TIME DELAYS IN LARGE-SCALE SYSTEMS

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ABSTRACT

The paper examines classes of large scale systems problems which have been tackled by methods associated with \( M \)-matrices and vector Lyapunov functions, and shows that the insertion of time delays into the interconnection terms will preserve exponential stability, assuming that such stability is present when no time delays exist. The key is to use new types of Lyapunov-like functions for analysing the behaviour of the delay-differential equations which are encountered. These functions allow the derivation of results for time-varying systems and systems including delay functions other than a pure delay.

1. Introduction

Among the various methods so far suggested for tackling large scale systems problems is one which brings to bear the ideas of \( M \)-matrices, often through the vehicle of vector Lyapunov functions.\(^1\)\(^-\)\(^4\). The basic idea is often to set up Lyapunov-like functions \( v_1, v_2 \ldots v_n \) for a collection of \( n \) separate subsystems and then to show that, after interconnection of the subsystems, the \( v_i \) are majorized by quantities \( w_i \) satisfying an equation set of the form

\[
v_i = -a_i w_i + a_{i1} w_1 + \ldots + a_{in} w_n + \alpha_i \quad (1.1)
\]

with the \( a_{ij} \) positive and \( a_{ii} \) nonnegative constants.

Stability of (1.1) can often be guaranteed by simple requirements, e.g. the row dominance property

\[
a_{ii} = \sum_{j=1}^{n} a_{ij} > 0
\]

for all \( i \), or the requirement that the negative of the matrix on the right side of (1.1) be an \( M \)-matrix.\(^1\)-\(^4\).

The majorization idea permits one to develop "connective stability" results, which mathematically are statements that the majorization is preserved under certain classes of perturbation in the \( v_i \) equations; these perturbations correspond to adjustment to the interconnection between the individual subsystems.

The main theme of this paper is that another form of perturbation is possible in terms of the large scale system paradigm; insertion of time delays in the interconnection does not destroy stability. In terms of (1.1), replacement of any term \( a_{ij} w_j \) by \( a_{ij} w_j(\tau_{ij}) \) for some \( \tau_{ij} \) does not destroy stability. Of course, for small time delays this result is a standard one.\(^5\)-\(^8\); But restrictions of smallness will not be applied here.

Some attempt has been made already, e.g.\(^7\)-\(^8\), to develop the types of ideas discussed in this paper with the aid of Lyapunov theory. However, because of the choice of Lyapunov-like functions\(^*\) in these works, the results appear to be more conservative. Another approach can be found in\(^9\), which in fact offers a proof of the result stated above, at least for the case when the coefficient matrix in (1.1) defined by the \( a_{ij} \) is irreducible. However, the approach of\(^9\) is inextricably tied to constant \( a_{ij} \) and \( a_{ii} \); a fact which is initially assumed for clarity in our treatment, and then easily dispensed with.

We turn now to the structure of the paper. Section 2 contains the main theorem, to the effect that with the \( a_{ij} \) positive and \( a_{ii} \) nonnegative constants, (1.1) and with (1.2) holding, constant time delays in the off-diagonal terms will not affect stability. The theorem is developed with the aid of a number of lemmas which introduce a majorization result and demonstrate a novel type of Lyapunov-like function.

In Section 3, we consider a number of variants on the ideas of the main theorem, illustrating for example possibilities of time variation, connective stability, and more complicated delay functions. Throughout this section, the time delays themselves are kept constant. In Section 4, we consider the issue of variable time delays. As one would expect, if the rate of time variation is slow enough, there is no difficulty. For larger rates of time variation however, we are also able to establish a result.

Section 5 confirms the general supposition that the ideas will apply to interconnected systems with delays in the interconnection, through the medium of vector Lyapunov functions. The calculations are straightforward, and are illustrative rather than exhaustive in that we focus on one of the same problems which reduce to looking at (1.2) or an equivalent in the no-time-delay case. The general pattern should be quite clear from the illustration.

Besides the general tenor of the results — that insertion of time delays in certain situations does not matter in a qualitative sense — perhaps the most relevant parts of this paper are those exhibiting the Lyapunov-like functions. These functions are both novel, and powerful in their application. The general conclusion about the tolerance of time delay is, in a sense, a welcome one, since large scale systems are often envisaged as involving time-delays between the subsystems, often on account of information processing requirements.

2. The Fundamental Result

The material of the later sections hinges mainly on the statement, but in part also on the techniques for proof, of the following main theorem. The theorem in effect states that the insertion of time delays into a differential equation system with connective exponential stability does not destroy the exponential stability property.

\* Certain functions are termed "Lyapunov-like" in this paper for one or two reasons. They fulfill all requirements of a Lyapunov function save for continuous differentiability everywhere, or they act as a Lyapunov function only for a restricted set of initial conditions. Naturally, Lyapunov theorems cannot be applied blindly, given such functions, although often modest adjustment suffice.
Theorem: For \( i, j = 1, 2, \ldots, n \), let the constants \( \alpha_{ij} \) satisfy
\[
\alpha_{ii} > 0 \quad \alpha_{ij} \geq 0 \quad \alpha_{ii} - \sum_{j \neq i} \alpha_{ij} > 0 \tag{2.1}
\]
and let the constants \( \tau_{ij} \) satisfy
\[
\tau_{ij} \geq 0 \tag{2.2}
\]
Then the equation set
\[
\begin{align*}
\frac{d}{dt} y_i(t) &= -\alpha_{ii} y_i(t) + \alpha_{ij} y_j(t - \tau_{ij}) + \ldots + \alpha_{in} y_n(t - \tau_{in}) \quad (2.3) \\
\frac{d}{dt} y_j(t) &= -\alpha_{jj} y_j(t) + \alpha_{ji} y_i(t - \tau_{ji}) + \ldots + \alpha_{jn} y_n(t - \tau_{jn}) \quad (2.4)
\end{align*}
\]
and with initial conditions for (2.4) and (2.5) identical. Then \( y_i(t) \) is nonnegative for all \( t \).

The preceding lemma shows that if we can prove exponential stability of (2.5), then exponential stability of (2.3) follows. Accordingly, we shall consider (2.3). Note that the initial condition assumptions on (2.5) actually limit the entries of \( y(t) \) over the appropriate initial time interval to being nonnegative, and that \( y_i(t) \) for all \( t \) is also nonnegative.

Lemma 3: With the same hypotheses as in Lemma 2, let \( y_i(t) \) satisfy (2.5), and define
\[
V(t) = \sum_{i=1}^{n} |y_i(t)| \tag{2.6}
\]
Then
\[
\dot{V} = -\sum_{i=1}^{n} z_i(t) - \sum_{i=1}^{n} |y_i(t)| \tag{2.7}
\]
Proof is by direct calculation.

The next task is to exhibit a bound away from zero of \( V/V \).

Lemma 4: Assume the same hypotheses as for Lemma 3. Also define positive constants \( k_1, k_2, k_3, k_4 \) by
\[
k_1 = \max_{i,j} \alpha_{ii} \quad k_2 = \max_{i,j} \alpha_{ij} \quad k_3 = \min_{i,j} \alpha_{ji} \quad k_4 = \min_{i,j} \alpha_{ij} \tag{2.8}
\]
and let
\[
W(t) = \sum_{i=1}^{n} |y_i(t)| \tag{2.9}
\]
Then for \( t \geq k_2 + t_0 \)
\[
\frac{d}{dt} W(t) \leq 1 + (n-1)k_3 \left[ e^{k_2 t} - 1 \right] W(t) \tag{2.10}
\]
and
\[
\dot{V} \leq \frac{k_4}{1 + (n-1)k_3 e^{k_2 t}} \tag{2.11}
\]
Proof: Omitted due to space limitations.

Lemma 5: With the same hypothesis as Lemma 2, all solutions of (2.3) with nonnegative initial conditions are exponentially stable.

Proof: Omitted due to space limitations.

1. Linear Equation Modifications Preserving Constant Time Delays

In this section, we illustrate a number of minor variations on the main theorem of the previous section.

Scaling of the \( z_i \). The main theorem is made to look a bit more mysterious in the event that the matrix
\[
A = \begin{bmatrix} -\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & -\alpha_{22} & \ldots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \ldots & -\alpha_{nn} \end{bmatrix} \tag{3.1}
\]
of (1.1) and (2.3) has all \( a_{ij} > 0 \), \( a_{ii} > 0 \), positive even order and negative odd order principal minors. (Thus \( -A \) is an \( N \)-matrix.) But then a scaling of the \( x_i \), equivalent to replacement of \( A \) by a matrix in which the row dominance property \( a_{ii} - \sum_{j \neq i} a_{ij} \geq 0 \) holds, will recover the theorem. This idea is widely used, see [1, 3].

Time variable \( \alpha_{ij} \). Only trivial adjustments are required to copy with time variable \( \alpha_{ij} \) in (2.3). Besides a smoothness and boundedness assumption, one requires uniform positivity of \( a_{ij} - \alpha_{ij} \). The exponential rate constructed in Lemma (2.5) of course reflects this time variation.

This tolerance of time variation emphasizes the comparative inappropriateness of frequency domain stability criteria for (2.3).

Connective Stability. If any \( \alpha_{ij} \) for \( i \neq j \) is replaced by \( \alpha_{ii} \) where \(-1 < \alpha_{ii}(t) \leq 1\) for all \( t \) in (2.3) but not in (1.1), the conclusion of the theorem as (1.1) is false. It is easy to show that the solutions of (1.1) which Moriarity the solutions of the defined (2.3). Moreover, stability of (2.3) easily follows as a necessary condition for connective stability.

Further time delay terms. One can conceive of more complicated ways of introducing delay time change than those reflected in (2.3). For example, the term \( a_{12}(t)z_i(t) \) could become 
\[- a_{11}(t) - a_{12}(t-\tau) \cdots - a_{12}(t-\tau) \cdots - a_{12}(t-\tau) \] 
and the term \( a_{21}(t-\tau) \) could become 
\[- a_{22}(t-\tau) - a_{22}(t-2\tau) \cdots - a_{22}(t-2\tau) \cdots - a_{22}(t-2\tau) \]
By defining
\[ a_{11} = a_{(1)}^{(1)} - a_{(2)}^{(1)} - \cdots - a_{(r)}^{(1)} \] 
\[ a_{12} = a_{(1)}^{(2)} + a_{(2)}^{(2)} + \cdots + a_{(r)}^{(2)} \]
and then requiring (2.1) to hold, the theorem becomes valid for the modified (2.3), this being shown with an easy modification of (2.6).

Perhaps the most interesting case is when all \( \alpha_{ij} \) and \( \alpha_{ij}^{(k)} \) are positive. Then the effect of having delay on the diagonal, i.e., delay in \( z_1 \) in the equation for \( z_2 \), is suggested by (3.2) to be destabilizing, a fact that is well known and can be studied for a simple equation of the form \( x = \lambda x \). On the other hand, delays in off diagonal terms, no matter how distributed, do not affect stability, i.e. in the \( z_1 \) equation, a term
\[ a_{(1)}^{(2)}(t-\tau)z_2(t-\tau) \]
will not introduce instability in a non-delayed version of the same term, viz., \[ a_{(11)}^{(2)}z_2(t) \] is consistent with stability.

Delay Functions. There has been some attempt to study delay functions, see e.g., [7, 8]. The results suggest that increase of time delay may lead to instability. This is not so, provided the "non-delayed equivalents" of the delay functions do not introduce instability, and provided the delay is not diagonal.

Both because of current interest and the somewhat greater complexity of the problem, we shall analyze the situation in a little more detail. Consider the equation
\[ z_1 = -a_1z_1 + \int_{-\tau}^{0} dz_{12}(s) z_2(s+\tau) \]  
\[ z_2 = \int_{-\tau}^{0} dz_{12}(s) z_1(s+\tau) - a_2z_2 \]  
(3.4)
(This \( n \)-dimensional case presents no extra difficulties in principle, and diagonal delay can be dealt with in the same manner as off diagonal delay, and will be seen to promote instability, as expected.) In (3.4), \( B_{12} \) and \( B_{21} \) are generating functions of bounded variation over intervals of length \( T_{12} \) and \( T_{21} \). They may include step, which will then single out terms of the type \( B_{12}(t-T_{12}) \). In view of this get the remarks made in the previous subsection we should be interested in comparing (3.4) with a non-delay system
\[ \dot{x_1} = -a_1x_1 + a_{12}x_2 \]  
\[ \dot{x_2} = a_{21}x_1 - a_{22}x_2 \]  
(3.5)
in which \( a_{12} \) is total variation of \( B_{12}(s) \) in \([-T_{12}, 0] \) and similarly for \( a_{21} \). In case \( B_{12}(s) \) is monotonic, this is of course nothing other than the difference in the values of \( B_{12}(s) \) at the end points, care being taken concerning any end-point discontinuities.)

Accordingly, suppose
\[ a_{11} > 0 \]  
\[ a_{22} > 0 \]  
\[ a_{12} > 0 \]  
\[ a_{21} > 0 \]  
(3.6)
with
\[ v(t) = \frac{\dot{x_1}}{x_1} + \frac{\dot{x_2}}{x_2} \]  
(3.7)
It is a standard result of analysis (12, p. 85) that we can write \( v(t) = v_1(t) + v_2(t) \) where \( v_1(t) \) and \( v_2(t) \) are both bounded, nonincreasing and nonoscillating with \( v_1(t) = v_2(t) = 0 \). Moreover, \( v(T_{12}) = v(T_{21}) = 0 \) and similarly for \( B_{12}(s) \). Equation (3.4) then yields
\[ \frac{d}{dt}(z_1 - a_{11}z_1 + \int_{-T_{12}}^{0} dz_{12}(s) z_2(s+\tau) - a_{12}z_2) \]  
(3.8)
and it is not hard to show that if we define \( y_1(t) \) by
\[ \dot{y_1} = -a_1y_1(t) + \int_{-T_{12}}^{0} \dot{z}_{12}(s) + a_{12}y_2(s+\tau) \]  
\[ \dot{y_2} = -a_{21}y_1(t) + \int_{-T_{12}}^{0} \dot{z}_{12}(s) + a_{22}y_2(s+\tau) \]  
(3.9)
with the same initial conditions, then \( y_1(t) \leq y(t) \) for all \( t \) and all \( T_{12}, T_{21} \). With \( \epsilon(t) \) defined as \( \epsilon(t) \) and \( \epsilon_{12} \) as step functions, this formula reduces to one obtained in the previous section.

Then \( V = -a_{11}y_1(t) + a_{12}y_2(t) \) is preserved, and we study the stability of (3.9) via the Lyapunov-like function
\[ V = y_1(t) + y_2(t) + \int_{-T_{12}}^{0} dz_{12}(s) \int_{-T_{12}}^{T_{12}} y_1(s) ds + \int_{-T_{12}}^{T_{12}} y_2(s) ds \]  
(3.9)
with the same initial conditions, then \( y(t) \leq y(t) \) for all \( t \) and all \( T_{12}, T_{21} \).

From this point on, the argument proceeds much as before. The conditions (3.6), just as they guarantee exponential stability of (3.5), guarantee exponential stability of the distributed version (3.9), and in turn of (3.4), which (3.9) majorizes.

4. Linear Equation Modification Permitting Nonconstant Time Delays
In the preceding two sections, all time delays have been considered constant. The question arises as to whether results can also be obtained for variable time delays. When the \( \alpha_{ij} \) are constant, it turns

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out that variable time delays pose no problem, provided that they are bounded, and provided that the coefficient matrix associated with the $\alpha_{ij}$ in (2.3) is irreducible, [9]. There seems no way the proof technique of [9] could be extended to, for example, remove the constancy requirement.

By setting up differential equations for $\dot{z}_t$, rather than $z_{t+1}$, one can with a little difficulty capture the result of [9]. However, two types of result can be more easily obtained. One shows that exponential stability is preserved if time delay variations are sufficiently slow. The other shows that if the delay-differential equations are adjusted appropriately, large rates of time delay variation are permitted. This adjustment will, on some occasions, make good physical sense.

Finally, we present an example of a system with time-varying $\alpha_{ij}$ and time-varying delays, and show that although the $\alpha_{ij}$ satisfy a dominance condition, the system is unstable. This shows the falsity of section any form of generalization, without some significant modification of the earlier results of this paper (for time-varying $\alpha_{ij}$ and constant delays) or of the results of [9] (for constant $\alpha_{ij}$ and variable delays) to cope with simultaneous $\alpha_{ij}$ and time delay variation.

To begin with, we shall prove a result for slow time delays. For convenience, we shall consider the case of $n=2$ and the simple equation

$$
\begin{align*}
\dot{z}_1 &= \alpha_{11}z_1(t) + \alpha_{12}z_2(t - \tau_{12}(t)) \\
\dot{z}_2 &= \alpha_{21}z_1(t - \tau_{11}(t)) - \alpha_{22}z_2(t)
\end{align*}
$$

(4.1)

We stress that the various refinements discussed in Section 3 can all be built in with little or no trouble, including time-variability of $\alpha_{11}$, etc. We assume that

$$
\alpha_{11} > 0, \quad \alpha_{12} > 0, \quad \alpha_{21} > 0, \quad \alpha_{22} > 0
$$

(4.2)

We shall also assume

$$
0 \leq \tau_{ij}(t) \leq L_{ij}, \quad \frac{d\tau_{ij}(t)}{dt} \leq L_{ij} \quad (4.3)
$$

for some constants $L_{11}, L_{22}$. Further restrictions on $\frac{d\tau_{ij}(t)}{dt}$ will be imposed subsequently.

There is of course no difficulty in introducing a majorizing equation. Let us assume this is done, so that we are now studying (4.1) with nonnegative initial conditions. Set

$$
\dot{V} = z_1(t)\dot{z}_2(t) + \int_{t-\tau_{12}(t)}^{t} \alpha_{12}z_2(s)ds + \int_{t-\tau_{11}(t)}^{t} \alpha_{21}z_1(s)ds \quad (4.4)
$$

Then

$$
\dot{V} = -[\alpha_{11} - \alpha_{12}]z_1(t) + \alpha_{12}\frac{d\tau_{12}(t)}{dt} \dot{z}_2(t - \tau_{12}(t)) + \alpha_{21}\frac{d\tau_{11}(t)}{dt} \dot{z}_1(t - \tau_{11}(t)) + \alpha_{12}z_2(t - \tau_{12}(t)) + \alpha_{22}z_2(t) \quad (4.5)
$$

For $t \leq \max(\tau_{12}(t), \tau_{11}(t))$, there is a maximum exponential rate of decay of $\{z_1, z_2\}$, e.g. $z_2(t - \tau_{12}(t)) \leq e^{\alpha_{12}\tau_{12}(t)}z_2(t)$, and $z_1(t - \tau_{11}(t)) \leq e^{\alpha_{21}\tau_{11}(t)}z_1(t)$. Thus

$$
\dot{V} \leq -[\alpha_{11} - \alpha_{12}]z_1(t) + \alpha_{12}\frac{d\tau_{12}(t)}{dt} \cdot \alpha_{12}L_{12}z_2(t) + \alpha_{21}\frac{d\tau_{11}(t)}{dt} \cdot \alpha_{21}L_{11}z_1(t)
$$

(4.6)

This will imply

$$
\dot{V} \leq \beta(z_1 + z_2) \quad (4.7)
$$

for some $\beta(t) > 0$ if $\frac{d\tau_{ij}(t)}{dt}$ are negative, or positive but suitably small. Arguments as in Section 2 will then establish stability. It is also evident that one could adjust the constant on $\frac{d\tau_{ij}(t)}{dt}$ to one involving average rates of variation of these quantities.

For the second result of this section, we postulate that $z_1(t)$ and $z_2(t)$ represent some form of physical entity with flows from one part to the other, via some delaying mechanism, see Fig. 1, together with an instantaneous flow to the rest of the universe. The delay is postulated to occur as a result of a physical storage mechanism, for example, a delay in the communicating of information. Assume for the moment the delays are constant. Thus, crudely, one can conceive of a pipe into which, at time $t$, $\alpha_{13}(t)$ units of $z_3$ flow per second, and at the other end, at the same time, $\alpha_{23}(t)$ units of $z_3$ flow out per second.

With this form of physical situation, (4.1) with constant delays is the correct model. But as soon as the delays are time-varying, a slight adjustment is required. This can be derived in the following way.

The pipe which stores entity $z_3$ has stored within it at time $t$

$$
\dot{z}_3(t) = \frac{\int_{t-\tau_{13}(t)}^{t} \alpha_{13}(s)\dot{z}_2(s)ds}{t - \tau_{13}(t)}
$$

where

$$
\dot{z}_3(t) = \alpha_{13}(t)z_2(t) - \alpha_{13}(t-\tau_{13}(t))z_2(t - \tau_{13}(t)) + \alpha_{13}(t - \tau_{13}(t)) \frac{d\tau_{13}(t)}{dt} \quad (4.8)
$$

in other words, while $\alpha_{13}(t)z_2(t)$ is still the inflow term, the outflow term in now $\alpha_{13}(t - \tau_{13}(t))z_2(t - \tau_{13}(t)) \frac{d\tau_{13}(t)}{dt} [t - \tau_{13}(t)]$ and (4.1) is accordingly replaced by

$$
\dot{z}_1 = -[\alpha_{11} - \alpha_{12}]z_1(t) + \alpha_{12}(t - \tau_{12}(t))\frac{d\tau_{12}(t)}{dt} \cdot \alpha_{12}L_{12}z_2(t - \tau_{12}(t)) + \alpha_{22}z_2(t) \quad (4.9)
$$

For the outflow term to make physical sense it must always correspond to a nonnegative quantity. More correctly therefore, the outflow term should be

$$
\alpha_{12}(t - \tau_{12}(t)) \max[1 - \frac{d\tau_{12}(t)}{dt}, 0]z_2(t - \tau_{12}(t))
$$

with corresponding adjustment in (4.8). To avoid notational complications, let us simply suppose that

$$
\frac{d\tau_{12}(t)}{dt} \leq 1 \quad (4.9)
$$

Now to study (4.8), under assumptions (4.2) and (4.9), we can again introduce a majorizing equation, and this is equivalent to studying (4.8) with nonnegative initial conditions. With the Lyapunov-like function $V$ of (4.6), we derive

$$
\dot{V} \leq -[\alpha_{11} - \alpha_{12}]z_1(t)z_1(t) - \alpha_{22}(t - \tau_{12}(t))z_2(t) \quad (4.10)
$$

and from this point on, arguments like those of Section 2 serve to demonstrate the required exponential stability.

Last, we shall illustrate an instability possibility. We shall exhibit a system with time-variable $\alpha_{ij}$ and time delays with exponential instability. Though the $\alpha_{ij}$ will not satisfy a dominance condition, on arbitrary
small perturbation will ensure satisfaction of the dominance condition. This suffices to guarantee the existence of an unstable system with the indicated time-variations and with the \( \alpha_{ij} \) satisfying the dominance condition.

Let \( z_1(0) = z_2(0) = 1 \), and over \( (0, 1) \),

\[
\begin{align*}
\dot{z}_1 &= z_2(0) \\
\dot{z}_2 &= -z_2 \\
\text{(Thus } & \alpha_{12} = 1, \alpha_{21} = 0, t - \tau, \tau(t) = 0 \text{ for all } t \in (0, 1). \text{ These result } z_1(1) = 0, z_2(1) = e^{-1}. \text{ Now over the interval } [1, T] \text{ with } T \text{ specified below suppose that:} \\
\dot{z}_1 &= -z_1 \\
\dot{z}_2 &= z_1(1) \end{align*}
\]

Since \( \alpha_{12} = 1, \alpha_{21} = 0 \), \( t - \tau, \tau(t) = 1 \). Then \( z_2(1) = 2e^{-1} - 1 \) and \( z_2(T) = e^{-1} + 2(T-1) \). By taking the appropriate limit we obtain \( z_1(T) = 1.27e^{-1} \).

Evidently, over an interval of length \( T \), both \( z_1 \) and \( z_2 \) have grown. Clearly, the growth can be amplified over \([T, 2T] \). \([2T, 3T] \), etc., and exponential instability is manifest. Also, it is clear that not only can the \( \alpha_{ij} \) be perturbed by an arbitrarily small amount to meet the dominance requirement, but they can be made to vary smoothly, as can the \( T_{ij} \) without the exponential instability being lost.

5. Modifications Involving Nonlinear Equations

The stability properties of interconnected separately stable systems can sometimes be examined with the aid of linear differential equations or inequalities for the Lyapunov functions of the separate systems; this idea has been a constant theme in a number of works, e.g., [2, 11]. We show here that when the stability of interconnected systems can be deduced in this way, then introduction of time delays in the interconnections will not affect the stability.

The earlier work has been concerned with pointing out that a connective stability property also holds, that is, the gain linking the separate systems can be varied between \(-1 \) and \( 1 \). In the light of this tolerance of both time delays and gain variation it is natural to ask what other forms of adjustment would be tolerable. No satisfactory consequence has suggested an asymptotic stability result which simply demands that any form of adjustment definable by an operator with norm ("gain") \( \leq 1 \), but this result so far does not extend to exponential stability.

Let us suppose that there are \( n \) linear systems with possibly nonlinear interconnection such that after application of a local stabilizing control to each system, one has

\[
\dot{x}_i = A_i x_i + h_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) i = 1, 2, \ldots, n
\]

(Each \( x_i \) is, in general, a vector rather than scalar.) For the system \( \dot{x}_j = A_j x_j \), there is a positive definite quadratic form \( x_j^T P_j x_j \) whose derivative along trajectories of \( \dot{x}_j = A_j x_j \) is a negative definite form \( -x_j^T Q_j x_j \). Set \( v_i(x_j) = (x_j^T P_j x_j)^{1/2} \) and observe that there exist positive constants \( \eta_{i1}, \eta_{i2}, \eta_{13} \) such that

\[
\eta_{i1} \| x_i \| \leq v_i(x_j) \leq \eta_{i2} \| x_i \|. \tag{5.3}
\]

To obtain results for the interconnected system, suppose that

\[
\| h_i \| \leq \sum_{j=1}^{\infty} \| h_{ij} \| \| x_j \|. \tag{5.4}
\]

Now evaluate \( \dot{v}_i(x_j) \) along trajectories of (5.1). At points where \( x_j^T x_j < \infty \) is nonzero,

\[
\frac{d}{dt} v_i(x_j) = \frac{\delta}{d} (x_j^T x_j)^{1/2} = \frac{1}{2} (x_j^T P_j x_j)^{1/2} \sum_{j=1}^{\infty} h_{ij} h_{jk} \| x_j \| \| x_k \| \|
\]

It is easy to show that the points where \( v_i \) is zero cause no essential problem, and so we have

\[
\dot{v}_i(x_j) = -\alpha_i v_i + \alpha_j v_i + \ldots + \sum_{j=1}^{\infty} v_i (x_j - \tau_i x_i) \tag{5.5}
\]

Then introduction of the \( \dot{v}_i(x_j) \) as before leads to

\[
\frac{d}{dt} v_i(x_j) \leq -\alpha_i v_i + \alpha_j v_i + \ldots + \sum_{j=1}^{\infty} v_i(x_j - \tau_i x_i) \tag{5.6}
\]

The material of Section 2 confirms the stability of this equation for fixed \( \tau_{ij} \). Of course, one can study variations on this theme, along the ideas of Sections 3 and 4. One further variation on (3.1) is to allow "overlapping" of subsystems. Thus (3.1) is replaced by

\[
\dot{x}_i = A_i x_i + h_i(x_1, x_2, \ldots, x_i, \ldots, x_n) \tag{5.8}
\]

It is straightforward to determine the variations to the no-time-delay calculation, see e.g., [11], and to conclude that delays are possible (without destroying stability) in the "nonoverlapping" coupling terms, i.e., \( x_i \) for \( j \neq i \) in (5.8).

Above we have illustrated but one of a number of possibilities of deducing results for nonlinear systems. The general pattern should be clear.

6. Conclusions

The basis for writing this paper has been on the effect of time delay in large scale systems. In fact, we have found that the notion of connective stability can be broadened out to include time delay as well as gain variation, this time delay being of a distributed sort, as
illustrated at the end of Section 3. What would be of great interest would be the establishing of an exponential stability result to cover more general forms of perturbations of the interconnecting gains than scaling or time delay.

References


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