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New Multivariable Frequency-Domain Design Techniques

N.T. HUNG and B.D.O. ANDERSON
Department of Electrical Engineering, The University of Newcastle

SUMMARY An efficient design procedure for the control of multivariable feedback systems must be evaluated against the following criteria: ease of design procedure, computational burden in design procedure, complexity of compensator, and quality of closed loop performance. Although the most direct design approach is perhaps to reduce the multivariable design problem to independent scalar design problems, the existing design techniques tend to use trial-and-error procedures to achieve this aim. The following paper presents two new techniques that have been developed to solve this problem systematically, the difference between the two techniques being in the structure of the compensated system: the complete decomposed diagonal form or the partly decomposed triangular form with a special property-namely triangular-diagonal-dominance. The new techniques are illustrated by two practical examples: an unstable chemical reactor and a pressurized flow-box.

1 INTRODUCTION

The paper studies systematic procedures which reduce a multivariable control problem to a collection of independent scalar design problems. Similar in aim to current techniques devised by Rosenbrock, MacFarlane, Mayne and Owens [1-4], the new design methods search for simple compensation schemes in the frequency domain in order to achieve the following design objectives: closed-loop stability, good tracking of step inputs, control over plant input magnitudes, low interaction, and security against sensor and actuator failures.

More precisely, the new techniques proceed by the diagonalization or triangularization of a multivariable plant using one or more compensators which are proper, stable and minimum phase [5]. When a diagonal matrix is obtained this way, classical design techniques may be applied independently to obtain a desirable multivariable closed-loop system. Even for a triangular matrix we shall argue that it is still possible to carry out independent scalar closed-loop designs provided that the matrix has a special property named triangular-diagonal-dominance (TDD). Roughly, a TDD matrix is a triangular matrix whose diagonal elements suffice to determine the stability properties of the system. (As an example, a triangular matrix with stable off-diagonal elements is TDD). The main result is that if a compensated plant transfer matrix is TDD, then only diagonal elements need to be involved in closed-loop design, i.e. off-diagonal elements can be neglected in the design process, in the knowledge that their presence in the real situation will not substantially degrade the closed-loop performance [6].

Therefore, two compensator design stages are necessary. The first set of compensators serves to diagonalize or triangularize the original plant, while the second set which is obtained by using classical design techniques aims at achieving the desirable design objectives. As a consequence, the new design approaches are essentially simple, systematic and require little prior design experience other than that necessary for classical (scalar) design.

The structure of the paper is as follows. Section 2 is concerned with the process of diagonalization and triangularization. Section 3 deals with the closed-loop design approach via diagonalization and Section 4 presents the TDD design approach. In Section 5, the techniques are illustrated by two practical examples: an unstable chemical reactor and a pressurized flow-box. Furthermore, a comparison with other techniques is made in terms of the simplicity of compensators, the difficulty of design procedures, and the closed-loop performance.

2 RESULTS ON DIAGONALIZATION AND TRIANGULARIZATION

In this section, we shall restrict attention to obtaining a diagonal or triangular structure for the compensated plant. We may require temporary use of a post compensator as well as a precompensator for the diagonal structure, and in the case of the triangular structure, a precompensator is sufficient.

In any multivariable design procedure, it is crucial that a compensator U(s) be realizable, and it is highly desirable that it be stable and minimum phase* to eliminate the possibility of unstable pole-zero cancellation between the compensator and the plant. The transfer matrix of this type of compensator may be singular at $s = \infty$. We shall denote the transfer matrix of such a compensator by V(s) and the class of all such transfer matrices by $V^{n\times n}$. A more appealing class of compensator $U^{n\times n}$ is a subset of the class $V^{n\times n}$ with transfer matrices which are proper, stable, minimum phase and nonsingular at $s = \infty$. (This last condition causes less overall phase lag than a compensator in the class $V^{n\times n}$.) Thus we seek a mechanism to design compensators to satisfy these constraints as well as to achieve diagonalization and triangularization.

The existence and the construction procedures of such compensators are given in the following theorems.

* Roughly speaking, a square compensator is minimum phase if its determinant has this property. More precisely, the compensator zeros [9] must be in Re[s] < 0.

Theorem 1: Let $G(s)$ be a proper, stable transfer matrix. Then there exists a compensator $U(s) \in U^{n\times n}$ such that

$$G(s)U(s) = T(s)$$

where $T(s)$ is proper, stable and triangular.

Theorem 2: Let $G(s)$ be a proper, stable transfer matrix. Then there exist compensator $U_1(s)$, $U_2(s) \in U^{n\times n}$ such that

$$U_1(s)G(s)U_2(s) = D(s)$$

where $D(s)$ is proper, stable and diagonal.

Theorem 3: Let $G(s) \in V^{n\times n}$. Then there exists a compensator $V(s) \in V^{n\times n}$ such that

$$G(s)V(s) = D(s)$$

where $D(s)$ is proper, stable, minimum phase and diagonal.

The proofs for these theorems are given in [5].

Remarks:

1. Theorems 1 and 2 can be extended to unstable $G(s)$.
2. Construction techniques for the compensators of the theorems are given in [6].

3 DIAGONALIZATION DESIGN TECHNIQUE

This design approach first searches for compensators so that the compensated plant has a diagonal structure. Then an additional compensator may be found by the use of classical design techniques in order to satisfy the design objectives.

Phase 1. Diagonalization

If $G(s) \in V^{n\times n}$, then by Theorem 3, we can find a precompensator $V(s) \in V^{n\times n}$ such that

$$D(s) = G(s)V(s)$$

is diagonal. If $G(s)$ is stable but does not necessarily belong to $V^{n\times n}$, by Theorem 2, we can find a precompensator $U_2(s)$ and a postcompensator $U_1(x) \in U^{n\times n}$ such that

$$D(s) = U_1(s)G(s)U_2(s)$$

is diagonal. If $G(s)$ is not stable, the following procedure is used. Find $L(s) = \frac{\ell(s)}{m(s)} I$ such that $L(s)G(s)$ is proper, and after cancellations, stable. Then find $U_1(s)$, $U_2(s) \in U^{n\times n}$ such that U_1LGU_2 is diagonal. Because $L(s)$ can commute with any other matrix, it is clear that U_1GU_2 is also diagonal.

Phase 2. Closed-loop Design

As the compensated plant $D(s)$ has a diagonal structure, the system is completely decomposed and classical design techniques may be used to obtain a precompensator $K(s)$ in order to have a nice closed-loop system.

The diagonalization approach often requires a post-compensator as well as a precompensator. Since postcompensators are often impractical in practice, the following procedure is used. Temporarily introduce a postcompensator U_1 in order to have the necessary diagonal structure, and at the end of the design, moving it back round the feedback loop [1]. As a consequence the final controller is

$$C = U_2KU_1$$

It is clear that this way we eliminate the need for postcompensators, but with a possibility of slightly inferior closed-loop performance.

Remark:

In general, the compensation scheme for the diagonalization design technique is more complicated than necessary. In our experi~~~e, the triangularization diagonal-dominant design technique in the next section will achieve comparable performance with less complicated compensators and greater design freedom. However, the diagonalization design approach is important in its own right as there are cases where this technique would give better results.

4 TRIANGULAR-DIAGONAL-DOMINANCE DESIGN TECHNIQUE

This design approach does not require complete decomposition in order to apply classical techniques for design purposes. By Theorem 1, we can construct a precompensator $U(s)$ so that the compensated system GU has a triangular structure. The crucial result for the method is that it is possible to carry out independent scalar closed-loop design techniques provided that the matrix GU has a special property, namely triangular diagonal dominance (TDD). This property may be defined as follows.

For any proper transfer matrix $X(s)$, let $X_+(s)$ denote the sum of those strictly proper terms in a partial fraction expansion of $X(s)$ with poles in $Re[s] \geq 0$ [7-8]. Let $\Delta(s)$ be the least common denominator monic polynomial of all minors of X, let $\delta[\Delta(s)]$ denote its polynomial degree and let $\delta[X(s)]$ denote the McMillan degree of $X(s)$ [9]

$$\delta[X(s)] = \delta[\Delta(s)] \quad (4.1)$$

For any square transfer function matrix $X(s)$, let

$$\tilde{X}(s) = \text{diag}[x_{11}(s), \ldots, x_{nn}(s)]$$

Definition: A triangular matrix $P(s)$ is triangular-diagonal-dominant (TDD) if

$$\delta[P_+(s)] = \delta[\tilde{P}_+(s)] \quad (4.2)$$

This is equivalent to the condition $\Delta_+(s) = \tilde{\Delta}_+(s)$ where $\Delta_+(s)$ and $\tilde{\Delta}_+(s)$ are the least common denominator monic polynomials of all minors of $P_+(s)$ and $\tilde{P}_+(s)$ respectively. Roughly speaking, the condition says <u>any instability off the diagonal of $P(s)$ must also show up on the diagonal</u>.

Phase 1. Triangularization:

If $G(s)$ is stable, by Theorem 1, we can find a precompensator $U(s) \in U^{n\times n}$ such that

$$P(s) = G(s)U(s)$$

is triangular. If $G(s)$ is not stable, the following procedure is used. Find $L(s) = \text{diag}[\ell_i(s)/m_i(s)]$ such that $L(s)G(s)$ is proper, and after cancellations, stable. Find $U(s)$ such that LGU is triangular, and since $L(s)$ is diagonal, GU is triangular.

Phase 2. Checking the TDD property and closed-loop design

The main results for this design technique are contained in the following lemmas and theorem, proved in [6].

Lemma 1: With $G(s)$ a square transfer matrix, either all $P(s)$ or no $P(s)$ obtained by the Phase 1 procedure are TDD.

Suppose that $G(s)$ can be precompensated to give a TDD $P(s)$. The following theorem shows that the off-diagonal elements can be ignored in the design process, and using classical design techniques, each loop is closed by a scalar compensator to give a good closed-loop performance for the scalar plant defined by $p_{ii}(s)$. It turns out that the resulting diagonal compensator also stabilizes $P(s)$.

Theorem 4: Suppose $G(s)$ can be precompensated to TDD form $P(s)$, and let $K(s) = \text{diag}[k_i(s)]$ be such that the compensator $K(s)$ stabilizes $P(s)$, so that $p_{ii}k_i(1 + p_{ii}k_i)^{-1}$ is stable for each i, and there are no unstable pole-zero cancellations between p_{ii} and k_i. Then $K(s)$ also stabilizes $P(s)$.

In phase 2, the TDD property of $P(s)$ may be checked using the definition. However, in order to facilitate the checking operations, sufficient conditions guaranteeing the TDD property are contained in the following Lemma.

Lemma 2: $G(s)$ can be precompensated to TDD form if any of the following conditions hold:

1. $G(s)$ is stable
2. The off-diagonal part of $P(s) = G(s)U(s)$ is stable.
3. Any unstable pole in the off-diagonal part of $P(s)$ occurs in only one off-diagonal entry and then also occurs in the diagonal element in either the same row or same column, with at least the same multiplicity.
4. $\delta[|G(s)|_+] = \delta[G_+(s)]$ (4.3)

Equation (4.3) suggests and experience bears out, that for practical examples of $G(s)$ selected from real life, the TDD property can be expected. However, one can readily continue examples where $G(s)$ can not be precompensated to TDD form, e.g.

$$G(s) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{s-2} & \frac{1}{s-3} \end{bmatrix} \quad (4.4)$$

The failure of TDD is associated with modes which are both unstable and not on the diagonal. It follows that they should be forced onto the diagonal. The following procedure has been found to work empirically. Premultiply $G(s)$ by a constant nonsingular matrix M to form $\hat{G}(s) = MG(s)$. Then $\hat{G}(s)$ can be precompensated to TDD form. Once compensators $U(s)$, $K(s)$ have been found for $\hat{G}(s)$, the compensator $U(s)K(s)M$ is then used for $G(s)$. This corresponds to temporarily introducing a post-compensator M and then at the end of the design, moving it back round the feedback loop.

Remarks:

1. The TDD design approach requires three stages in general. The first stage is concerned with achieving a triangular structure for the precompensated plant. The second stage requires an additional compensator if necessary (followed by retriangularization) in order to obtain the TDD property. Then using classical design techniques, the last stage aims to satisfy the design objectives.

2. Interaction is caused by signals from one loop entering another loop; accordingly, high loop gains, which serve to reject unwanted signal disturbances, must reduce interaction. Some gains are more important than others. For example, consider the 2×2 case; large k_2, rather than large k_1, helps reduce interaction. High k_2 cuts down the effect of unwanted signals in loop 2, the unwanted signal here being coupling from loop 1. On the other hand, there is minor coupling from loop 2 to loop 1 because of the triangular structure.

5 DESIGN EXAMPLES

In order to illustrate the new design techniques, we carry out the basic procedures for the controller design of two practical plants: an unstable chemical reactor and a pressurized flow-box. The problem is to search for a compensator so that the closed-loop system is stable, has a good tracking property for step inputs, and reasonable magnitudes for plant inputs. In addition, interactions are kept as low as possible and some security against sensor and actuator failures can be easily incorporated in the new design approaches.

Example 1. (Unstable Chemical Reactor)

The control problem of an unstable chemical reactor provided by ICI was studied in [10] using the Sequential Return Difference Technique [3].

With the organic material flow (u_1) and the cooling water flow (u_2) as the plant inputs and the reaction pressure (y_1) and the jacket temperature (y_2) as the plant outputs, the open loop transfer function $G(s)$ for the reactor is

$$\frac{\begin{bmatrix} -(2.63+11.11s+2.11s^2) & 233.5+29.2s \\ 1.515+0.943s & -96.5-58.25s+43.83s^2+5.68s^3 \end{bmatrix}}{\rho_0(s)}$$

with

$$\rho_0(s) = (s-0.0974)(s-1.766)(s+5.061)(s+8.673)$$

Using the Sequential Return Difference approach, the following precompensator is selected in [10]:

$$C(s) = \begin{bmatrix} \frac{-50(s+0.4)}{s} & 0 \\ 0 & \frac{10(s+1.2)}{s} \end{bmatrix}, \quad \delta(C) = 2$$

Diagonalization Technique: In order to completely decouple the system, a postcompensator as well as a precompensator must be used for Phase 1.

$$U_1(s)G(s)U_2(s) = D(s) = \begin{bmatrix} d_1(s) & 0 \\ 0 & d_2(s) \end{bmatrix}$$

Using the approach of [5] and [6], one obtains

$$U_1(s) = \begin{bmatrix} 1 & 0 \\ \frac{0.4469(s+1.607)}{(s+0.2484)(s+5.017)} & 1 \end{bmatrix}$$

$$U_2(s) = \begin{bmatrix} 1 & \frac{13.839(s+7.997)}{(s+0.2484)(s+5.017)} \\ 0 & 1 \end{bmatrix}$$

$$d_1(s) = \frac{-2.11(s+0.2484)(s+5.017)}{(s-0.0974)(s-1.766)(s+5.061)(s+8.673)}$$

$$d_2(s) = \frac{5.68(s+1.104)}{(s+0.2484)(s+5.017)}$$

Now classical design techniques may be used and the following choice of feedback loop gains ensures that the steady state error for the first loop is less than 3.8% and the error for the second loop is less than 2%.

$$K = \begin{bmatrix} -70 & 0 \\ 0 & 70 \end{bmatrix}$$

The final design for the controller is obtained by moving the postcompensator $U_1(s)$ round the feedback loop

$$C = U_2KU_1 = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

where

$$C_{11} = \frac{-70(s+0.8352)(s+0.9645)[(s+5.201)^2+(0.8746)^2]}{[(s+0.2484)(s+5.017)]^2}$$

$$C_{12} = \frac{140(S+8)}{(S+\frac{1}{4})(S+5)}$$

$$C_{21} = \frac{4.469(s+1.607)}{(s+0.2484)(s+5.017)}$$

$$C_{22} = 0$$

$$\delta(C) = 4$$

TDD Technique: A precompensator is required to obtain the triangularization structure as per [6]:

$$G(s)U(s) = \begin{bmatrix} \frac{-2.11(s+0.2484)(s+5.017)}{\rho_0(s)} & 0 \\ \frac{0.943(s+1.607)}{\rho_0(s)} & \frac{5.68(s+1.104)}{(s+0.2484)(s+5.017)} \end{bmatrix}$$

where

$$U(s) = \begin{bmatrix} 1 & \frac{13.839(s+7.997)}{(s+0.2484)(s+5.017)} \\ 0 & 1 \end{bmatrix}$$

As the precompensated system is TDD, classical design techniques may be used and similar to the previous technique, the following feedback loop gains are chosen to ensure that the steady state errors for the two loops are 3.8% and 2% respectively.

$$K = \begin{bmatrix} -70 & 0 \\ 0 & 10 \end{bmatrix}$$

Consequently, the following precompensator is chosen

$$C(s) = U(s)K = \begin{bmatrix} -70 & \frac{138.39(s+7.997)}{(s+0.2484)(s+5.017)} \\ 0 & 10 \end{bmatrix}, \delta(C) = 2$$

The step responses and plant inputs for the closed-loop systems designed by these three methods are shown in Figures 1 and 2. The precompensators designed by the diagonal and TDD techniques perform comparably, however the TDD compensator is much simpler (McMillan degree = 2). Steady state errors for the first compensator are zero because of the integral feedback, and for the new compensators, the error is 3.8% for the first loop, 2.0% for the second. Overshoots in the 1-1 term, and interaction in general are better for the new compensators, but plant input magnitudes for step reference inputs are greater by a factor of approximately 2.

Example 2 (Pressurized Flow-Box)

The control problem of a pressurized flow-box was studied in [11] using the Characteristic Locus Technique [2].

With the stock inflow and the air inflow as the plant inputs and the total head and liquid level as the plant outputs, the transfer function matrix for this pressurized flow-box is given by:

$$G(s) = \begin{bmatrix} \frac{0.0336}{(s+0.395)} & \frac{1.03s}{\alpha(s)} \\ \frac{(9.66s+0.117)10^{-4}}{\alpha(s)} & \frac{-0.0114}{\alpha(s)} \end{bmatrix}$$

where

$$\alpha(s) = (s+0.3947)(s+3.1925 \times 10^{-4})$$

Using the Characteristic Locus approach, the following precompensator is selected in [11]:

$$C(s) = \begin{bmatrix} \frac{117.2}{s} & 1030 \\ 99.6 + \frac{0.12}{s} & -33.6 - \frac{1.1}{s} \end{bmatrix}, \delta(C) = 2$$

Diagonalization Technique and TDD Technique: It is possible to diagonalize the system using only precompensators because the plant is minimum phase. The approximated compensators designed by the diagonalization technique and TDD technique are respectively.

$$\begin{bmatrix} \frac{385}{s+0.3972} & 3065 \\ \frac{37.62(s+0.01211)}{(s+0.3972)} & -100 \end{bmatrix} \text{ and } \begin{bmatrix} 1000 & 3065 \\ 0 & -100 \end{bmatrix}$$

The step responses and plant inputs for the closed-loop systems designed by the three techniques are shown in Figures 3 and 4. For all three designs interaction is very small and overshoot is non-existent. The compensators designed by the new techniques perform comparably, except that the simpler TDD compensator gives slightly better plant input magnitudes and the diagonalization compensator produces smaller interaction. Steady state errors for all compensators are comparable and very small ($\leq$ 1%).

6 CONCLUSIONS

We have presented two methods of reducing a multivariable design problem to a collection of independent scalar design problems using an appropriate precompensation scheme. The diagonalization technique deals with the possibility of achieving exact decoupling and with this diagonal structure, feedback controllers may be designed independently. The TDD technique relies on the triangular-diagonal-dominance property of a compensated system, since feedback controllers may be obtained independently using classical design techniques when the property is satisfied. The additional advantage of using the diagonal or triangular structure is that certain knowledge on the step response, interaction, steady state errors, and plant input magnitudes for the multivariable system is readily available when performing classical design techniques, thus simplifying the design task greatly.

The new methods can be compared against other methods using at least four different criteria: ease of use in terms of requirements for designer experience, computational burden, complexity of controller, and quality of performance. By eliminating some of the trial-and-error in existing frequency domain approaches, the method scores well on the requirements for designer experience. It is hard to compare the computational burden of graphically based schemes like Characteristic Locus and "analytic" schemes like Sequential Return Difference and those of this paper. To the extent that a comparison is possible it appears that for low order systems, the analytic schemes may be advantageous. For the two examples chosen, the new methods perform acceptably in relation to complexity of controller and quality of performance. These conclusions are consistent with the authors' experience in other examples.

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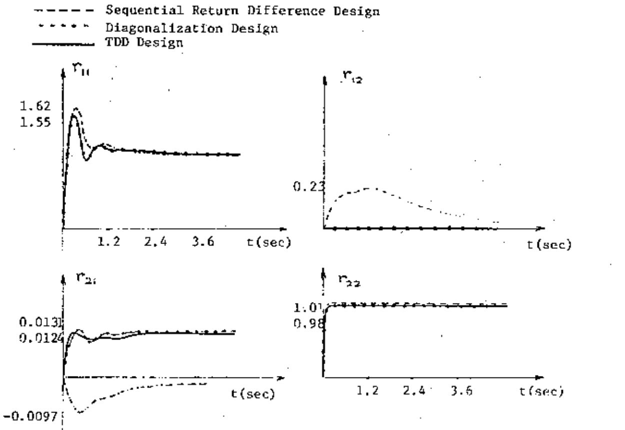

Figure 1 Output step responses for design Example I (r_{ij}: Plant output i for step on reference input j)

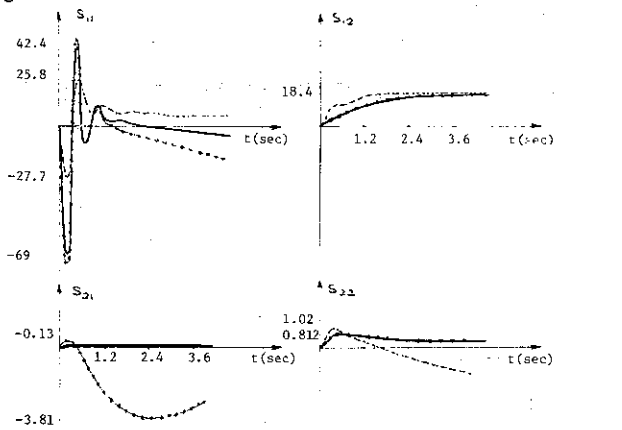

Figure 2 Plant inputs for design Example I (s_{ij}: Plant input i for step on reference input j)

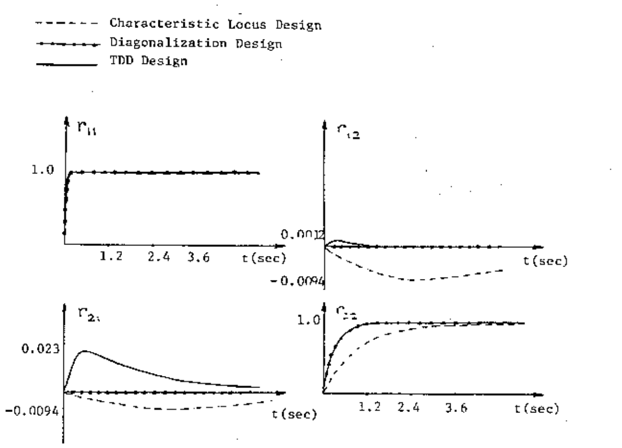

Figure 3 Output step response for design Example 2 (r_{ij}: Plant output i for step on reference input j)

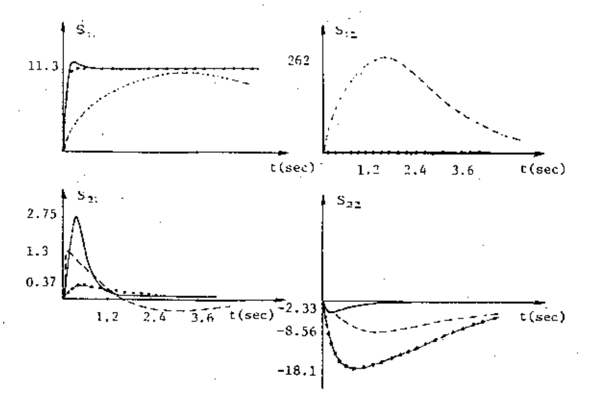

Figure 4 Plant inputs for design Example 2 (s_{ij}: Plant input i for step on reference input j)