TIME DELAYS IN LARGE-SCALE SYSTEMS

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ABSTRACT

The paper examines classes of large scale systems problems which have been tackled by methods associated with M-matrices and vector Lyapunov functions, and shows that the insertion of time delays into the interconnection terms will preserve exponential stability, assuming that such stability is present when no time delays exist. The key is to use new types of Lyapunov-like functions for analysing the behaviour of the delay-differential equations which are encountered. These functions allow the derivation of results for time-varying systems and systems including delay functions other than a pure delay.

1. Introduction

Among the various methods so far suggested for tackling large scale systems problems is one which brings to bear the ideas of M-matrices, often through the vehicle of vector Lyapunov functions,[1-4]. The basic idea is often to set up Lyapunov like functions $v_1, v_2, \ldots, v_n$ for a collection of n separate subsystems and then to show that, after interconnection of the subsystems, the v_i are majorized by quantities w_i satisfying an equation set of the form

$$
\begin{aligned}
\dot{w}_1 &= -\alpha_{11}w_1 + \alpha_{12}w_2 + \ldots + \alpha_{1n}w_n \\
\dot{w}_2 &= \alpha_{21}w_1 - \alpha_{22}w_2 + \ldots + \alpha_{2n}w_n \\
&\vdots \\
\dot{w}_n &= \alpha_{n1}w_1 + \alpha_{n2}w_2 + \ldots - \alpha_{nn}w_n
\end{aligned}
\tag{1.1}
$$

with the α_{ii} positive and α_{ij} nonnegative constants. Stability of (1.1) can often be guaranteed by simple requirements, e.g. the row dominance property

$$\alpha_{ii} - \sum_{j\neq i} \alpha_{ji} > 0 \tag{1.2}$$

for all i, or the requirement that the negative of the matrix on the right side of (1.1) be an M-matrix [1-4].

The majorization idea permits one to develop "connective stability" results, which mathematically are statements that the majorization is preserved under certain classes of perturbation in the v_i equations; these perturbations correspond to adjustment to the interconnection between the individual subsystems.

The main theme of this paper is that another form of perturbation is possible in terms of the large scale system paradigm: introduction of time delays in the interconnection does not destroy stability. In terms of (1.1), replacement of any term $\alpha_{ij}w_j(t)$ $(i \neq j)$ by $\alpha_{ij}w_j(t-\tau_{ij})$ for some $\tau_{ij} > 0$ does not destroy stability. Of course, for small time delays this result is a standard one[5,6]. But restrictions of smallness will not be applied here.

Some attempt has been made already, e.g.[7,8], to develop the types of ideas discussed in this paper with the aid of Lyapunov theory. However, because of the choice of Lyapunov-like functions* in these works, the results appear to be more conservative. Another approach can be found in [9], which in fact offers a proof of the result stated above, at least for the case when the coefficient matrix in (1.1) defined by the α_{ij} is irreducible. However, the approach of [9] is inextricably tied to constant α_{ii} and α_{ij}, a fact which is initially assumed for clarity in our treatment, and then easily dispensed with.

We turn now to the structure of the paper. Section 2 contains the main theorem, to the effect that with the α_{ii} positive and α_{ij} nonnegative constants in (1.1) and with (1.2) holding, constant time delays in the off-diagonal terms will not affect stability. The theorem is developed with the aid of a number of Lemmas which introduce a majorization result and demonstrate a novel type of Lyapunov-like function.

In Section 3, we consider a number of variants on the ideas of the main theorem, illustrating for example possibilities of time variation, connective stability, and more complicated delay functions. Throughout this section, the time delays themselves are kept constant. In Section 4, we consider the issue of variable time delays. As one would expect, if the rate of time variation is slow enough, there is no difficulty. For larger rates of time variation however, we are also able to establish a result.

Section 5 confirms the national supposition that the ideas will apply to interconnected systems with delays in the interconnection, through the medium of vector Lyapunov functions. The calculations are straightforward, and are illustrative rather than exhaustive in that we focus on one of the many problems which reduce to looking at (1.2) or an equivalent in the no-time-delay case. The general pattern should be quite clear from the illustration.

Besides the general tenor of the results - that insertion of time delays in certain situations does not matter in a qualitative sense - perhaps the most relevant parts of the paper are those exhibiting the Lyapunov-like functions. These functions seem both novel, and powerful in their application. The general conclusion about the tolerance of time delay is, in a sense, a welcome one, since large scale systems are often envisaged as involving time-delays between the subsystems, often on account of information processing requirements.

2. The Fundamental Result

The material of the later sections hinges mainly on the statement, but in part also on the techniques for proof, of the following main theorem. The theorem in effect states that the insertion of time delays into a differential equation system with connective exponential stability does not destroy the exponential stability property.

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* Certain functions are termed "Lyapunov like" in the paper for one or two reasons. They fulfill all requirements of a Lyapunov function save for continuous differentiability everywhere, or they act as a Lyapunov function only for a restricted set of initial conditions. Naturally, Lyapunov theorems cannot be applied blindly, given such functions, although often modest adjustment suffice.

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Theorem: For $i, j = 1, 2, \ldots, n$, let the constants α_{ij} satisfy

$$\alpha_{ii} > 0 \quad \alpha_{ij} \geq 0 \quad \alpha_{ii} - \sum_{j \neq i} \alpha_{ji} > 0 \tag{2.1}$$

and let the constants τ_{ij} satisfy

$$\tau_{ij} \geq 0 \tag{2.2}$$

Then the equation set

$$\begin{aligned} \dot{z}_1 &= -\alpha_{11} z_1(t) + \alpha_{12} z_2(t-\tau_{12}) + \ldots + \alpha_{1n} z_n(t-\tau_{1n}) \\ \dot{z}_2 &= \alpha_{21} z_1(t-\tau_{21}) - \alpha_{22} z_2(t) + \ldots + \alpha_{2n} z_n(t-\tau_{2n}) \\ &\vdots \\ \dot{z}_n &= \alpha_{n1} z_1(t-\tau_{n1}) + \alpha_{n2} z_2(t-\tau_{n2}) + \ldots - \alpha_{nn} z_n(t) \end{aligned} \tag{2.3}$$

is exponentially stable.

Remark: The notion of exponential stability is a standard one for linear differential delay equations,[5,6]. Solutions of (2.3) for $t > t_0$ are uniquely determined by initial points

$$\lim_{t \to t_0 + 0} z_i(t) = z_i^0(t+)$$

and initial functions

$$z_i(t) = z_i^0(t) \qquad t_0 - \max_j \tau_{ji} \leq t \leq t_0$$

with the initial functions assumed continuous and bounded. It is helpful to take $z_i^0(t_0+) = z_i^0(t_0-)$. We denote solutions of the equation by $z(t, Z^0)$, where Z^0 denotes the initial point and initial function data. Let

$$\rho[z(t)] = \sup |z_i(t-\tau), \ 0 \leq \tau \leq \max_j \tau_{ji}, \ 1 \leq i \leq n|$$

Exponential stability is then equivalent to the condition

$$\rho[z(t, Z^0)] \leq b\rho[z(t_0)]\exp^{-a(t-t_0)}$$

for some positive a and b.

The proof of the theorem depends on several lemmas. The first two are concerned with developing a majorizing set of equations.

Lemma 1: Under the same hypotheses as in the theorem statement equation (2.3) imply

$$\begin{aligned} \frac{d}{dt}|z_1(t)| &\leq -\alpha_{11}|z_1(t)| + \alpha_{12}|z_2(t-\tau_{12})| + \ldots + \alpha_{1n}|z_n(t-\tau_n)| \\ &\vdots \\ \frac{d}{dt}|z_n(t)| &\leq \alpha_{n1}|z_1(t-\tau_{n1})| + \alpha_{n2}|z_2(t-\tau_{n2})| + \ldots - \alpha_{nn}|z_n(t)|. \end{aligned} \tag{2.4}$$

where $\frac{d}{dt}|z_i(t)|$ is $\dot{z}_i$ if $z_i \geq 0$, and $-\dot{z}_i$ if $z_i < 0$.

Proof is by direct calculation. We remark that if all initial conditions in (2.3) are nonnegative, $z_i(t) = |z_i(t)|$ for all t and i, and equality holds in (2.4).

Next, we set up a delay-differential equation set to obtain a comparison result.

Lemma 2: Assume the same hypotheses as in the theorem statement, and consider the following equation set, obtained by replacing inequalities in (2.4) by equalities:

$$\begin{aligned} \frac{d}{dt} y_1(t) &= -\alpha_{11} y_1 + \alpha_{12} y_2(t-\tau_{12}) + \ldots \alpha_{1n} y_n(t-\tau_{1n}) \\ &\vdots \\ \frac{d}{dt} y_n(t) &= \alpha_{n1} y_1(t-\tau_{n1}) + \alpha_{n2} y_2(t-\tau_{n2}) + \ldots - \alpha_{nn} y_n(t) \end{aligned} \tag{2.5}$$

and with initial conditions for (2.4) and (2.5) identical. Then $y_i(t) \geq |z_i(t)|$ for all i.

The preceding lemma shows that if we can prove exponential stability of (2.5), then exponential stability of (2.3) follows. Accordingly, we shall consider (2.5). Note that the initial condition assumptions on (2.5) actually limit the entries of $y(\cdot)$ over the appropriate initial time interval to being nonnegative, and that $y_i(t)$ for all t is also nonnegative.

Lemma 3: With the same hypotheses as in Lemma 2, let $y_i(t)$ satisfy (2.5), and define

$$V = \sum_i y_i + \sum_{\substack{i,j \\ i \neq j}} \int_{t-\tau_{ij}}^{t} \alpha_{ij} y_j(s) ds \tag{2.6}$$

Then

$$\dot{V} = -\sum_i [\alpha_{ii} - \sum_{j \neq i} \alpha_{ji}] y_i \tag{2.7}$$

Proof is by direct calculation.

The next task is to exhibit a bound away from zero of $\dot{V}/V$.

Lemma 4: Assume the same hypotheses as for Lemma 3. Also define positive constants k_1, k_2, k_3, k_4 by

$$k_1 = \max_i \alpha_{ii} \quad k_2 = \max_{\substack{i,j \\ i \neq j}} \tau_{ij}$$

$$k_3 = \max_{\substack{i,j \\ i \neq j}} \alpha_{ij} \quad k_4 = \min_i [\alpha_{ii} - \sum_{j \neq i} \alpha_{ji}] \tag{2.8}$$

and let

$$W(t) = \sum_i y_i(t) \tag{2.9}$$

Then for $t \geq k_2 + t_0$

$$0 \leq W \leq V \leq [1 + (n-1)k_3(e^{k_1 k_2} - 1)]W \tag{2.10}$$

and

$$\dot{V}/V \leq -\frac{k_4}{1 + (n-1)k_3(e^{k_1 k_2} - 1)} \tag{2.11}$$

Proof: Omitted due to space limitations.

Lemma 5: With the same hypothesis as Lemma 2, all solutions of (2.5) with nonnegative initial conditions are exponentially stable.

Proof: Omitted due to space limitations.

3. Linear Equation Modifications Preserving Constant Time Delays

In this section, we illustrate a number of minor variations on the main theorem of the previous section.

Scaling of the z_i. The main theorem is made to look a bit more mysterious in the event that the matrix

$$A = \begin{bmatrix} -\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & -\alpha_{22} & \ldots & \alpha_{2n} \\ \alpha_{n1} & \alpha_{n2} & \ldots & -\alpha_{nn} \end{bmatrix} \tag{3.1}$$

of (1.1) and (2.3) has all $\alpha_{ij} \geq 0$, $\alpha_{ii} > 0$, positive even order and negative odd order principal minors. (Thus $-A$ is an M-matrix). But then a scaling of the z_i, equivalent to replacement of A by a matrix in which the row dominance property $\alpha_{ii} - \sum_{j \neq i} \alpha_{ji} > 0$ holds, will recover the theorem. This idea is widely used, see [1, 3].

Time variable α_{ij}. Only trivial adjustments are required to copy with time variable α_{ij} in (2.3). Besides a smoothness and boundedness assumption, one requires uniform positivity of $\alpha_{ii} - \sum_{j \neq i} \alpha_{ji}$. The exponential rate constructed in Lemma 4 of course reflects this time variation.

This tolerance of time variation emphasises the comparative inappropriateness of frequency domain stability criteria for (2.3).

Connective Stability. If any α_{ij} for $i \neq j$ is replaced by $e_{ij}(t)\alpha_{ij}$ where $-1 \leq e_{ij}(t) \leq 1$ for all t in (2.3) but not in (2.1), the conclusion of the theorem is still valid. It is easy to show that the solutions of (2.5) again majorize the solutions of the defined (2.3). Moreover, stability of (2.5) easily follows as a necessary condition for connective stability.

Further time delay terms. One can conceive of more complicated ways of introducing time delay than those reflected in (2.3). For example, the term $-\alpha_{11}z_1(t)$ could become $-\alpha_{11}^{(1)}z_1(t) - \alpha_{11}^{(2)}z_1(t-\tau_{11}^{(2)}) \dots -\alpha_{11}^{(r)}z_1(t-\tau_{11}^{(r)})$ and the term $\alpha_{12}z_2(t-\tau_{12})$ could become $\alpha_{12}^{(1)}z_2(t-\tau_{12}^{(1)}) + \alpha_{12}^{(2)}z_2(t-\tau_{12}^{(2)}) + \dots + \alpha_{12}^{(s)}z_2(t-\tau_{12}^{(s)})$. By defining

$$\alpha_{11} = \alpha_{11}^{(1)} - |\alpha_{11}^{(2)}| - \dots - |\alpha_{11}^{(r)}| \tag{3.2}$$

$$\alpha_{12} = |\alpha_{12}^{(1)}| + |\alpha_{12}^{(2)}| + \dots + |\alpha_{12}^{(s)}| \tag{3.3}$$

and then requiring (2.1) to hold, the Theorem becomes valid for the modified (2.3), this being shown with an easy modification of (2.6).

Perhaps the most interesting case is when all $\alpha_{ii}^{(k)}$ and $\alpha_{ij}^{(k)}$ are positive. Then the effect of having delay on the diagonal, i.e. delay in z_i in the equation for z_i, is suggested by (3.2) to be destabilizing, a fact that is well known and can be studied for a simple equation like $\dot{x} = ax + bx(t-h)$. On the other hand, delays in off diagonal terms, no matter how distributed, do not affect stability, i.e. in the z_1 equation, a term $\sum \alpha_{12}^{(i)} z_2(t-\tau_{12}^{(i)})$ will not introduce instability if a non-delayed version of the same term, viz., $[\sum \alpha_{12}^{(i)}]z_2(t)$ is consistent with stability.

Delay Functionals. There has been some attempt to study delay functionals, see e.g. [7, 8]. The results suggest that increase of time delay may lead to instability. This is not so, provided the "nondelayed equivalent" of the delay functional does not introduce instability, and provided the delay is not diagonal.

Both because of current interest and the somewhat greater complexity of the problem, we shall analyse the situation in a little more detail. Consider the equation

$$\begin{aligned} \dot{z}_1 &= -\alpha_{11}z_1 + \int_{t-\tau_{12}-}^{0+} d\beta_{12}(s) z_2(s+t) \\ \dot{z}_2 &= \int_{-\tau_{21}-}^{0+} d\beta_{21}(s) z_1(s+t) - \alpha_{22}z_2 \end{aligned} \tag{3.4}$$

(The n-dimentional case presents no extra difficulties in principle, and diagonal delay can be dealt with in the same manner as off diagonal delay, and will be seen to promote instability, as expected). In (3.4), β_{12} and β_{21} are generating functions of bounded variation over intervals of length τ_{12} and τ_{21}. They may include steps, which will then single out terms of the type $\alpha_{12}z_2(t-\tau_{12})$. In view of this and the remarks made in the previous subsection we should be interested in comparing (3.4) with a nondelay system

$$\begin{aligned} \dot{x}_1 &= -\alpha_{11}x_1(t) + \alpha_{12}x_2(t) \\ \dot{x}_2 &= \alpha_{21}x_1(t) - \alpha_{22}x_2(t) \end{aligned} \tag{3.5}$$

in which α_{12}= total variation of $\beta_{12}(s)$ in $[-\tau_{12}, 0]$ and similarly for α_{21}. [In case $\beta_{22}(s)$ is monotonic, this is of course nothing other than the difference in the values of $\beta_{12}(s)$ at the end points, care being taken concerning any end-point discontinuities.]

Accordingly, suppose

$$\alpha_{11} > 0 \quad \alpha_{22} > 0 \quad \alpha_{11} - \alpha_{21} > 0 \quad \alpha_{22} - \alpha_{12} > 0 \tag{3.6}$$

with

$$\alpha_{12} = V(\beta_{12}) \Big|_{-\tau_{12}-}^{0+} \qquad \alpha_{21} = V(\beta_{21}) \Big|_{-\tau_{21}-}^{0+} \tag{3.7}$$

It is a standard result of analysis [12, p. 85] that we can write $\beta_{12}(s) = \gamma_{12}(s) - \delta_{12}(s)$ where $\gamma_{12}(s)$, $\delta_{12}(s)$ are both bounded, nonnegative and nondecreasing with $\gamma_{12}(-\tau_{12}-) = \delta_{12}(-\tau_{12}-) = 0$. Moreover, $V(\beta_{12}) \Big|_{-\tau_{21}-}^{0+} = \gamma_{12}(0+) + \delta_{12}(0+) = \alpha_{12}$ and similarly for $\beta_{21}(s)$. Equation (3.4) then yields

$$\frac{d}{dt}|z_1| \leq -\alpha_{11}|z_1| + \int_{-\tau_{12}-}^{0+} d\gamma_{12}(s)|z_2(s+t)| + \int_{-\tau_{12}-}^{0+} d\delta_{12}(s)|z_2(s+t)|ds$$

$$\frac{d}{dt}|z_2| \leq \int_{-\tau_{21}-}^{0+} d\gamma_{21}(s)|z_1(s+t)| + \int_{-\tau_{21}-}^{0+} d\delta_{21}(s)|z_1(s+t)|] - \alpha_{22}|z_2| \tag{3.8}$$

and it is not hard to show that if we define $y_i(t)$ by

$$\begin{aligned} \frac{dy_1}{dt} &= -\alpha_{11}y_1(t) + \int_{-\tau_{21}-}^{0+} d[\gamma_{12}(s) + \delta_{12}(s)]y_2(s+t) \\ \frac{dy_2}{dt} &= \int_{-\tau_{21}-}^{0+} d[\gamma_{21}(s) + \delta_{21}(s)]y_1(s+t) - \alpha_{22}y_2(t) \end{aligned} \tag{3.9}$$

with the same initial conditions, then $|z_i(t)| \leq y_i(t)$ for $i = 1, 2$ and all t. With $\varepsilon_{ij}(s) = \gamma_{ij}(s) + \delta_{ij}(s)$, we study the stability of (3.9) via the Lyapunov-like function

$$V = y_1(t) + y_2(t) + \int_{-\tau_{12}-}^{0+} d\varepsilon_{12}(s) \int_{t+s}^{t} y_2(s_1)ds_1 + \int_{-\tau_{21}-}^{0+} d\varepsilon_{21}(s) \int_{t+s}^{t} y_1(s_1)ds_1$$

(In case $\varepsilon_{12}(s)$, $\varepsilon_{21}(s)$ are step functions, this formula reduces to one obtained in the previous section). Then $\dot{V} = -(\alpha_{11} - \alpha_{21})y_1(t) - (\alpha_{22} - \alpha_{12})y_2(t)$

From this point on, the argument proceeds much as before. The conditions (3.6), just as they guarantee exponential stability of (3.5), guarantee exponential stability of the distributed version (3.9), and in turn of (3.4), which (3.9) majorizes.

4. Linear Equation Modification Permitting Nonconstant Time Delays

In the preceding two sections, all time delays have been considered constant. The question arises as to whether results can also be obtained for variable time delays. When the α_{ii} and α_{ij} are constant, it turns out that variable time delays pose no problem, provided that they are bounded, and provided that the coefficient matrix associated with the α_{ij} in (2.3) is irreducible, [9]. There seems no way the proof technique of [9] could be extended to, for example, remove the constancy requirement.

By setting up differential equations for $\dot{x}_i$, rather than x_i, one can with a little difficulty capture the result of [9]. However, two types of result can be more easily obtained. One shows that exponential stability is preserved if time delay variations are sufficiently slow. The other shows that if the delay-differential equations are adjusted appropriately, large rates of time delay variation are permitted. This adjustment will, on some occasions, make good physical sense.

Finally, we present an example of a system with time-varying α_{ij} and time-varying delays, and show that although the α_{ij} satisfy a dominance condition, the system is unstable. This shows the futility of seeking any form of generalization, without some significant modification of the earlier results of this paper (for time-varying α_{ij} and constant delays) or of the results of [9] (for constant α_{ij} and variable delays) to cope with simultaneous α_{ij} and time delay variation.

To begin with, we shall prove a result for slow time delays. For convenience, we shall consider the case of $n = 2$ and the simple equation

$$\begin{aligned} \dot{z}_1 &= \alpha_{11}z_1(t) + \alpha_{12}z_2[t - \tau_{12}(t)] \\ \dot{z}_2 &= \alpha_{21}z_1[t - \tau_{21}(t)] - \alpha_{22}z_2(t) \end{aligned} \tag{4.1}$$

We stress that the various refinements discussed in Section 3 can all be built in with little or no trouble, including time-variability of α_{11}, etc. We assume that

$$\alpha_{11} > 0 \quad \alpha_{22} > 0 \quad \alpha_{11} - \alpha_{21} > 0 \quad \alpha_{22} - \alpha_{12} > 0 \tag{4.2}$$

We shall also assume

$$0 \le \tau_{ij}(t) \le L_1, \left|\frac{d\tau_{ij}(t)}{dt}\right| < L_2 \tag{4.3}$$

for some constants L_1, L_2. Further restrictions on $\frac{d\tau_{ij}(t)}{dt}$ will be imposed subsequently.

There is of course no difficulty in introducing a majorizing equation. Let us assume this is done, so that we are now studying (4.1) with nonnegative initial conditions. Set

$$V = z_1(t) + z_2(t) + \int_{t-\tau_{12}(t)}^{t} \alpha_{12}z_2(s)ds + \int_{t-\tau_{21}(t)}^{t} \alpha_{21}z_1(s)ds \tag{4.4}$$

Then

$$\begin{aligned} \dot{V} &= -(\alpha_{11} - \alpha_{21})z_1(t) - (\alpha_{22} - \alpha_{12})z_2(t) \\ &\quad + \alpha_{12}\frac{d\tau_{12}(t)}{dt} z_2[t-\tau_{12}(t)] \\ &\quad + \alpha_{21}\frac{d\tau_{21}(t)}{dt} z_1[t-\tau_{21}(t)] \end{aligned} \tag{4.5}$$

For $t \ge t_0 + \max(\tau_{12}, \tau_{21})$, there is a maximum exponential rate of decay (α_{11} and α_{22} are the exponents) for z_1 and z_2, e.g. $z_2[t-\tau_{21}(t)] \le e^{\alpha_{11}\tau_{21}(t)}z_2(t) \le e^{\alpha_{11}L_1}z_2(t)$, and $z_1[t-\tau_{12}(t)] \le e^{\alpha_{22}L_1}z_1(t)$. Thus

$$\begin{aligned} \dot{V} &\le -(\alpha_{11} - \alpha_{21} - \alpha_{21}\frac{d\tau_{21}(t)}{dt} e^{\alpha_{11}L_1})z_1 \\ &\quad -(\alpha_{22} - \alpha_{12} - \alpha_{12}\frac{d\tau_{12}(t)}{dt} e^{\alpha_{22}L_1})z_2 \end{aligned} \tag{4.6}$$

This will imply

$$\dot{V} \le -\beta[z_1 + z_2] \tag{4.7}$$

for some $\beta(t) > \varepsilon > 0$ if $\frac{d\tau_{ij}(t)}{dt}$ are negative, or positive but suitably small. Arguments as in Section 2 will then establish stability. It is also evident that one could adjust the constraint on $\frac{d\tau_{ij}(t)}{dt}$ to one involving average rates of variation of these quantities.

For the second result of this section, we postulate that $z_1(\cdot)$ and $z_2(\cdot)$ represent some form of physical entity with flows from one part to the other, via some delaying mechanism, see Fig. 1, together with an instantaneous flow to the rest of the universe. The delay is postulated to occur as a result of a physical storage mechanism, rather than, for example, a delay in the communicating of information. Assume for the moment the delays are constant. Thus, crudely, one can conceive of a pipe into which, at time t, $\alpha_{12}(t)z_2(t)$ units of z_2 flow per second, and at the other end, at the same time, $\alpha_{12}(t-\tau_{12})z_2(t-\tau_{12})$ units of z_2 flow out per second.

With this form of physical situation, (4.1) with constant delays is the correct model. But as soon as the delays are time-varying, a slight adjustment is required. This can be derived in the following way. The pipe which stores entity z_2 has stored within it at time t

$$S(t) = \int_{t-\tau_{12}(t)}^{t} \alpha_{12}(s)z_2(s)ds$$

whence

$$\begin{aligned} \dot{S} &= \alpha_{12}(t)z_2(t) - \alpha_{12}[t-\tau_{12}(t)]z_2[t-\tau_{12}(t)] + \alpha_{12}[t-\tau_{12}(t)] \\ &\quad \times \frac{d\tau_{12}}{dt} z_2[t-\tau_{12}(t)] \end{aligned}$$

In other words, while $\alpha_{12}(t)z_2(t)$ is still the inflow term, the outflow term is now $\alpha_{12}[t-\tau_{12}(t)]z_2[t-\tau_{12}(t)]$ $-\alpha_{12}[t-\tau_{12}(t)]\frac{d\tau_{12}}{dt}[t-\tau_{12}(t)]$ and (4.1) is accordingly replaced by

$$\begin{aligned} \dot{z}_1 &= -\alpha_{11}(t)z_1(t) + \alpha_{12}[t-\tau_{12}(t)][1 - \frac{d\tau_{12}(t)}{dt}]z_2[t-\tau_{12}(t)] \\ \dot{z}_2 &= \alpha_{21}[t-\tau_{21}(t)][1 - \frac{d\tau_{21}(t)}{dt}]z_1[t-\tau_{21}(t)] - \alpha_{22}(t)z_2(t) \end{aligned} \tag{4.8}$$

For the outflow term to make physical sense it must always correspond to a nonnegative quantity. More correctly therefore, the outflow term should be

$$\alpha_{12}[t-\tau_{12}(t)]\max\{[1 - \frac{d\tau_{12}(t)}{dt}], 0\}z_2[t-\tau_{12}(t)]$$

with corresponding adjustment in (4.8). To avoid notational complications, let us simply suppose that

$$-L_2 \le \frac{d\tau_{ij}(t)}{dt} \le 1 \tag{4.9}$$

Now to study (4.8), under assumptions (4.2) and (4.9), we can again introduce a majorizing equation, and this is equivalent to studying (4.8) with nonnegative initial conditions. With the Lyapunov-like function V of (4.4), we derive

$$\dot{V} = -[\alpha_{11}(t) - \alpha_{21}(t)]z_1(t) - [\alpha_{22}(t) - \alpha_{12}(t)]z_2(t) \tag{4.10}$$

and from this point on, arguments like those of Section 2 serve to demonstrate the required exponential stability.

Last, we shall illustrate an instability possibility. We shall exhibit a system with time-variable α_{ij} and time delays with exponential instability. Though the α_{ij} will not satisfy a dominance condition, an arbitrary small perturbation will ensure satisfaction of the dominance condition. This suffices to guarantee the existence of an unstable system with the indicated time-variations and with the α_{ij} satisfying the dominance condition.

Let $z_1(0) = z_2(0) = 1$, and over $[0, 1)$,

$$\dot{z}_1 = z_2(0)$$

$$\dot{z}_2 = -z_2$$

[Thus $\alpha_{12} = \alpha_{22} = 1$, $\alpha_{11} = \alpha_{12} = 0$, $t - \tau_{12}(t) = 0$ for all $t \in [0, 1)$]. There results $z_1(1) = 2$, $z_2(1) = e^{-1}$. Now over the interval $[1, T)$ with T specified below suppose that

$$\dot{z}_1 = -z_1$$

$$\dot{z}_2 = z_1(1)$$

[Now, $\alpha_{11} = \alpha_{12} = 1$, $\alpha_{12} = \alpha_{22} = 0$, $t - \tau_{21}(t) = 1$. Then $z_1(T) = 2e^{-(T-1)}$ and $z_2(T) = e^{-1} + 2(T-1)$. By taking $T = 1.452$ we obtain $z_1(T) = z_2(T) = 1.272$.

Evidently, over an interval of length T, both z_1 and z_2 have grown. Clearly, the growth can be duplicated over $[T, 2T]$, $[2T, 3T]$ etc, and exponential instability is manifest. Also, it is clear that not only can the α_{ij} be perturbed by an arbitrarily small amount to meet the dominance requirement, but they can be made to vary smoothly, as can the τ_{ij} without the exponential instability being lost.

5. Modifications Involving Nonlinear Equations

The stability properties of interconnected separately stable systems can sometimes be examined with the aid of linear differential equations or inequalities for the Lyapunov functions of the separate systems; this idea has been a constant theme in a number of works, e.g. [2, 11]. We show here that when the stability of interconnected systems can be deduced in this way, then introduction of time delays in the interconnections will not affect the stability.

The earlier work has been concerned with pointing out that a connective stability property also holds, that is the gain linking the separate systems can be varied between -1 and $+1$. In the light of this tolerance of both time delays and gain variation it is natural to ask what other forms of adjustment would be tolerable. Moylan [pivate communication] has suggested an asymptotic stability result which simply demands that any form of adjustment definable by an operator with norm ("gain") ≤ 1, but this result so far does not extend to exponential stability.

Let us suppose that there are n linear systems with possibly nonlinear interconnection such that after application of a local stabilizing control to each system, one has

$$\dot{x}_i = A_i x_i + h_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots x_n) \quad i = 1, 2, \ldots, n \tag{5.1}$$

(Each x_i is, in general, a vector rather than scalar) For the system $\dot{x}_i = A_i x_i$, there is a positive definite quadratic form $x_i' P_i x_i$ whose derivative along trajectories of $\dot{x}_i = A_i x_i$ is a negative definite form $-x_i' Q_i x_i$.

Set $v_i(x_i) = (x_i' P_i x_i)^{\frac{1}{2}}$ and observe that there exist positive constants η_{i_1}, η_{i_2} and η_{i_3} such that

$$\eta_{i_1} \| x_i \| \leq v_i(x_i) \leq \eta_{i_2} \| x_i \| \tag{5.2}$$

$$\frac{d}{dt} v_i(x_i) \bigg|_{\dot{x}_i = A_i x_i} \leq -\eta_{i_3} \| x_i \| \tag{5.3}$$

To obtain results for the interconnected system, suppose that

$$\| h_i \| \leq \sum_{j \neq i} \| h_{ij} \| \; \| x_j \| \tag{5.4}$$

Now evaluate $\frac{d}{dt} v_i(x_i)$ along trajectories of (5.1). At points where $x_i' P_i x_i$ is nonzero,

$$\frac{d}{dt}(x_i' P_i x_i)^{\frac{1}{2}} = \frac{1}{2(x_i' P_i x_i)^{\frac{1}{2}}}[-x_i' Q_i x_i + 2x_i' P_i h_i]$$

$$\leq -\eta_{i_3} \| x_i \| + \sum_{j \neq i} \frac{\eta_{i_2}^2}{\eta_{i_1}} \| h_{ij} \| \; \| x_j \|$$

$$\leq -\frac{\eta_{i_3}}{\eta_{i_2}} v_i(x_i) + \sum_{j \neq i} \frac{\eta_{i_2}^2}{\eta_{i_1} \eta_{j_1}} \| h_{ij} \| v_j(x_j)$$

It is easy to show that the points where v_i is zero cause no essential problem, and so we have

$$\frac{dv_1}{dt} \leq -\alpha_{11} v_1 + \alpha_{12} v_2 + \ldots + \alpha_{1n} v_n$$

$$\frac{dv_2}{dt} \leq \alpha_{21} v_1 - \alpha_{22} v_2 + \ldots + \alpha_{2n} v_n \tag{5.5}$$

$$\vdots$$

$$\frac{dv_n}{dt} \leq \alpha_{n1} v_1 + \alpha_{n2} v_2 + \ldots - \alpha_{nn} v_n$$

where all α_{ij} are positive. If then the coefficient matrix on the right side of this differential inequality is the negative of an M-matrix, exponential stability of the v_i's and therefore $\| x_i \|$ follows. Of course, for sufficiently small $\| h_{ij} \|$ the M-matrix property (in fact a row-dominance property) can be guaranteed. Now observe the effect of time delays in the coupling. Thus suppose (5.1) becomes

$$\dot{x}_i = A_i x_i + \sum_{j \neq i} h_{ij} x_j (t - \tau_{ij}) \tag{5.6}$$

Then introduction of the $v_i(x_i)$ as before leads to

$$\frac{dv_1}{dt} \leq -\alpha_{11} v_1 + \alpha_{12} v_2(t - \tau_{12}) + \ldots + \alpha_{1n} v_n(t - \tau_{1n})$$

$$\vdots \tag{5.7}$$

$$\frac{dv_n}{dt} \leq \alpha_{n1} v_1(t - \tau_{n1}) + \alpha_{n2} v_2(t - \tau_{n2}) + \ldots - \alpha_{n} v_n(t)$$

The material of Section 2 confirms the stability of this equation for fixed τ_{ij}. Of course, one can study variations on this theme, along the ideas of Sections 3 and 4. One further variation on (5.1) is to allow "overlapping" of subsystems. Thus (5.1) is replaced by

$$\dot{x}_i = A_i x_i + h_i(x_1, x_2, \ldots, x_i, \ldots, x_n) \tag{5.8}$$

It is straightforward to determine the variations to the no-time-delay calculation, see e.g. [11], and to conclude that delays are possible (without destroying stability) in the "nondiagonal" coupling terms, i.e. x_j for $j \neq i$ in (5.8).

Above we have illustrated but one of a number of possibilities of deducing results for nonlinear systems. The general pattern should be clear.

6. Conclusions

The focus through this paper has been on the effect of time delay in large scale systems. In fact, we have found that the notion of connective stability can be broadened out to include time delay as well as gain variation, this time delay being of a distributed sort, as illustrated at the end of Section 3. What would be of great interest would be the establishing of an exponential stability result to cover more general forms of perturbations of the interconnecting gains than scaling or time delay.

References

[1] D. Siljak and M.B. Vukcevic, "Large-scale Systems: Stability Complexity, Reliability", *J. Frank. Inst.* Vol. 301, Nos. 1 and 2, January-February 1976, pp. 49-69.

[2] L.T. Grujic and D.D. Siljak, "Asymptotic Stability and Instability of Large-scale Systems", *IEEE Trans. Auto. Control,* Vol. AC-18, No.6, December 1973, pp. 636-645.

[3] J.C. Willems, "Lyapunov functions for diagonally dominant systems", *Automatica,* Vol. 12, No. 5, September 1976, pp. 519-523.

[4] A. Araki, "Stability of Large-Scale Nonlinear Systems-Quadratic-Order Theory of Composite-Systems Method using M-Matrices", *IEEE Trans. Auto. Control,* Vol. AC-23, No. 2, April 1978, pp. 129-142.

[5] W.H. Hahn, Theory and Application of Lyapunov's Direct Method, Prentice-Hall, New Jersey, 1963.

[6] N.N. Krasovskii, Stability of Motion, Stanford University Press, California, 1963.

[7] G.S. Ladde, "Stability of Model Ecosystems with Time-delays", *J. Theory. Biol.,* Vol. 61, 1976, pp. 1-13.

[8] G.S. Ladde, "Time-Lag versus Stability", *IEEE Trans. Auto. Control,* Vol. AC-23, No. 1, February 1978, pp. 84-85.

[9] J. Tokumaru, N. Adachi and T. Amemiya, Macroscopic Stability of Interconnected Systems", *Proc. 6th World Congress,* Paper 44-4, Boston, 1975.

[10] J.K. Hale, Ordinary Differential Equations, Wiley-International, New York, 1969.

[11] D.D. Siljak, "On Decentralized Control of Large-Scale Systems", *Preprints 7th IFAC World Congress,* Helsinki, pp. 1849-1756.

[12] G.E. Shilov and B.L. Gurevich, Integral, Measure and Derivative: A Unified Approach, Prentice-Hall, Inc. Englewood Cliffs, N.J., 1966.

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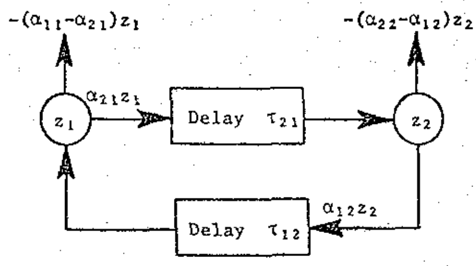

Figure 1: Physical set-up giving rise to Equation (4.1). The delays are produced by a physical storage mechanism.