ P_p(L) \end{bmatrix} = \begin{bmatrix} M_{1p} & M_{2p} \\ M_{3p} & M_{4p} \end{bmatrix} \begin{bmatrix} B_p \\ P_p \end{bmatrix}$$

and

$$P_{p-1}(Z) = Z^{-(p-1)} b_{N-L}(Z^{-1})$$

and

$$M_4 = ZM_{2p} t(Z^{-1}) = ZM_{1p}$$
$$M_3 = ZM_{2p} t(Z^{-1}) = ZM_{1p}$$

where $M^* = M^t(Z^{-1})$. Hence

$$\begin{bmatrix} P_{p-1} \\ P_{p-1} \end{bmatrix} = M_{1p} M_{2p} \begin{bmatrix} B_p \\ P_p \end{bmatrix}$$

Equation (46) can be rewritten as

$$\begin{bmatrix} P_{p-1} \\ P_{p-1} \end{bmatrix} = \frac{1}{v_p} \begin{bmatrix} M_{1p} - M_{2p} Z^{-1} \\ -M_{2p} M_{1p} Z^{-1} \end{bmatrix} \begin{bmatrix} B_p \\ P_p \end{bmatrix}$$

where $v_p$ is a constant such that

$$M_{1p} M_{2p} - M_{2p} M_{1p} = v_p I$$
Now we can obtain $A_{p-1}$ from $A_p$ by the relation

$$A_{p-1} = A_p - H_p P \cdot P$$

where

$$H_p = P \cdot P^{-1} A_p P$$

(49)

It can be proved that $P_{p-1}$ is invertible. This process can be repeated giving the relation

$$A_p = \sum_{i=0}^{p} H_i P_i$$

(50)

The output block $Y$ will therefore be obtained from the input block $X$ as

$$Y = \sum_{i=0}^{1} H_i P_i B_{i-1} X$$

where $P_i B_{i-1} X$ is obtained from the equation

$$P_i B_{i-1} X = \frac{1}{2} \begin{bmatrix} M_{1i}^* & M_{zi} Z^{-1} \\ M_{zi} & M_{2i} Z^{-1} \end{bmatrix} \begin{bmatrix} B_{i-1} P_{i-1} \end{bmatrix}$$

(52)

for $i=1, \ldots, p$, where $M_{1i}^*$ and $M_{2i}$ are matrix polynomials of order 1 and $B_0 = P_0 = I$. Equation (52) gives the lattice structure and the output is obtained as a weighted sum of $P_i B_{i-1} X$, $i=1, \ldots, p$.

Equation (41) can also be written as

$$\begin{bmatrix} B_p \\ P_p \end{bmatrix} = \begin{bmatrix} I & \gamma_1 J^{-1} \end{bmatrix} \begin{bmatrix} B_p \\ P_p \end{bmatrix}$$

(53)

where

$$J^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & Z^{-1} \\ 1 & 0 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

(54)

This process can be repeated giving us

$$\begin{bmatrix} B_p \\ P_p \end{bmatrix} = \begin{bmatrix} I & \gamma_1 J^{-1} \\ \gamma_1 I & J^{-1} \end{bmatrix} \cdots \begin{bmatrix} I & \gamma_L J^{-1} \\ \gamma_L I & J^{-1} \end{bmatrix} \begin{bmatrix} B_{p-1} \\ P_{p-1} \end{bmatrix}$$

(55)

If, instead of multiplying these matrices we implement (55) as a cascaded section, each of the subsections will be a lattice.

This formulation will give scaled internal variables as these are block versions of the internal variables in the scalar structure. This can be shown to give a stable filter provided $|\gamma_1| < 1$.

V. Conclusion

In this paper we have presented some new algorithms to obtain lattice and ladder structures of a block digital filter. Section II gives a general method of synthesizing a block digital filter in lattice and ladder form. In section III lattice and ladder structures similar to the scalar Gray and Markel structures are discussed. Both of these structures are unscaled structures. But the formulation is easy. Finally, in section IV we have developed an algorithm to obtain a structure in which the internal variables are scaled.

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